Modelling and Control of Distributed Parameter Systems: A port-Hamiltonian Approach

Abstract Differential Equations

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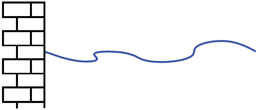
February 10, 2018

Introduction

In this first part we have seen models of physical systems, like that of the vibrating string



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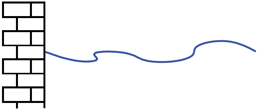

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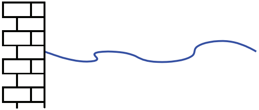

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In this part we investigate existence of solutions for (linear, time-invariant) partial differential equations.
We begin by identifying the state and state space.

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Idea behind the state: The state is that which you have to know now to predict/know the future behaviour.

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What is the state for vibrating string

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta, t)\right]?$$

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$$L^2((a,b);\mathbb{R}^n) = \left\{ f : (a,b) \mapsto \mathbb{R}^n \mid \int_a^b \|f(\zeta)\|^2 d\zeta < \infty \right\}.$$

More on $L^2((a,b);\mathbb{R}^n)$

$L^2((a,b);\mathbb{R}^n)$ is a Hilbert space. That is

- There exists an inner product $\langle \cdot, \cdot \rangle$ given by

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An inner product is a mapping from $X \times X$ to $\mathbb{R}$ satisfying

- $\langle \alpha f + \beta h, g \rangle = \alpha \langle f, g \rangle + \beta \langle h, g \rangle$, $\alpha, \beta \in \mathbb{R}$.
- $\langle f, g \rangle = \langle g, f \rangle$
- For $f \neq 0$, $\langle f, f \rangle > 0$.

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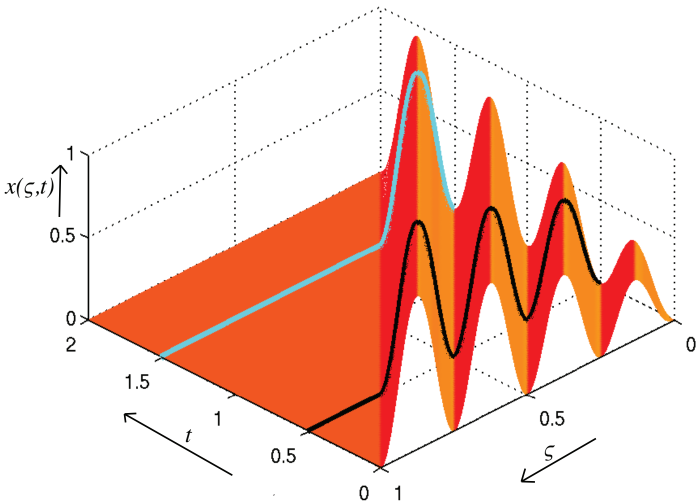
  - For $f \neq 0$, $\langle f, f \rangle > 0$.
- The norm $\|\cdot\|$ on an inner product space is given as $\|f\|^2 = \langle f, f \rangle$.
- If $\|f_n - f_m\| \to 0$ as $n, m \to \infty$, then there exists an $f \in X$ such that $\|f - f_n\| \to 0$ as $n \to \infty$.

States



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For the finite-dimensional systems

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0,$$

with A a $n \times n$ matrix, this solution map is e^{At}, since

$$x(t) = e^{At} x_0.$$

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What can we say about this mapping when the underlying differential equation is linear and time-invarying?

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We assume that X, our state space, is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

Semigroups

We introduce some notation. $\mathcal{L}(X)$ denotes the set of linear and bounded operators from X to X. Thus if $Q \in \mathcal{L}(X)$, then

- $Q(\alpha x_0 + \beta \tilde{x}_0) = \alpha Q(x_0) + \beta Q(\tilde{x}_0)$, and
- there exists a $q \geq 0$ such that for all $x_0 \in X$,

$$\|Q(x_0)\| \leq q\|x_0\|.$$

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Definition

A strongly continuous semigroup (C_0-semigroup) is an operator valued function, $(T(t))_{t\geq 0}$, from $[0, \infty)$ to $\mathcal{L}(X)$ which satisfies

- $T(0) = I$
- $T(t)T(s) = T(t+s), \quad t, s \in [0, \infty)$
- For all $x_0 \in X$ there holds

$$\lim_{t\downarrow 0} T(t)x_0 = x_0.$$

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It tells that the solution becomes more and more the initial state when time get smaller and smaller.

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- Question: How to formulate state/state space for a partial differential equation?
  We study an example first.

Example (Transport equation)

On the spatial domain $[0, 1]$ consider the p.d.e.

$$\begin{aligned} \frac{\partial w}{\partial t}(\zeta, t) &= \frac{\partial w}{\partial \zeta}(\zeta, t), \qquad \zeta \in [0, 1],\ t \geq 0, \\ w(1, t) &= 0 \\ w(\zeta, 0) &= w_0(\zeta) \qquad \text{(given)}. \end{aligned}$$

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- So $x(t) = w(\cdot,t)$, or $(x(t))(\zeta) = w(\zeta,t)$.
- As state space we choose $L^2(0,1)$.
- If we now introduce $\dot{x}(t) = \frac{\partial w}{\partial t}(\cdot,t)$ and $Ax(t) = \frac{\partial w}{\partial \zeta}(\cdot,t)$, then the p.d.e. becomes

$$\dot{x}(t) = Ax(t).$$

State differential equation

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can with $x(t) = w(\cdot,t)$, $\dot{x}(t) = \frac{\partial w}{\partial t}(\cdot,t)$, and $Ax(t) := \frac{\partial w}{\partial \zeta}(\cdot,t)$, be written as abstract differential equation:

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Another problem: The derivative does not exists for all $x(t) \in L^2(0,1)$.

More on A

We see that A is a mapping working for a fixed t, i.e., so for $f \in L^2(0,1)$ we can define Af as

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We want that A maps into X, and so we only take the derivative of $f \in X$ when the answer lies in X again. So

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Summary on A

So the p.d.e.

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is written as the abstract differential equation:

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 = w_0$$

with $x(t) = w(\cdot,t) \in X = L^2(0,1)$, and

$$(Af)(\zeta) = \frac{df}{d\zeta}(\zeta)$$

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A and $T(t)$

Now we have of a p.d.e. two concept

- $(T(t))_{t \geq 0}$; solution map, i.e., $x(t) = T(t)x_0$ is the solution, and
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To answer this we look at the finite-dimensional case once more.

Finding A

Let A be given as

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix},$$

then

$$e^{At} = \begin{pmatrix} e^t & e^{3t} - e^t \\ 0 & e^{3t} \end{pmatrix}.$$

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Answer Evaluate the derivative of the semigroup at $t = 0$.

Since $\frac{d}{dt}e^{At} = Ae^{At}$, we have

$$\frac{d}{dt}e^{At} \mid_{t=0} = A.$$

A and $T(t)$

Theorem

Assume that $(T(t))_{t\geq 0}$ is the solution map of our p.d.e., then for those $x_0 \in X$ for which the following limit exists

$$\lim_{t\downarrow 0} \frac{T(t)x_0 - x_0}{t},$$

we have that

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Furthermore, $D(A)$ consists of precisely those $x_0 \in X$ for which the limit exists.
A is named the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$. □

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Lemma

If $x_0 \in D(A)$, then for $t > 0$, $T(t)x_0$ is differentiable, and

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So $x(t) := T(t)x_0$ is a solution (classical) of

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

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Lemma

If $x_0 \in D(A)$, then for $t > 0$, $T(t)x_0$ is differentiable, and

$$\frac{d}{dt}\left(T(t)x_0\right) = AT(t)x_0.$$

So $x(t) := T(t)x_0$ is a solution (classical) of

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

For $x_0 \in X$, $T(t)x_0$ is called a weak solution. □

A and $T(t)$

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We do this for a special class of C_0-semigroups

Contraction semigroup

Definition

The C_0-semigroup $(T(t))_{t \geq 0}$ is contraction semigroup if

$$\|T(t)x_0\| \leq \|x_0\| \qquad \text{for all } t \geq 0 \text{ and for all } x_0 \in X.$$

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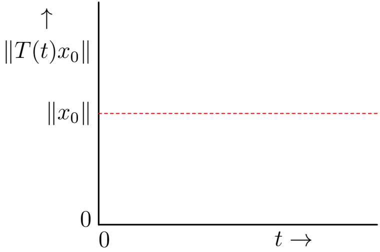

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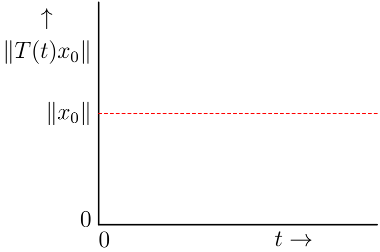

What can we say about these semigroups?

Contraction semigroup

We known that

$$\|T(t)x_0\|^2 = \langle T(t)x_0, T(t)x_0\rangle.$$

For $x_0 \in D(A)$, we have that the derivative of $T(t)x_0$ equals $AT(t)x_0$.
So if we differentiate $\|T(t)x_0\|^2$, we find

$$\frac{d}{dt}\|T(t)x_0\|^2 = \langle AT(t)x_0, T(t)x_0\rangle + \langle T(t)x_0, AT(t)x_0\rangle.$$

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Now we choose $t = 0$.

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Now we choose $t = 0$. We know that $T(0)x_0 = x_0$. Thus at time equal to zero, we find

$$\frac{d}{dt}\left(\|T(t)x_0\|^2\right)\bigg|_{t=0} = \langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle.$$

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So if $T(t)$ is a contraction semigroup, then

$$\langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle = \frac{d}{dt}\|T(t)x_0\|^2 \,|_{t=0} \leq 0.$$

This has to hold for all $x_0 \in D(A)$.

Contraction semigroup

Theorem (Lumer-Phillips)

Let A be a densely defined operator, then A generates a contraction semigroup on X if and only if

1. $\langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle \leq 0$ *for all* $x_0 \in D(A)$.
2. *The range of* $A - I$ *is the whole of* X.

□

Contraction semigroup

Example

Consider on the state space $X = L^2(0,1)$ the operator A which is given as

$$Af = \frac{df}{d\zeta}, \qquad \zeta \in [0,1]$$

with the domain

$$D(A) = \Big\{ f \in L^2(0,1) \mid f \text{ is absolutely continuous}, \frac{df}{d\zeta} \in L^2(0,1) \text{ and } f(1) = 0 \Big\}.$$

Let us check the properties:

Example: Contraction semigroup

- A is densely defined in $L^2(0, 1)$.

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$$=$$

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- $$\begin{aligned}\langle Az, z\rangle + \langle z, Az\rangle \\ &= \int_0^1 \frac{dz}{dx}(x)\overline{z(x)}dx + \int_0^1 z(x)\overline{\frac{dz}{dx}(x)}dx\end{aligned}$$

Example: Contraction semigroup

- A is densely defined in $L^2(0,1)$.
- $$\begin{aligned}\langle Az, z\rangle + \langle z, Az\rangle & \\ &= \int_0^1 \frac{dz}{dx}(x)\overline{z(x)}dx + \int_0^1 z(x)\overline{\frac{dz}{dx}(x)}dx \\ &= \int_0^1 \frac{d}{dx}\left[z(x)\overline{z(x)}\right]dx \\ &= |z(x)|^2\Big|_0^1 \\ &= 0 - |z(0)|^2 \leq 0.\end{aligned}$$

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&= \int_0^1 \frac{dz}{dx}(x)\overline{z(x)}dx + \int_0^1 z(x)\overline{\frac{dz}{dx}(x)}dx \\
&= \int_0^1 \frac{d}{dx}\left[z(x)\overline{z(x)}\right]dx \\
&= |z(x)|^2\Big|_0^1 \\
&= 0 - |z(0)|^2 \leq 0.
\end{aligned}$$

- To see if the range of $(A - I)$ is everything, we have for every $f \in L^2(0,1)$ to solve $(A - I)z = f$.

Example: Contraction semigroup

Solving $(A - I)z = f$ means solving

$$\frac{dz}{dx}(x) - z(x) = f(x), \qquad x \in (0,1)$$

with boundary condition $z(1) = 0$.

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Conclusion:

$$Af = \frac{df}{d\zeta}, \qquad \zeta \in [0,1]$$

with the domain

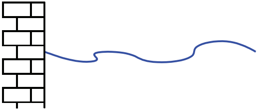
$$D(A) \;=\; \left\{ f \in L^2(0,1) \mid \frac{df}{d\zeta} \in L^2(0,1) \text{ and } f(1) = 0 \right\}$$

generates a contraction semigroup on $X = L^2(0,1)$. □

Port-Hamiltonian Systems

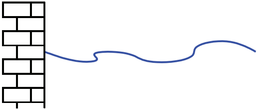
Homogeneous equation

The wave equation



$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left[T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) \right].$$

The wave equation



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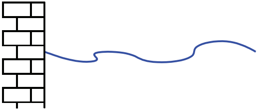
We want to write this p.d.e. as a state differential equation, $\dot{x}(t) = Ax(t)$. Therefor we need

- The state x
- The state space X.
- The infinitesimal generator A with its domain $D(A)$.

The wave equation



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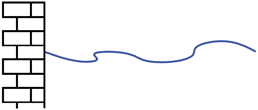
To answer the first two questions, we look at the energy associated to vibrating string.

The wave equation, energy



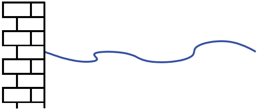
$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right].$$

The energy is given by

$$H(t) = \frac{1}{2}\int_0^1 \rho(\zeta)\left(\frac{\partial w}{\partial t}(\zeta,t)\right)^2 + T(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta,t)\right)^2 d\zeta$$

with ρ is the mass density, and T is Young's modulus.

The wave equation, energy



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This looks like an $L^2((0,1);\mathbb{R}^2)$-norm (squared) in the variables $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial \zeta}$.

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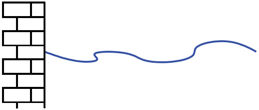

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This indicates a choice for the state variables. We choose $x_1 := \rho\frac{\partial w}{\partial t}$ (the momentum), $x_2 := \frac{\partial w}{\partial \zeta}$ (the strain).

The wave equation, state

With the choice $x_1 := \rho \frac{\partial w}{\partial t}$ (the momentum), $x_2 := \frac{\partial w}{\partial \zeta}$ (the strain), the energy

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becomes

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becomes

$$H(t) = \frac{1}{2} \int_0^1 \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} d\zeta.$$

The wave equation, state and state space

Based on the (quadratic) expression of the energy

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The wave equation, state and state space

With the inner product

$$\langle f, g \rangle_X = \frac{1}{2} \int_0^1 f(\zeta)^\top \mathcal{H}(\zeta) g(\zeta) d\zeta$$

we see that $\|f\|_X^2$ is precisely the energy.

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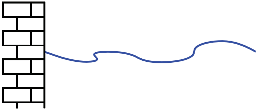
Next we rewrite the p.d.e. model of the vibrating string in our state variables.

The wave equation, state differential equation

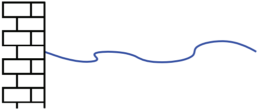


$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left[T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) \right]$$

With the state variables $x_1 = \rho \frac{\partial w}{\partial t}$ and $x_2 = \frac{\partial w}{\partial \zeta}$ we can write the above p.d.e. as

$$\frac{\partial}{\partial t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (\zeta, t)$$

The wave equation, state differential equation



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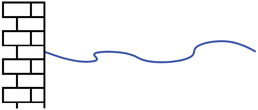
The wave equation, state differential equation



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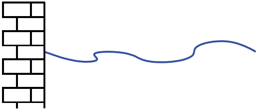
The wave equation, state differential equation



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$$\begin{aligned}
\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(\zeta,t) &= \begin{bmatrix} \rho\frac{\partial^2 w}{\partial t^2}(\zeta,t) \\ \frac{\partial^2 w}{\partial t \partial \zeta} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right] \\ \frac{\partial}{\partial \zeta}\left[\frac{\partial w}{\partial t}\right](\zeta,t) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial \zeta}\left(\underbrace{\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}}_{=\mathcal{H}} x(\zeta,t)\right).
\end{aligned}$$

The wave equation, state differential equation

With the state variables $x_1 = \rho \frac{\partial w}{\partial t}$ and $x_2 = \frac{\partial w}{\partial \zeta}$ we can write the above p.d.e. as

$$\frac{\partial}{\partial t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (\zeta, t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{= P_1} \frac{\partial}{\partial \zeta} \left(\underbrace{\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}}_{= \mathcal{H}} x(\zeta, t) \right).$$

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We generalise this to our class of first order port-Hamiltonian equations.

Port-Hamiltonian partial differential equations

Our model class are p.d.e.'s of the form

$$\frac{\partial x}{\partial t}(\zeta, t) = \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right) [\mathcal{H} x(\zeta, t)]$$

with

- $x(\zeta, t) \in \mathbb{R}^n$, $\zeta \in [a, b]$, $t \geq 0$
- P_1 is an invertible, symmetric real $n \times n$-matrix,
- P_0 is a skew-symmetric real $n \times n$-matrix,
- $\mathcal{H}(\zeta)$ is a symmetric, invertible $n \times n$-matrix with $mI \leq \mathcal{H}(\zeta) \leq MI$ for some $m, M > 0$.

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- $\mathcal{H}(\zeta)$ is a symmetric, invertible $n \times n$-matrix with $mI \leq \mathcal{H}(\zeta) \leq MI$ for some $m, M > 0$.

The energy/Hamiltonian is defined as

$$H(t) = H(x(\cdot, t)) = \frac{1}{2} \int_a^b x(\zeta, t)^T \mathcal{H}(\zeta) x(\zeta, t) d\zeta.$$

Power balance

For the Port-Hamiltonian p.d.e. with energy/Hamiltonian

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$$\dot{H}(t) = \frac{dH}{dt}(x(\cdot,t)) = \frac{1}{2}\left[(\mathcal{H}x)^T(\zeta,t) P_1 (\mathcal{H}x)(\zeta,t)\right]_a^b$$

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Thus the change of internal energy goes via the boundary of the spatial domain, i.e. power balance.

Port-Hamiltonian partial differential equations

Given our port-Hamiltonian partial differential equation

$$\frac{\partial x}{\partial t}(\zeta, t) = \left(P_1 \frac{\partial}{\partial \zeta} + P_0 \right) [\mathcal{H}(\zeta) x(\zeta, t)]$$

with the properties on P_0, P_1 and $\mathcal{H}$.

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We write these boundary conditions as

$$W_B \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix} = 0.$$

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We answer this question by using semigroup theory. However, we do it only for contraction semigroups.

Port-Hamiltonian p.d.e., state space

Given our port-Hamiltonian partial differential equation with boundary conditions

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)[\mathcal{H}(\zeta)x(\zeta, t)] \\ 0 &= W_B \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix} \end{aligned}$$

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- As state we choose $x(t)=x(\cdot,t)$.
- As state space we choose the energy space, i.e., $X=L^2((0,1);\mathbb{R}^n)$ with inner product

$$
\langle f,g\rangle_X=\frac{1}{2}\int_a^b f(\zeta)^\top\mathcal{H}(\zeta)g(\zeta)d\zeta.
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Port-Hamiltonian p.d.e., state space formulation

With the state $x(t) = x(\cdot, t)$ and $X = L^2((0,1); \mathbb{R}^n)$ our port-Hamiltonian p.d.e. with boundary conditions;

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becomes

$$\dot{x}(t) = Ax(t),$$

where

$$Ax = \left(P_1 \frac{d}{d\zeta} + P_0\right)[\mathcal{H}x]$$

with domain

$$D(A) = \left\{ x \in X \mid \frac{d}{d\zeta}(\mathcal{H}x) \in X, W_B \begin{bmatrix} \mathcal{H}(b)x(b) \\ \mathcal{H}(a)x(a) \end{bmatrix} = 0 \right\}.$$

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Theorem (Lumer-Phillips)

Let A be a densely defined operator, then A generates a contraction semigroup on X if and only if

1. $\langle Ax_0, x_0 \rangle + \langle x_0, Ax_0 \rangle \leq 0$ *for all* $x_0 \in D(A)$.
2. *The range of* $A - I$ *is the whole of* X.

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For our class of port-Hamiltonian p.d.e.'s we now have

Port-Hamiltonian p.d.e., existence of solutions

Theorem (Le Gorrec, Maschke & Zwart '05, Jacob & Zwart '11)

Assume the (standard) conditions on P_0, P_1 and $\mathcal{H}$. Assume further that W_B is a $n \times 2n$ matrix of full rank.

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The latter is equivalent to $\dot{H} \leq 0$.
A generates a unitary C_0-group (i.e. $\|T(t)x_0\| = \|x_0\|$, $\forall x_0, \forall t$) if and only if

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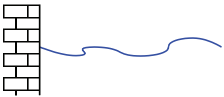
Hence only the simple condition of L-P theorem needs to be checked.

Example: the wave equation



$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left[T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) \right]$$

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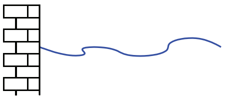

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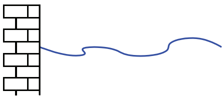

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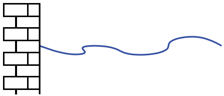

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- W_B has rank 2.
- $\dot{H} = 0$.

Thus A generates a unitary group on the energy space.