



Modeling and Control of Distributed Parameter Systems: the Port Hamiltonian Approach

Part3: Input/Output

Yann Le Gorrec

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Outline



1. Introduction
2. Considered PDEs
3. C_0 Semigroups
4. PHS and generator of C_0 semigroups
5. Boundary control systems



In the first part of this course we focused on properties of Distributed Parameter Systems *i.e.*

$$f = \mathcal{J}e, e_{\partial}, f_{\partial}, s.t.$$

$$(f, f_{\partial}, e, e_{\partial}) \in \mathcal{D}_{\mathcal{J}}$$

In Parts 2-3 we focus on solutions associated with the PDE:

- *proving existence of solutions by using the semigroup theory*
- *studying the conditions for asymptotic or exponential stability*
- *designing stabilizing controllers*

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Considered class of PDEs (1D)

$$\frac{\partial x}{\partial t}(t, z) = \mathcal{J} \delta_x \mathcal{H}(x(t, z)), \quad \text{with } \mathcal{J} \text{ skew sym. diff. operator}$$



Considered class of PDEs (1D)

$$\frac{\partial x}{\partial t}(t, z) = P_1 \frac{\partial}{\partial z} (\mathcal{L}(z)x)(t, z) + (P_0 - G_0)\mathcal{L}(z)x(t, z)$$

$P_1 = P_1^\top$, $P_0 = -P_0^\top$, $G_0 \geq 0$, $x \in \mathbb{R}^n$, $z \in (a, b)$, $\mathcal{L}(z) = \mathcal{L}(z)^\top > 0$. State space $X = L_2(a, b; \mathbb{R}^n)$ with $\langle x_1, x_2 \rangle_{\mathcal{L}} = \langle x_1, \mathcal{L}x_2 \rangle$ and the norm $\|x_1\|_{\mathcal{L}}^2 = \langle x_1, x_1 \rangle_{\mathcal{L}}$.



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The norm $\|\cdot\|_{\mathcal{L}}^2$ is equivalent to the **energy** of the system

Applications

- Mechanical systems, magneto-electro-mechanical, chemical, etc...
- Some beam and wave equations, Maxwell equations, transmission lines, vibrating strings, Saint-Venant equations, ...
- But also by using appropriate extension + closure relations: heat transmission, diffusion systems, tubular reactors, etc...

Boundary controlled port Hamiltonian systems



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Boundary port variables

Let $\mathcal{L}x \in H^1(a, b; \mathbb{R}^n)$. Then the boundary port variables are the vectors $e_{\partial, \mathcal{L}x}, f_{\partial, \mathcal{L}x} \in \mathbb{R}^n$,

$$\begin{bmatrix} f_{\partial, \mathcal{L}x} \\ e_{\partial, \mathcal{L}x} \end{bmatrix} = U \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{bmatrix} = R \begin{bmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{bmatrix}.$$

Where

$$U^\top \Sigma U = \Sigma, \quad \Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \Sigma \in M_{2n}(\mathbb{R})$$



In finite dimension, linear systems can be described using first-order differential equation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

with solutions expressed through:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

The idea of semigroups ([Curtain and Zwart, 1995, Jacob and Zwart, 2012]) is to generalize the notion of e^{At} to abstract systems defined on Hilbert space by:

$$\dot{x}(z, t) = Ax(z, t), x(z, t) \in D(A), x(z, 0) = x_0$$

In what follows the semigroup associated to the generator A is noted $T(t)$.

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Definition of C_0 Semi group

Let X be a Hilbert space. $(T(t))_{t \geq 0}$ is called a strongly continuous semigroup (or C_0) semigroup if the following Holds:

1. For all $t \geq 0$, $T(t)$ is a bounded linear operator on X , i.e., $T(t) \in \mathcal{L}(X)$;
2. $T(0) = I$;
3. $T(t + \tau) = T(t)T(\tau)$ for all $t, \tau \geq 0$;
4. For all $x_0 \in X$, we have that $\|T(t)x_0 - x_0\|_X$ converges to zero, when $t \downarrow 0$ i.e. $t \mapsto T(t)$ is strongly continuous at zero.

Even if it has been defined for infinite dimensional systems it can be used in \mathbb{R}^n . In this case $T(t) = e^{At}$. Properties can be checked using

$$T(t)x = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle x, \phi_n \rangle \phi_n$$



Properties of C_0 Semi group

A strongly continuous semigroup $(T(t))_{t \geq 0}$ on X has the following properties:

1. $\|T(t)\|$ is bounded on every finite sub-interval of $[0, \infty)$;
2. The mapping $t \mapsto T(t)$ is strongly continuous on the interval $[0, \infty)$;
3. For all $x \in X$ we have that $\frac{1}{t} \int_0^t T(s)x ds \rightarrow x$ as $t \downarrow 0$;
4. If $\omega_0 = \inf \left(\frac{1}{t} \log \|T(t)\| \right)$ then $\omega_0 = \lim \left(\frac{1}{t} \log \|T(t)\| \right) < \infty$
5. For every $\omega > \omega_0$, $\exists M_\omega$ such that for every $t \geq 0$ we have $\|T(t)\| \leq M_\omega e^{\omega t}$.

Definition of infinitesimal generator

Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup on the Hilbert space X . If the following limit exists

$$\lim_{t \downarrow 0} \frac{T(t)x_0 - x_0}{t} \Rightarrow x_0 \in D(A)$$

we define A the infinitesimal generator of the strongly continuous semigroup by

$$Ax_0 = \lim_{t \downarrow 0} \frac{T(t)x_0 - x_0}{t}$$



Theorem

Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup on X with infinitesimal generator A . Then the following results hold:

1. For $x_0 \in D(A)$ and $t \geq 0$ we have $T(t)x_0 \in D(A)$;
2. $\frac{d}{dt}(T(t)x_0) = AT(t)x_0 = T(t)Ax_0$ for $x_0 \in D(A)$, $t \geq 0$;
3. A is a closed linear operator;

It means that for $x_0 \in D(A)$ the function $x(t) = T(t)x_0$ is a solution of the abstract differential equation:

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \quad (1)$$

Definition

A differentiable function $x : [0, \infty) \rightarrow X$ is called classical solution of (1) if $\forall t \geq 0$ we have $x(t) \in D(A)$ and equation (1) is satisfied.

Lemma

Let A be the infinitesimal generator of C_0 semigroup $(T(t))_{t \geq 0}$. Then for every $x_0(t) \in D(A)$ the map $t \mapsto T(t)x_0$ is the unique classical solution of (1).

C_0 Semi group

Definition

Let $(T(t))_{t \geq 0}$ be a C_0 semigroup on X . Then $(T(t))_{t \geq 0}$ is called a contraction semigroup if $\|T(t)\| \leq 1$ and unitary semigroup if $\|T(t)\| = 1$ for every $t \geq 0$.

Definition

A linear operator $A : D(A) \subset X \rightarrow X$ is called dissipative if

$$\operatorname{Re}\langle Ax, x \rangle \leq 0, \quad x \in D(A)$$

Lumer-Phillips Theorem

Let A be a linear operator with domain $D(A)$ on X . Then A is the infinitesimal generator of a contraction semigroup $(T(t))_{t \geq 0}$ on X if and only if A is dissipative and $\operatorname{ran}(I - A) = X$

Theorem

Let A be a linear, densely defined and closed operator on X . Then A is the infinitesimal generator of a contraction semigroup $(T(t))_{t \geq 0}$ on X if and only if A and A^* are dissipative.

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Port Hamiltonian systems



Port Hamiltonian systems

Let W be a $n \times 2n$ real matrix. If W has full rank and satisfies $W\Sigma W^T \geq 0$ ($W\Sigma W^T = 0$), then the operator $\mathcal{A}x = P_1(\partial/\partial z)(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain

$$D(\mathcal{A}) = \left\{ \mathcal{L}x \in H^1(a, b; \mathbb{R}^n) \mid \begin{bmatrix} f_{\partial, \mathcal{L}x}(t) \\ e_{\partial, \mathcal{L}x}(t) \end{bmatrix} \in \ker W \right\}$$

generates a contraction semigroup (unitary semigroup) on X .

Sketch of proof

We use the property

$$\langle e, \mathcal{J}e \rangle + \langle \mathcal{J}e, e \rangle = \begin{pmatrix} f_{\partial}^T & e_{\partial}^T \end{pmatrix}^T \Sigma \begin{pmatrix} f_{\partial} \\ e_{\partial} \end{pmatrix}$$

to prove that with $D(\mathcal{A})$ with rank and positivity condition the operator and its adjoint are dissipative.

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Boundary control systems



We are interested in abstract control systems of the form:

$$\begin{aligned}\dot{x} &= \mathcal{A}x(t), \quad x(0) = x_0, \\ \mathcal{B}x(t) &= u(t),\end{aligned}\tag{2}$$

Definition

The control system 2 is a boundary control system if the following hold:

1. The operator $A : D(A) \rightarrow X$ with $D(A) = D(\mathcal{A}) \cap \ker(\mathcal{B})$ and

$$\mathcal{A}x = Ax \text{ for } x \in D(A)$$

is the infinitesimal generator of a C_0 semigroup.

2. There exists an operator $B \in \mathcal{L}(U, X)$ such that for all $u \in U$ we have $Bu \in D(\mathcal{A})$, $\mathcal{A}B \in \mathcal{L}(U, X)$ and

$$\mathcal{B}Bu = u, \quad u \in U$$

Boundary controlled port Hamiltonian systems



Boundary controlled port Hamiltonian systems

Let W be a $n \times 2n$ real matrix. If W has full rank and satisfies $W\Sigma W^T \geq 0$, then the system $\frac{\partial x}{\partial t} = \mathcal{A}x$ with $\mathcal{A}x = P_1(\partial/\partial z)(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain

$$D(\mathcal{A}) = \left\{ \mathcal{L}x \in H^1(a, b; \mathbb{R}^n) \mid \begin{bmatrix} f_{\partial, \mathcal{L}x}(t) \\ e_{\partial, \mathcal{L}x}(t) \end{bmatrix} \in \ker W \right\}$$

and input

$$u(t) = W \begin{bmatrix} f_{\partial, \mathcal{L}x}(t) \\ e_{\partial, \mathcal{L}x}(t) \end{bmatrix}$$

is a Boundary Control System on X .

Sketch of proof

The operator $\mathcal{A}x = P_1(\partial/\partial z)(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain $D(\mathcal{A})$ generates a contraction semigroup on X . It remains to show that $\exists \mathcal{B}$ such that $\mathcal{B}Bu = u$, $u \in U$

Boundary controlled port Hamiltonian systems



Boundary controlled port Hamiltonian systems [Le Gorrec et al., 2005]

Let \tilde{W} be a full rank matrix of size $n \times 2n$ with $\begin{bmatrix} W \\ \tilde{W} \end{bmatrix}$ invertible and let $P_{W, \tilde{W}}$ be given by

$$P_{W, \tilde{W}} = \left(\begin{bmatrix} W \\ \tilde{W} \end{bmatrix} \Sigma \begin{bmatrix} W \\ \tilde{W} \end{bmatrix}^\top \right)^{-1} = \begin{bmatrix} W \Sigma W^\top & W \Sigma \tilde{W}^\top \\ \tilde{W} \Sigma W^\top & \tilde{W} \Sigma \tilde{W}^\top \end{bmatrix}^{-1}.$$

Define the output of the system as the linear mapping $C : \mathcal{L}^{-1} H^1(a, b; \mathbb{R}^n) \rightarrow \mathbb{R}^n$,

$$y = Cx(t) := \tilde{W} \begin{bmatrix} f_{\partial, \mathcal{L}x}(t) \\ e_{\partial, \mathcal{L}x}(t) \end{bmatrix}.$$

Then for $u \in C^2(0, \infty; \mathbb{R}^k)$, $\mathcal{L}x(0) \in H^1(a, b; \mathbb{R}^n)$, and $u(0) = W \begin{bmatrix} f_{\partial, \mathcal{L}x}(0) \\ e_{\partial, \mathcal{L}x}(0) \end{bmatrix}$ the following balance equation is satisfied:

$$\frac{1}{2} \frac{d}{dt} \|x(t)\|_{\mathcal{L}}^2 = \frac{1}{2} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^\top P_{W, \tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} - \langle G_0 \mathcal{L}x(t), \mathcal{L}x(t) \rangle \leq \frac{1}{2} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^\top P_{W, \tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}.$$

Specific cases



Using

$$\begin{aligned}W &= S(I + V, I - V) \\ \tilde{W} &= \tilde{S}(I - V, -I - V)\end{aligned}$$

We obtain for:

$$V = 0 \begin{cases} \dot{x}(t) &= \mathcal{J}x(t), \\ u(t) &= \frac{1}{2}(f_{\partial}(t) + e_{\partial}(t)) \\ y(t) &= \frac{1}{2}(f_{\partial}(t) - e_{\partial}(t)) \end{cases} \implies$$

Scattering system:

Boundary control system with the associated semigroup a contraction

$$\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2.$$

$$V = I \begin{cases} \dot{x}(t) &= \mathcal{J}x(t) \\ u(t) &= f_{\partial}(t) \\ y(t) &= -e_{\partial}(t) \end{cases} \implies$$

Impedance passive system

Boundary control system with the associated semigroup unitary

$$\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = u(t)^T y(t)$$

Back to the vibrating string



The PDE is given by:

$$\underbrace{\frac{\partial}{\partial t} \begin{pmatrix} \epsilon \\ \rho \end{pmatrix}}_f = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{P_1} \frac{\partial}{\partial z} \underbrace{\begin{pmatrix} T(z)\epsilon \\ \frac{1}{\mu(z)}\rho \end{pmatrix}}_e, Q = P_1$$

The boundary port variables are defined by:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} e(b) \\ e(a) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\rho(b)}{\mu(b)} - \frac{\rho(a)}{\mu(a)} \\ T(b)\epsilon(b) - T(a)\epsilon(a) \\ T(a)\epsilon(a) + T(b)\epsilon(b) \\ \frac{\rho(a)}{\mu(a)} + \frac{\rho(b)}{\mu(b)} \end{pmatrix}$$

By using the transformation

$$U = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \text{ s.t. } U^T \Sigma U = \Sigma$$

Back to the vibrating string



One can also choose:

$$\begin{pmatrix} f_{\partial} \\ e_{\partial} \end{pmatrix} = \frac{1}{\sqrt{2}} U \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} e(b) \\ e(a) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} T(b)\epsilon(b) \\ T(a)\epsilon(a) \\ \frac{\rho(b)}{\mu(b)} \\ -\frac{\rho(a)}{\mu(a)} \end{pmatrix}$$

Impedance passive case:

$$V = I \Rightarrow u = \frac{1}{\sqrt{2}} \begin{pmatrix} T(b)\epsilon(b) \\ T(a)\epsilon(a) \end{pmatrix} \quad \text{and} \quad y = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{\rho(b)}{\mu(b)} \\ \frac{\rho(a)}{\mu(a)} \end{pmatrix}$$

$$\frac{dH(t)}{dt} = y(t)^T u(t)$$



Let now consider systems with dissipation:

$$\frac{dx}{dt}(t, z) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*) \mathcal{L}x(t, z), \quad x(0, z) = x_0(z),$$

\Leftrightarrow

$$\begin{pmatrix} f \\ f_p \end{pmatrix} = \mathcal{J}_e \begin{pmatrix} e \\ e_p \end{pmatrix} = \begin{pmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{pmatrix} \begin{pmatrix} e \\ e_p \end{pmatrix}$$

with $e_p = S f_p$ where S is a coercive operator

$$\begin{pmatrix} f \\ f_p \end{pmatrix} \in \mathcal{F}, \quad \begin{pmatrix} e \\ e_p \end{pmatrix} \in \mathcal{E} \text{ and } \mathcal{E} = \mathcal{F} = L_2((a, b), \mathbb{R}^n) \times L_2((a, b), \mathbb{R}^n)$$

Systems with dissipation



From geometrical point of view:

$$f_e = \mathcal{J}_e e_e$$

\mathcal{J}_e is formally skew symmetric and can be parametrized by:

$$\mathcal{J}_e \tilde{e} = \sum_1^N \tilde{P}_k \frac{\partial^k}{\partial z^k} \tilde{e} \quad \text{with} \quad \tilde{P}_k = (-1)^{k+1} \tilde{P}_k^T$$

In this case \tilde{P}_N is not full rank and the bilinear product is defined on quotient space. The extended boundary port variables are defined by:

$$\begin{pmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{Q}_1 & -\tilde{Q}_1 \\ I & I \end{pmatrix} \begin{pmatrix} M_Q & 0 \\ 0 & M_Q \end{pmatrix} \begin{pmatrix} \tilde{e}(b) \\ \tilde{e}(a) \end{pmatrix}$$

M spanning the column of \tilde{Q} , $\tilde{Q}_1 = M^T \tilde{Q} M$ and $M_Q = (M^T M)^{-1} M^T$ with

$$\tilde{Q} = \begin{pmatrix} \tilde{P}_1 & \tilde{P}_2 & \dots & \tilde{P}_N \\ -\tilde{P}_2 & -\tilde{P}_3 & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ (-1)^{N-1} \tilde{P}_N & 0 & \dots & 0 \end{pmatrix}$$



Let W be full rank such that $W\Sigma W^T \geq 0$,

$$\frac{dx}{dt}(t) = \mathcal{J}_e x(t)$$

with input

$$u(t) = W \begin{pmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{pmatrix}$$

is a **boundary control system**. The operator $A_{\text{ext}} = \mathcal{J}_e$ with domain

$$D(A_{\text{ext}}) = \left\{ \begin{pmatrix} \tilde{e} \\ \tilde{e}_r \end{pmatrix} \in \begin{pmatrix} H^N((a, b), \mathbb{R}^n) \\ H^N((a, b), \mathbb{R}^n) \end{pmatrix} \mid \begin{pmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{pmatrix} \in \ker W \right\}, \quad (3)$$

generates a **contraction semigroup**.

Balance equation



Let \tilde{W} be full rank such that $\begin{pmatrix} W \\ \tilde{W} \end{pmatrix}$ invertible. Let define $C : H^N((a, b), \mathbb{R}^{2n}) \rightarrow \mathbb{R}^{2nN}$ as,

$$Cx(t) := \tilde{W} \begin{pmatrix} f_{e,\partial}(t) \\ e_{e,\partial}(t) \end{pmatrix} \quad (4)$$

and the output as

$$y(t) = Cx(t), \quad (5)$$

then for $u \in C^2((0, \infty); \mathbb{R}^{2nN})$, $x(0) \in H^N((a, b), \mathbb{R}^{2n})$, and $Bx(0) = u(0)$:

$$\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = \frac{1}{2} \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_{W, \tilde{W}} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \quad (6)$$

where

$$P_{W, \tilde{W}}^{-1} = \begin{pmatrix} W \Sigma W^T & W \Sigma \tilde{W}^T \\ \tilde{W} \Sigma W^T & \tilde{W} \Sigma \tilde{W}^T \end{pmatrix}. \quad (7)$$

Dissipative operator



Now the feedback is closed *i.e.*

$$f = \mathcal{J}e - \mathcal{G}_R S \mathcal{G}_R^* e,$$

The port variables become :

$$\begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} e(b) \\ (-S\mathcal{G}_R^* e)(b) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(b) \\ \frac{d^{N-1}(-S\mathcal{G}_R^* e)}{dz^{N-1}}(b) \\ e(a) \\ (-S\mathcal{G}_R^* e)(a) \\ \vdots \\ \frac{d^{N-1}(-S\mathcal{G}_R^* e)}{dz^{N-1}}(a) \end{pmatrix}, \quad (8)$$

Dissipative operator



Consider the operator

$$A = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$$

with domain

$$D(A) = \left\{ e \in H^N((a, b); \mathbb{R}^n) \mid S \mathcal{G}_R^* e \in H^N((a, b); \mathbb{R}^n), \right. \quad (9)$$

$$\left. \left(\begin{array}{c} g_{f, \partial} \\ g_{e, \partial} \end{array} \right) \in \ker W \right\}. \quad (10)$$

If W has full rank and satisfies $W \Sigma W^T \geq 0$, then A generates a contraction semigroup.



Let W be full rank and satisfies $W\Sigma W^T \geq 0$, then

$$\frac{dx}{dt}(t) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*) x(t) \quad (11)$$

with input

$$u(t) = \mathcal{B}x(t) = W \begin{pmatrix} g_{f,\partial}(t) \\ g_{e,\partial}(t) \end{pmatrix} \quad (12)$$

is a boundary control system. Furthermore, the operator $A = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$ with domain

$$D(A) = \left\{ e \in H^N((a, b); \mathbb{R}^n) \mid S \mathcal{G}_R^* e \in H^N((a, b); \mathbb{R}^n), \right. \quad (13)$$

$$\left. \begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} \in \ker W \right\}. \quad (14)$$

generates a contraction semigroup.

Dissipative operator



Let \tilde{W} be full rank such that $\begin{pmatrix} W \\ \tilde{W} \end{pmatrix}$ invertible. Define the linear mapping $C : H^N((a, b), \mathbb{R}^{2n}) \rightarrow \mathbb{R}^{2nN}$ as,

$$Cx(t) := \tilde{W} \begin{pmatrix} g_{f,\partial}(t) \\ g_{e,\partial}(t) \end{pmatrix} \quad (15)$$

and the output as

$$y(t) = Cx(t), \quad (16)$$

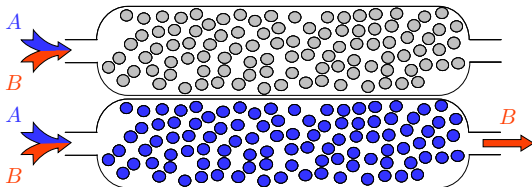
then for $u \in C^2((0, \infty); \mathbb{R}^{2nN})$, $x(0) \in H^N((a, b), \mathbb{R}^{2n})$, and $Bx(0) = u(0)$ the following balance equation is satisfied:

$$\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 \leq \frac{1}{2} \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_{W, \tilde{W}} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \quad (17)$$

where

$$P_{W, \tilde{W}}^{-1} = \begin{pmatrix} W \Sigma W^T & W \Sigma \tilde{W}^T \\ \tilde{W} \Sigma W^T & \tilde{W} \Sigma \tilde{W}^T \end{pmatrix}. \quad (18)$$

Chemical reactor



Let consider a chemical tubular reactor $z \in [a, b]$ with reaction $A \rightarrow B$

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial z} \left(-D_a \frac{\partial C}{\partial z} + vC \right) - k_0 C + \text{Boundary conditions}$$

where $D_a > 0$ and v is a positive constant.



Let consider a chemical tubular reactor $z \in [a, b]$ with reaction $A \rightarrow B$

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial z} \left(-D_a \frac{\partial C}{\partial z} + vC \right) - kC + \text{Boundary conditions}$$

where $D_a > 0$ and v is a positive constant. By choosing

$$\mathcal{J} = -\frac{\partial}{\partial z}, \quad \mathcal{G} = \frac{\partial}{\partial z} + \sqrt{\frac{k}{D_a}}, \quad \mathcal{G}^* = -\frac{\partial}{\partial z} + \sqrt{\frac{k}{D_a}}, \quad \mathcal{S} = \frac{D_a}{v}$$

$$\begin{pmatrix} \frac{\partial C}{\partial t} \\ f \end{pmatrix} = \begin{pmatrix} -\frac{\partial}{\partial z} & \frac{\partial}{\partial z} + \sqrt{\frac{k}{D_a}} \\ \frac{\partial}{\partial z} - \sqrt{\frac{k}{D_a}} & 0 \end{pmatrix} \begin{pmatrix} vC \\ e \end{pmatrix}$$

Boundary port variables



$$\begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} vC(b) \\ D_a \frac{\partial}{\partial z} C(b) - \sqrt{D_a k} C(b) \\ vC(a) \\ D_a \frac{\partial}{\partial z} C(a) - \sqrt{D_a k} C(a) \end{pmatrix}$$

Then

$$\begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} vC(a) - vC(b) + D_a \left(\frac{\partial}{\partial z} C(b) - \frac{\partial}{\partial z} C(a) \right) - \sqrt{D_a k} (C(b) - C(a)) \\ v(C(b) - C(a)) \\ v(C(b) + C(a)) \\ D_a \left(\frac{\partial}{\partial z} C(b) + \frac{\partial}{\partial z} C(a) \right) - \sqrt{D_a k} (C(b) + C(a)) \end{pmatrix}$$

Dankwert conditions



Usually the boundary conditions for tubular reactors are chosen as Dankwert Boundary Conditions:

- input total flow is imposed,
- output dispersive flow is equal to zero.

Dankwert conditions can be written as :

$$vC(t, a) - D_a \frac{\partial C}{\partial z}(t, a) = vC_{in}(t), \quad \text{and} \quad D_a \frac{\partial C}{\partial z}(t, b) = 0, \quad (19)$$

\Leftrightarrow

$$\begin{pmatrix} vC_{in} \\ 0 \end{pmatrix} = W \begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix}$$

$$W = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & \sqrt{\frac{k_0 D_a}{v}} & 1 - \sqrt{\frac{k_0 D_a}{v}} & -1 \\ 1 & 1 + \sqrt{\frac{k_0 D_a}{v}} & \sqrt{\frac{k_0 D_a}{v}} & 1 \end{pmatrix},$$

Conclusion



One can check that

$$W\Sigma W^T \geq 0$$

iif

$$\sqrt{\frac{kD_a}{v}} \leq \frac{1}{2}$$

It means that the system is a **Boundary Control System** with associated C_0 semigroup **unitary or a contraction** **if and only if the condition is satisfied**.

Otherwise it is not a contraction semigroup.

Outline



1. Introduction
2. Considered PDEs
3. C_0 Semigroups
4. PHS and generator of C_0 semigroups
5. Boundary control systems

Conclusion



In this part we have:

- defined C_0 semigroups, Boundary Control Systems,
- parametrized all the boundary port variables such that the system is a Boundary Control System,
- specified the impedance passive and the scattering cases,
- generalized the result to systems with dissipation

In the next part we will be interested in stability of open/closed loop boundary control systems.



Thank you for your attention !



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