ÉCOLE DOCTORALE
Sciences mécaniques et énergétiques, matériaux et géosciences (SMEMAG)

EX-MODELI
EXPLOITATION & MODÉLISATION des DYNAMIQUES NON LINÉAIRES
GdR CNRS 2134

Reduced order modelling for shell finite element structures using the direct parametrisation of invariant manifolds

16-17 octobre 2025
Les journées annuelles du GDR EX-MODELI à Lille dans les locaux de l'ENSAM

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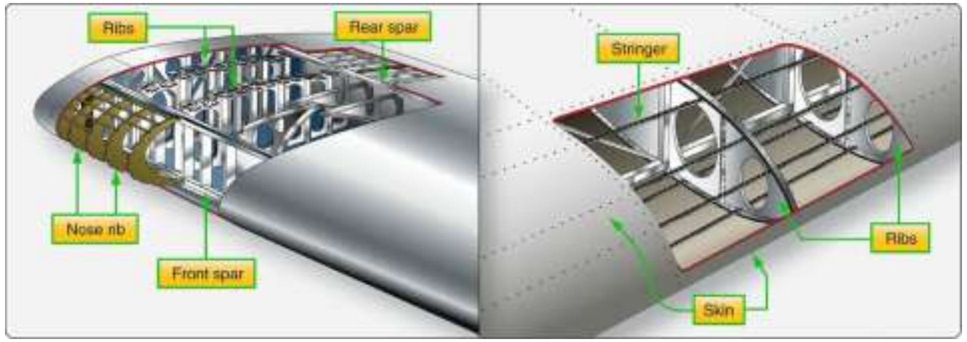
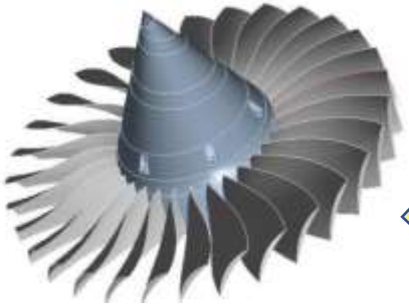
**LMSIA, ENSTA

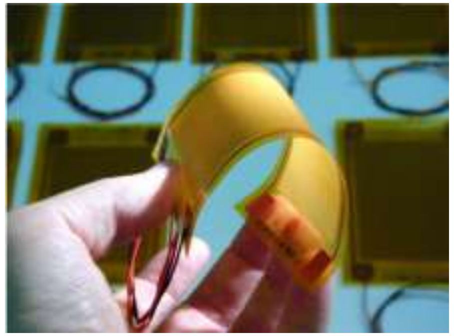
Abundant application scenarios of thin structures

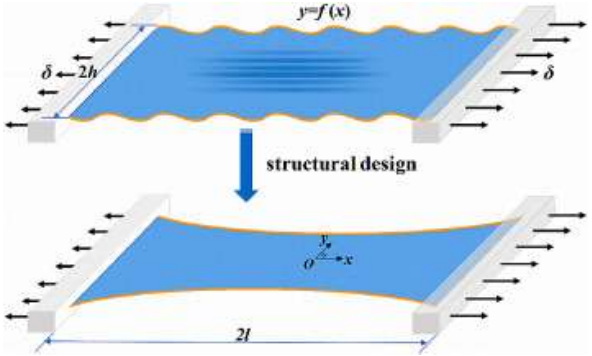
Engineering modeling



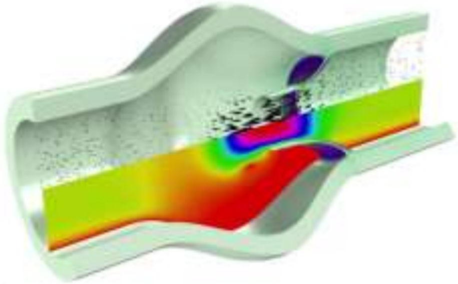


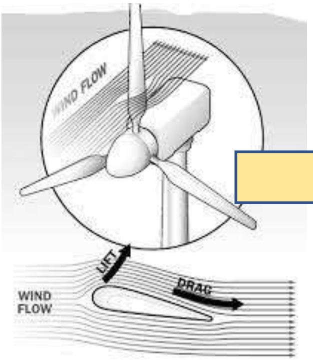
Thin films system





Fluid-structure interaction





Research highlights

1. How to establish an appropriate finite element method for modeling thin structures ?
2. An appropriate nonlinear dynamics reduced-order method to lower the cost of numerical simulations

Where do we stand?

Normal form method
- Touzé C. A **normal form approach** for nonlinear normal modes
- Vizzaccaro A, et al. **Direct computation** of nonlinear mapping via **normal form** for reduced-order models of finite element nonlinear structures

DPIM (Direct parametrisation of invariant manifold)
- Opreni A, et al. High-order **direct parametrisation of invariant manifolds** for model order reduction of finite element structures: application to generic forcing terms and parametrically excited systems
- Vizzaccaro A, et al. Direct parametrisation of invariant manifolds for generic **non-autonomous** systems including superharmonic resonances

Shell finite element modeling
- **Fewer degrees of freedom** compared to solid elements
- More convenient definition of the **kinematic description** of thin structures in the transverse direction
- Introduce **assumed natural strain** to prevent Poisson locking

Nonlinear dynamics solution methods
- Increment harmonic balance method (For **full order models**)
- Collocation method (For **reduced order models**)

These theories lay the foundation for our development of a framework for solving thin structures !

Curved shell structure modeling: governing equations

Positional relationship

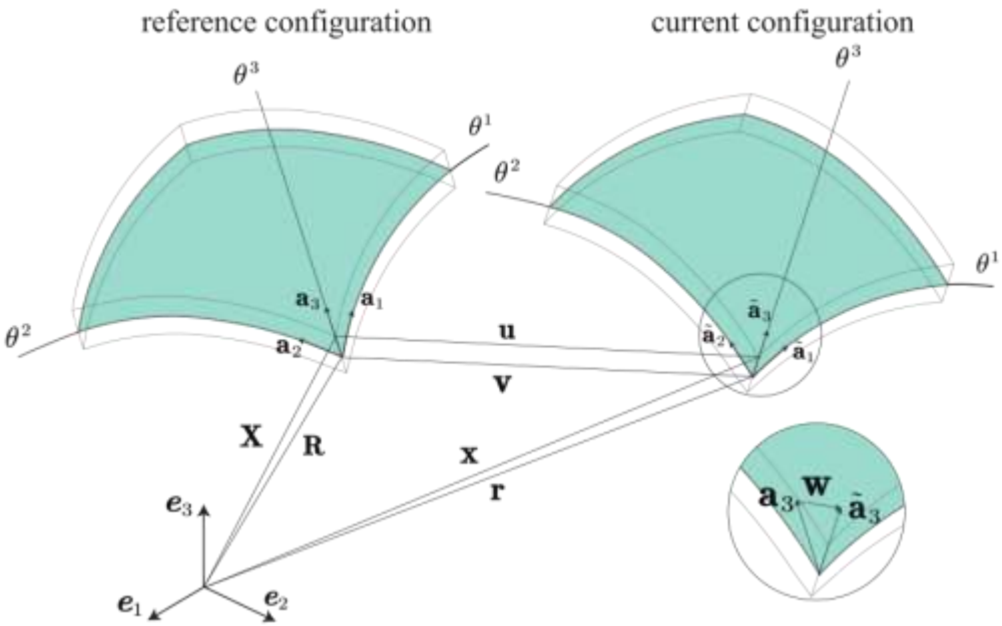
$$\mathbf{X}(\theta^\alpha,\theta^3)=\mathbf{R}(\theta^\alpha)+\theta^3\mathbf{a}_3(\theta^\alpha),\alpha=1,2$$

$$\mathbf{x}(\theta^\alpha,\theta^3)=\mathbf{r}(\theta^\alpha)+\theta^3\tilde{\mathbf{a}}_3(\theta^\alpha),\alpha=1,2$$

Covariant base tensor

$$\mathbf{G}_\alpha=\mathbf{X}_{,\alpha}=\mathbf{a}_\alpha+\theta^3\mathbf{a}_{3,\alpha} \qquad \mathbf{g}_\alpha=\mathbf{x}_{,\alpha}=\tilde{\mathbf{a}}_\alpha+\theta^3\tilde{\mathbf{a}}_{3,\alpha}$$

$$\mathbf{G}_3=\mathbf{X}_3=\mathbf{a}_3 \qquad \mathbf{g}_3=\mathbf{x},3=\tilde{\mathbf{a}}_3$$



Green-Lagrange strain

$$\begin{cases} E_{ij}=\dfrac{1}{2}(\mathbf{g}_i\cdot\mathbf{g}_j-\mathrm{G}_i\cdot\mathrm{G}_j) \\ E_{ij}=\dfrac{1}{2}\left(\mathbf{G}_i\cdot\dfrac{\partial\mathbf{u}}{\partial\theta^j}+\mathbf{G}_j\cdot\dfrac{\partial\mathbf{u}}{\partial\theta^i}+\dfrac{\partial\mathbf{u}}{\partial\theta^i}\dfrac{\partial\mathbf{u}}{\partial\theta^j}\right) \end{cases}$$

Constitutive relation

$\mathbf{S}=\mathbb{D}:\mathbf{E}$ Traditional shell elements lead to locking issues

$$E_{33}^{(0)}+\theta^3E_{33}^{(1)}\simeq-\frac{D^{33ij}}{D^{3333}}(E_{ij}^{(0)}+\theta^3E_{ij}^{(1)})$$

Enhanced assumed strain

$$\mathbf{E}^{\text{full}}=\mathbf{E}+\tilde{\mathbf{E}} \quad \text{with} \quad \int_\Omega\mathbf{S}:\delta\tilde{\mathbf{E}}\mathrm{d}\Omega=0$$

Hu-Washizu functional

$$\Pi_{HW}(\mathbf{u},\mathbf{E}^{\text{full}},\mathbf{S})=\int_\Omega\frac{1}{2}\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\mathrm{d}\Omega+\int_\Omega\frac{1}{2}\mathbf{E}^{\text{full}}:\mathbb{D}:\mathbf{E}^{\text{full}}\mathrm{d}\Omega$$

$$-\int_\Omega\mathbf{S}:(\mathbf{E}^{\text{full}}-\mathbf{E})\mathrm{d}\Omega-\mathbf{F}(t)\cdot\mathbf{u}$$

$$\begin{cases} \displaystyle\int_\Omega\rho\ddot{\mathbf{u}}\cdot\delta\mathbf{u}\mathrm{d}\Omega+\int_\Omega\mathbf{S}:\delta\mathbf{E}\mathrm{d}\Omega-\mathbf{F}(t)\cdot\delta\mathbf{u}=0 \\ \displaystyle\int_\Omega\mathbf{S}:\delta\tilde{\mathbf{E}}\mathrm{d}\Omega=0 \end{cases}$$

Traditional shell elements

Additional equations

Extend traditional variational equations to avoid locking issues

Invariant manifold techniques

Second-order dynamic equation

$$[\mathbf{M}]\{\ddot{\mathbf{U}}\}+[\mathbf{C}]\{\dot{\mathbf{U}}\}+[\mathbf{K}_l]\{\mathbf{U}\}+\{\mathbf{G}(\mathbf{U},\mathbf{U})\}+\{\mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U})\}=\{\mathbf{F}(t)\}$$

First-order dynamic equation

$$[\mathbf{B}]\{\dot{\mathbf{y}}\}=[\mathbf{A}]\{\mathbf{y}\}+\{\boldsymbol{Q}(\mathbf{y},\mathbf{y})\}+\{\boldsymbol{\Upsilon}\}\tilde{z}\quad \text{and} \quad \dot{\tilde{z}}=\tilde{\lambda}\tilde{z}$$

Nonlinear mapping

$$\{\mathbf{y}\}=\mathbf{W}(\mathbf{z})$$

ROM equations

$$\{\dot{\mathbf{z}}\}=\mathbf{f}(\mathbf{z})$$

$$\mathbf{W}(\mathbf{z})=\sum_{p=1}^{o}\langle\,\mathbf{W}(\mathbf{z})\rangle_p$$

$$\mathbf{f}(\mathbf{z})=\sum_{p=1}^{o}\langle\,\mathbf{f}(\mathbf{z})\rangle_p$$

$$[\mathbf{B}]\,\langle\frac{\partial\mathbf{W}(\mathbf{z})}{\partial\mathbf{z}}\mathbf{f}(\mathbf{z})\,\rangle_p=[\mathbf{A}]\langle\mathbf{W}(\mathbf{z})\rangle_p+\langle\boldsymbol{Q}(\mathbf{W}(\mathbf{z}),\mathbf{W}(\mathbf{z}))\rangle_p+\langle\boldsymbol{\Upsilon}\tilde{z}\rangle_p$$

$$[\mathbf{B}]\,\langle\frac{\partial\mathbf{W}(\mathbf{z})}{\partial\mathbf{z}}\mathbf{f}(\mathbf{z})\,\rangle_1=[\mathbf{A}]\langle\mathbf{W}(\mathbf{z})\rangle_1+\langle\boldsymbol{\Upsilon}\tilde{z}\rangle_1\qquad p=1$$

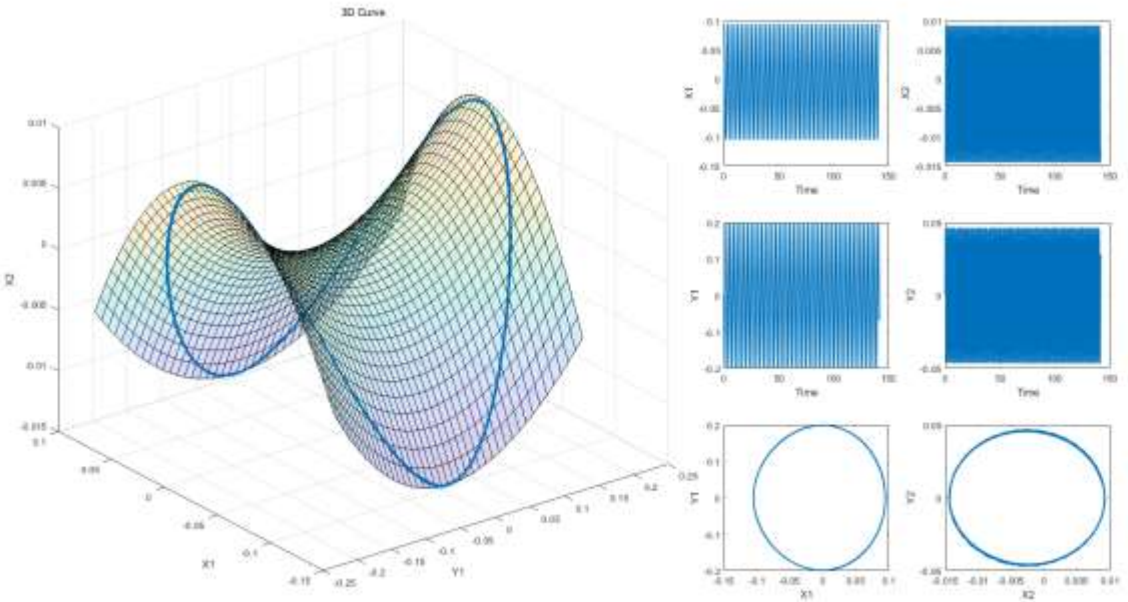
$$[\mathbf{B}]\,\langle\frac{\partial\mathbf{W}(\mathbf{z})}{\partial\mathbf{z}}\mathbf{f}(\mathbf{z})\,\rangle_p=[\mathbf{A}]\langle\mathbf{W}(\mathbf{z})\rangle_p+\langle\boldsymbol{Q}(\mathbf{W}(\mathbf{z}),\mathbf{W}(\mathbf{z}))\rangle_p\qquad p>1$$

Order-1 homological equation

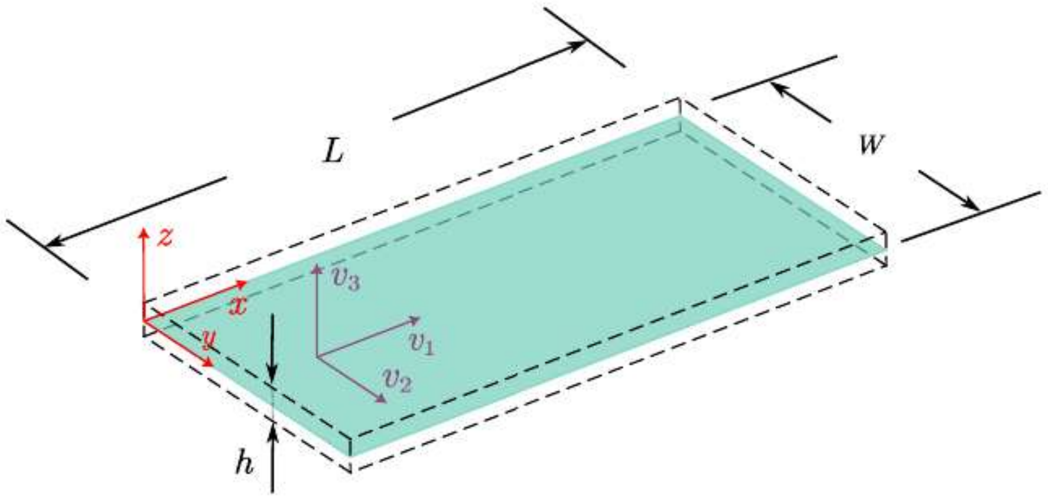
$$\begin{bmatrix}\tilde{\lambda}\mathbf{B}-\mathbf{A} & \mathbf{B}\boldsymbol{\Phi}_{\mathcal{R}} & 0\\ \boldsymbol{\Phi}_{\mathcal{R}}^{\dagger}\mathbf{B} & 0 & 0\\ 0 & 0 & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{W}^{(1,d+1)}\\ \mathbf{f}_{\mathcal{R}}^{(1,d+1)}\\ \mathbf{f}_{\not{\mathcal{R}}}^{(1,d+1)}\end{bmatrix}=\begin{bmatrix}\boldsymbol{\Upsilon}\\ 0\\ 0\end{bmatrix}$$

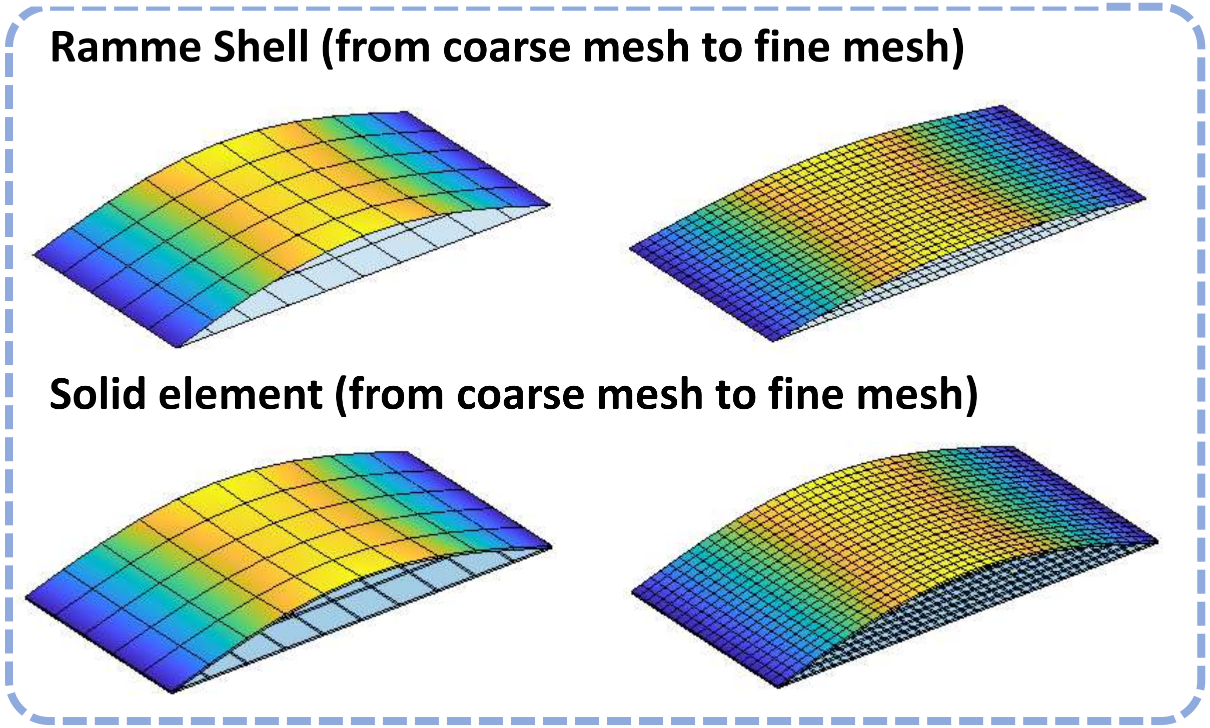
Order-p homological equation

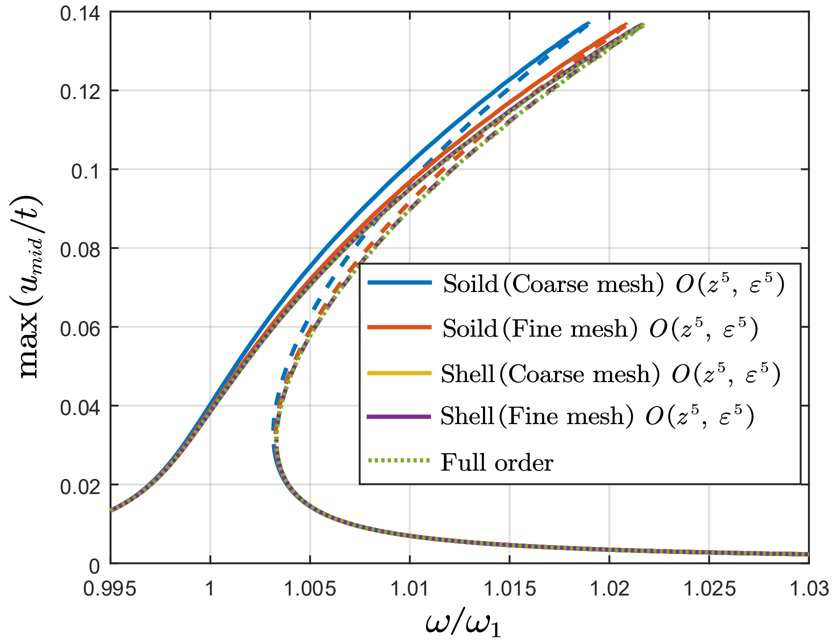
$$\begin{bmatrix}\sigma^{(p,k)}\mathbf{B}-\mathbf{A} & \mathbf{B}\boldsymbol{\Phi}_{\mathcal{R}} & 0\\ \boldsymbol{\Phi}_{\mathcal{R}}^{\dagger}\mathbf{B} & 0 & 0\\ 0 & 0 & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{W}^{(p,k)}\\ \mathbf{f}_{\mathcal{R}}^{(p,k)}\\ \mathbf{f}_{\not{\mathcal{R}}}^{(p,k)}\end{bmatrix}=\begin{bmatrix}\mathbf{R}^{(p,k)}\\ 0\\ 0\end{bmatrix}$$

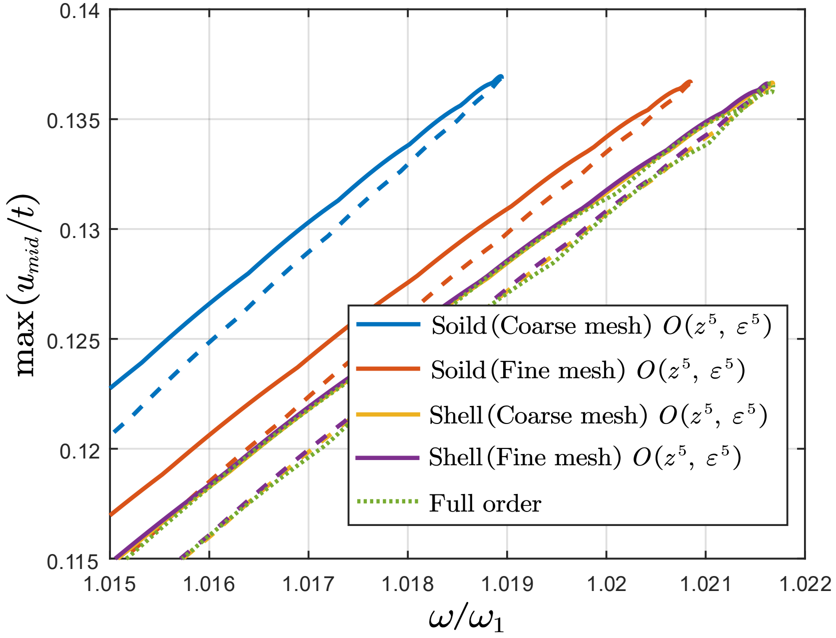


From Flat Plates to Curved Shells



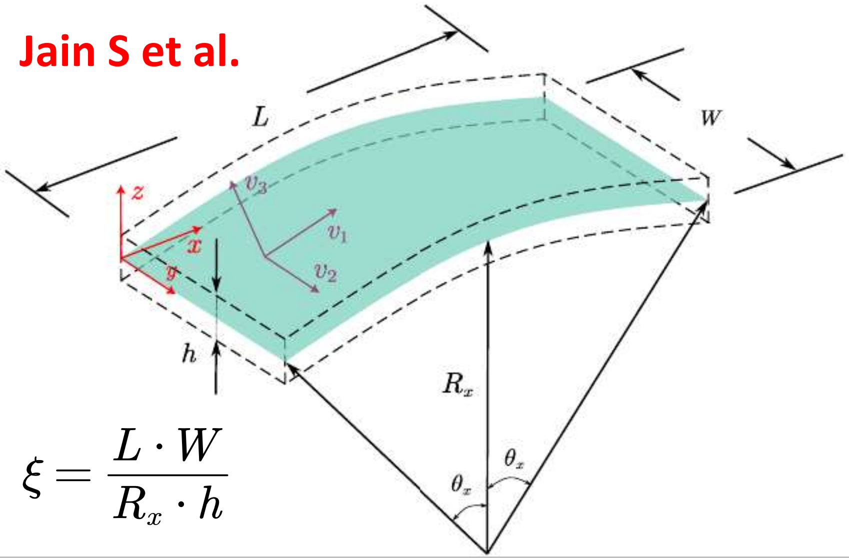






From Flat Plates to Curved Shells

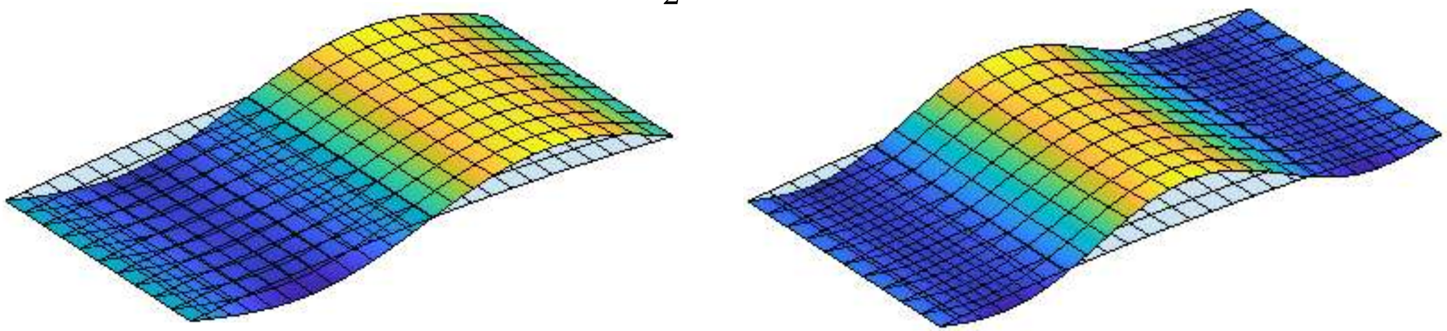
Jain S et al.



$$\xi = \frac{L \cdot W}{R_x \cdot h}$$

Exact 1:2 internal resonance $\xi_0 = 16.4609$

$\omega_1 = 23.71136$ $\omega_2 = 47.36333$



Normal Form-Based Single-Mode ROM

$$\dot{z}_1 = z_2 \cdot \Omega(z_1^2 + z_2^2)$$
$$\dot{z}_2 = -z_1 \cdot \Omega(z_1^2 + z_2^2)$$

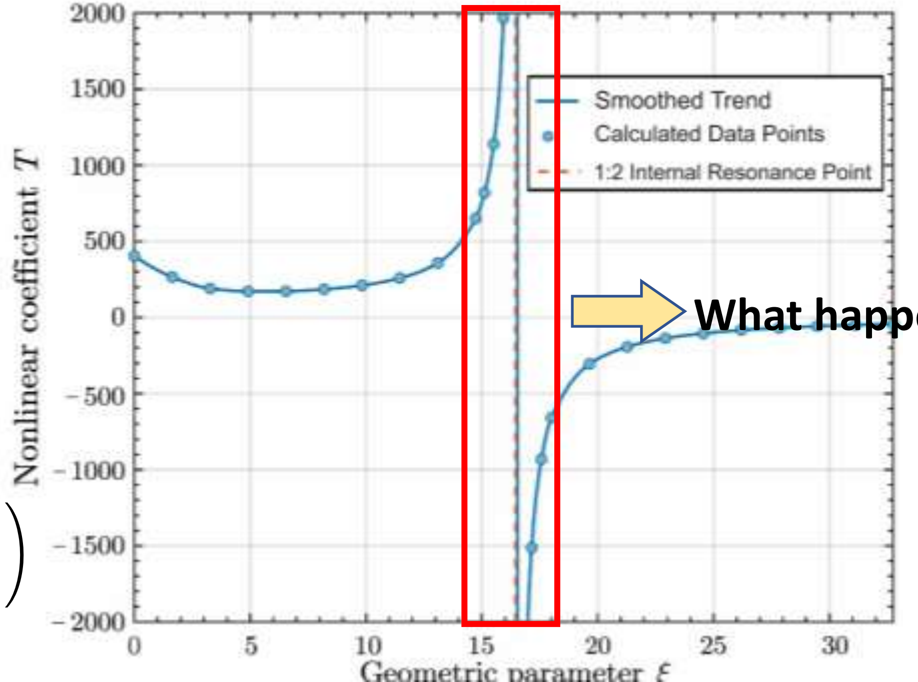
Stabile et al.

Polynomial Expansion of the Nonlinear Frequency

$$\Omega = K_0 + K_1 I + K_2 I^2 + K_3 I^3 + K_4 I^4 + \cdots$$

Backbone Curve Expression

$$\omega_{NL} = \Omega(A^2) = K_0\left(1 + \frac{K_1}{K_0} A^2 + \frac{K_2}{K_0} A^4 + \frac{K_3}{K_0} A^6 + \frac{K_4}{K_0} A^8\right)$$



What happened here?

$\xi \in [0, 2\xi_0]$

From Flat Plates to Curved Shells

Dual-Master-Mode Reduced-Order Model

$$\dot{z}_1 = i\omega_1 z_1 + i\alpha_{23} z_2 z_3 + i\beta_{113} z_1^2 z_3 + i\beta_{124} z_1 z_2 z_4$$

$$\dot{z}_2 = i\omega_2 z_2 + i\alpha_{11} z_1^2 + i\beta_{123} z_1 z_2 z_3 + i\beta_{224} z_2^2 z_4$$

$$\dot{z}_3 = -i\omega_1 z_3 - i\alpha_{23} z_4 z_1 - i\beta_{113} z_3^2 z_1 - i\beta_{124} z_3 z_4 z_2$$

$$\dot{z}_4 = -i\omega_2 z_4 - i\alpha_{11} z_3^2 - i\beta_{123} z_3 z_4 z_1 - i\beta_{224} z_4^2 z_2$$

The backbone curve is defined by a hyperbolic

$$\omega_{NL}^{p+} = \omega_1 + \frac{\alpha_{23}}{2}\rho_2 + \frac{\beta_{113}}{4}\rho_1^2 + \frac{\beta_{124}}{4}\rho_2^2$$

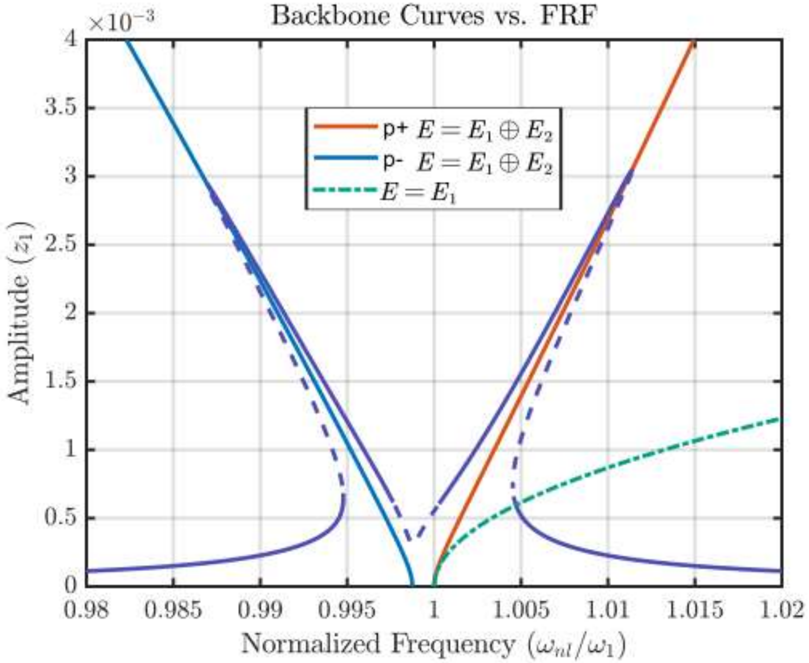
$$\omega_{NL}^{p-} = \omega_1 - \frac{\alpha_{23}}{2}\rho_2 + \frac{\beta_{113}}{4}\rho_1^2 + \frac{\beta_{124}}{4}\rho_2^2$$

Gobat et al.

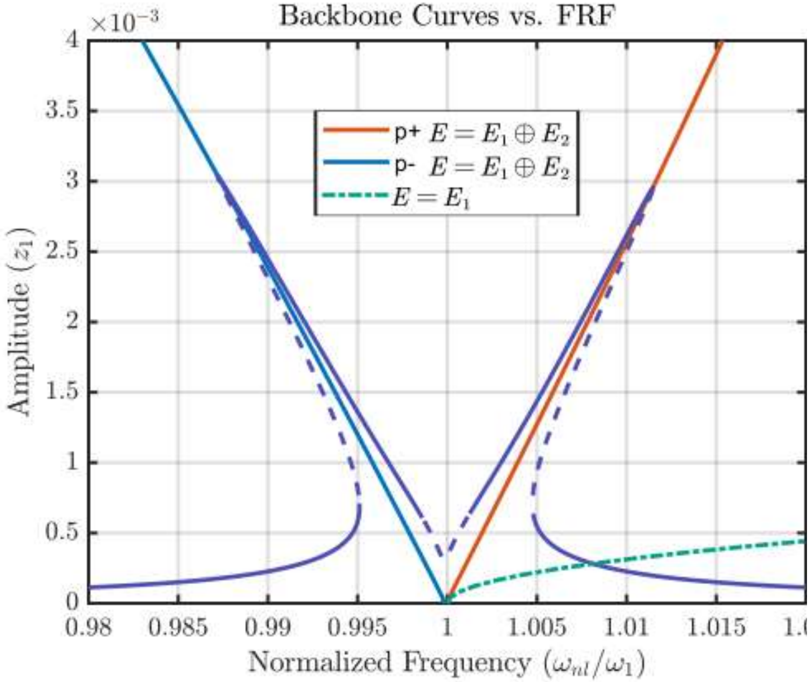
The amplitudes are subject to a constraint relation

$$\frac{(\beta_{224} - 2\beta_{124})}{4}\rho_2^3 - p\alpha_{23}\rho_2^2 + \left[\frac{(\beta_{123} - 2\beta_{113})}{4}\rho_1^2 + \omega_2 - 2\omega_1\right]\rho_2 + p\frac{\alpha_{11}}{2}\rho_1^2 = 0$$

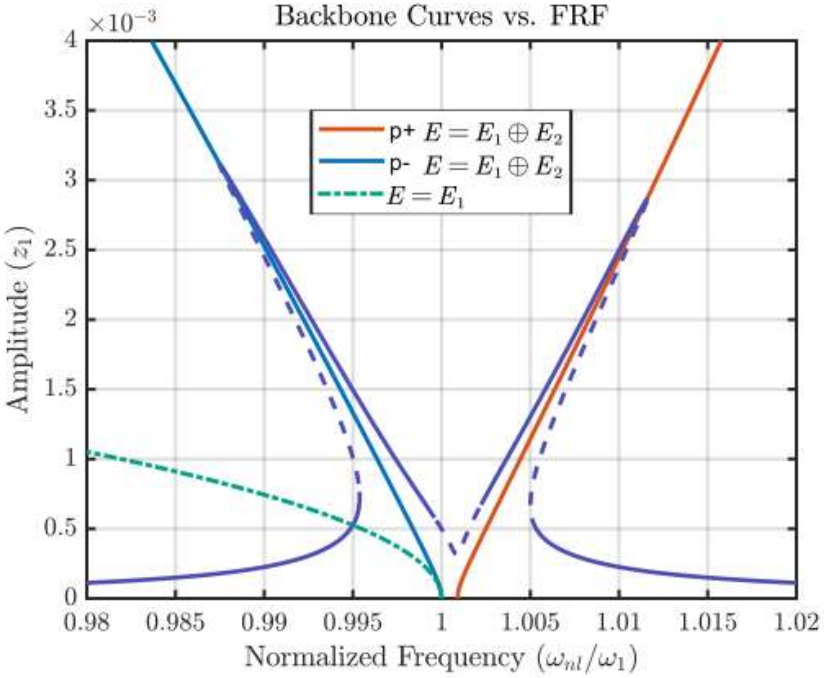
$\xi = 16.3725$



$\xi = 16.4609$

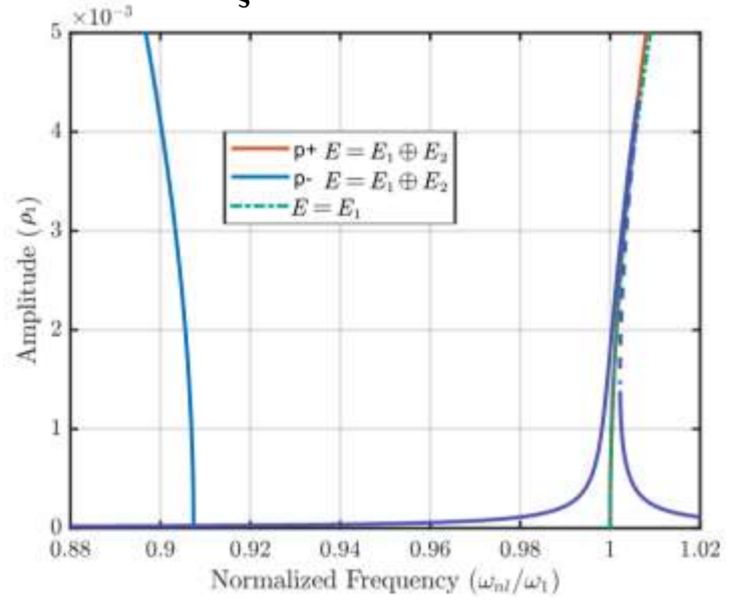


$\xi = 16.5503$

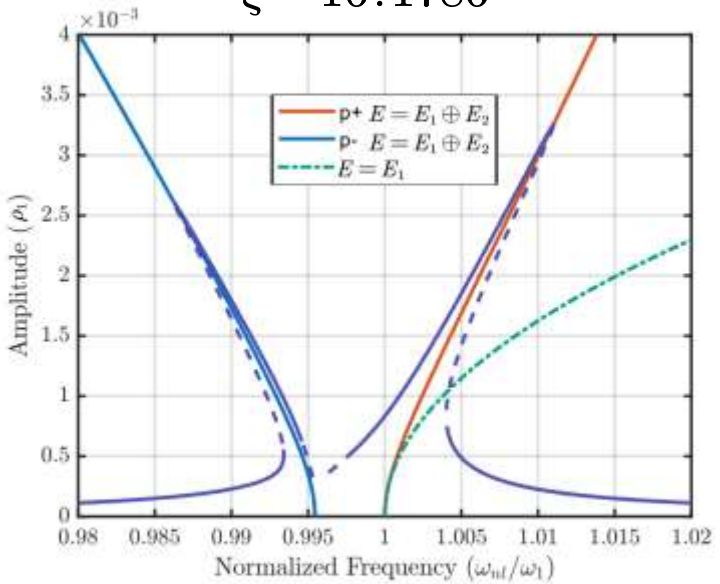


From Flat Plates to Curved Shells

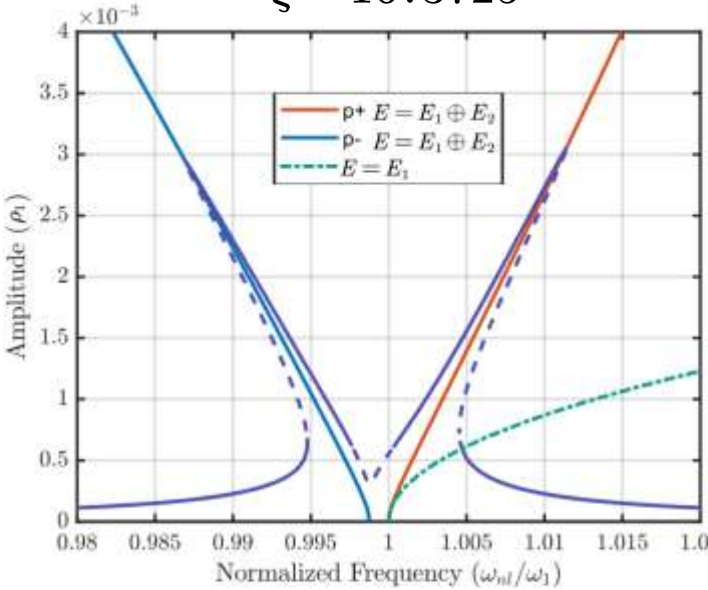
$\xi = 13.0980$



$\xi = 16.1786$

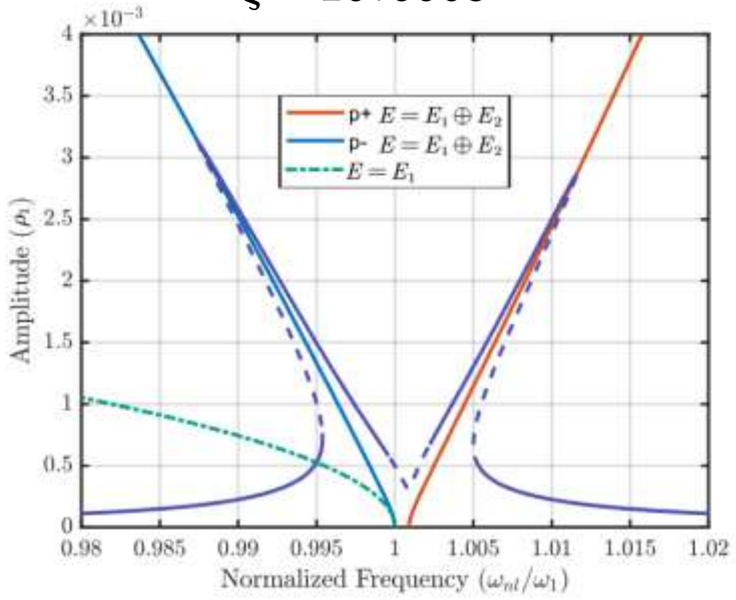


$\xi = 16.3725$

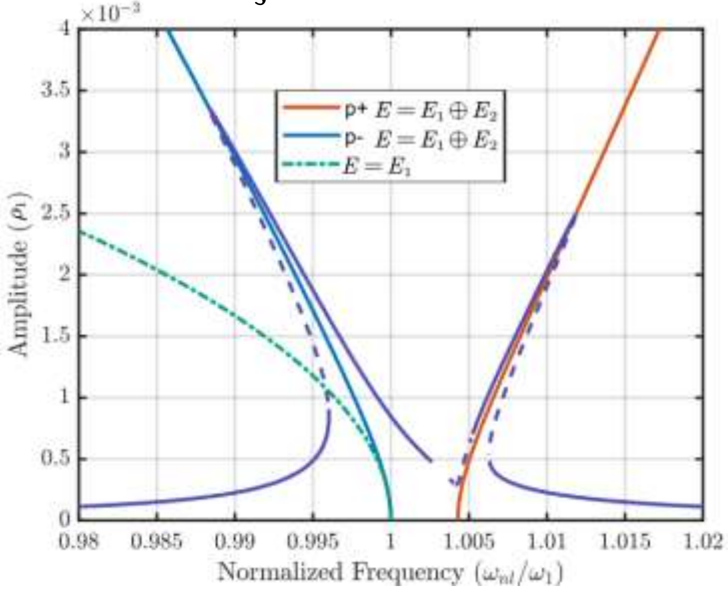


$\xi \in (\xi_0 - \epsilon, \xi_0)$

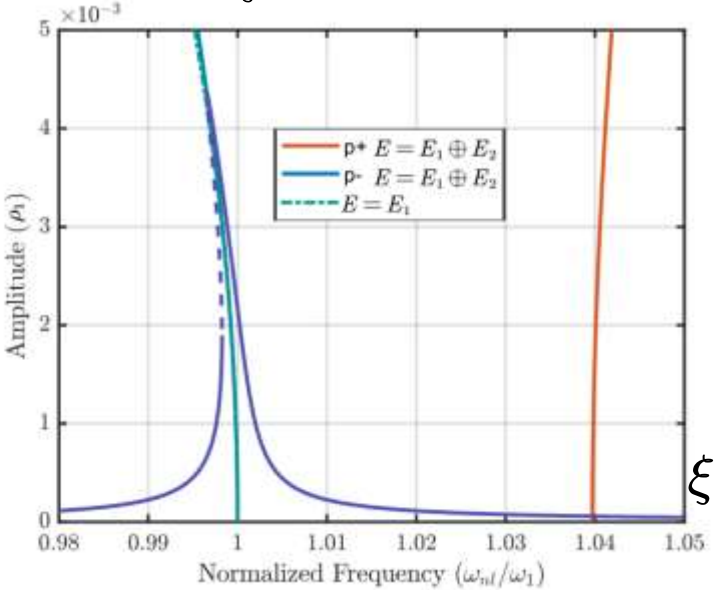
$\xi = 16.5503$



$\xi = 16.7532$

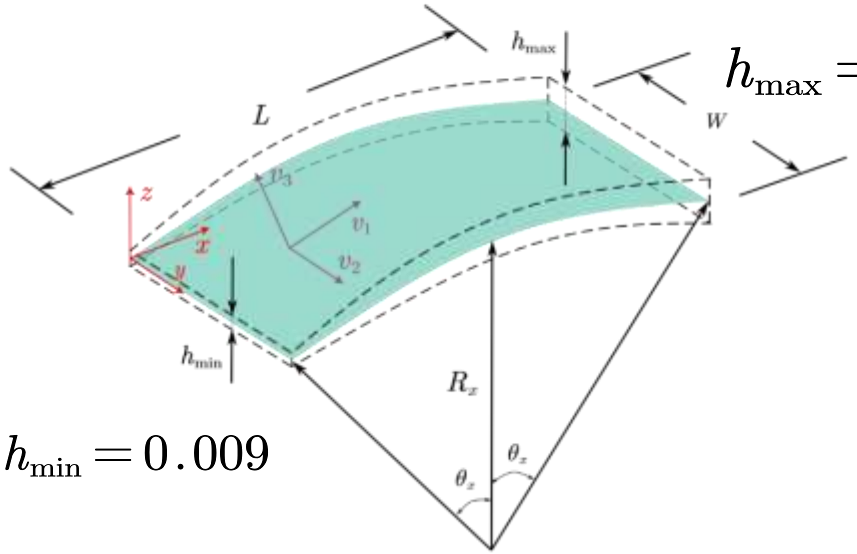


$\xi = 21.2843$



$\xi \in (\xi_0, \xi_0 + \epsilon)$

From Uniform to Variable-thickness



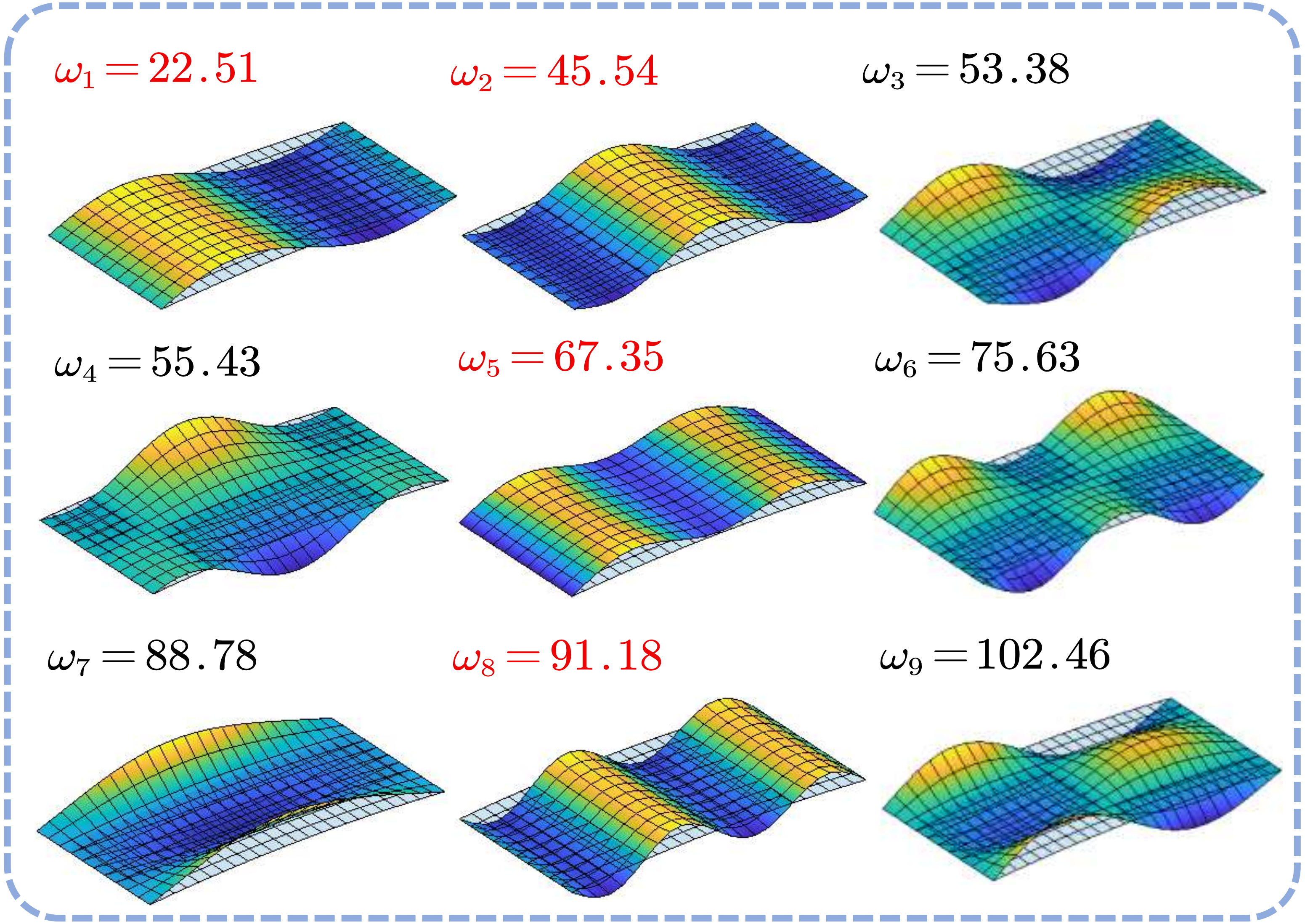
$$h_{\max} = 0.01$$

$$h_{\min} = 0.009$$

$$\mathbf{M\ddot{U}} + \mathbf{C\dot{U}} + \mathbf{KU} + \mathbf{G}(\mathbf{U},\ \mathbf{U}) + \mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U}) = \kappa \mathbf{M} \boldsymbol{\phi}_{\mathrm{B}1} \cos(\Omega t)$$

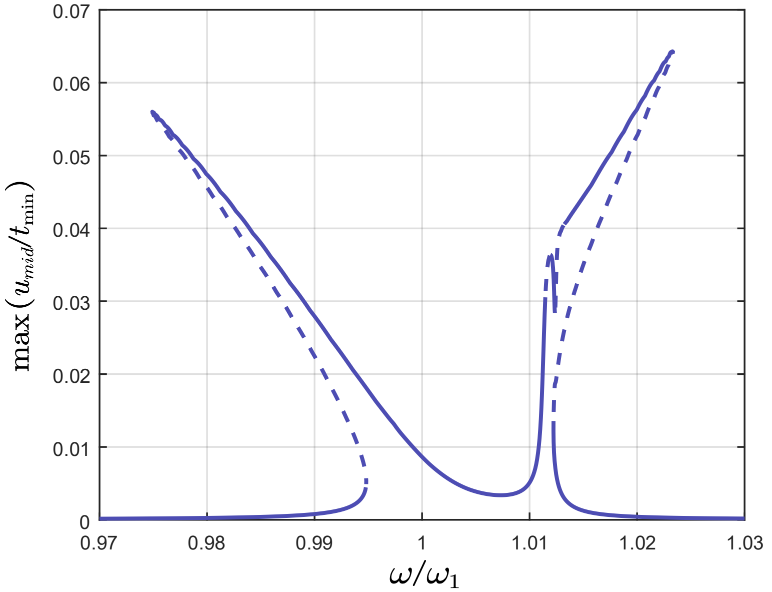
$$h(x) = h_{\min} + \frac{h_{\max} - h_{\min}}{x_{\max} - x_{\min}}(x - x_{\min})$$

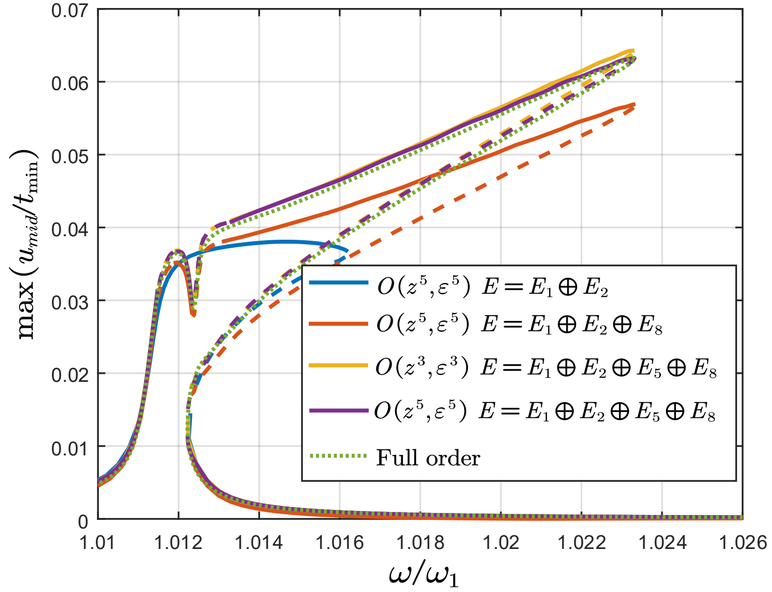
- **It has been confirmed that capturing the system's dynamics accurately in the reduced-order model necessitates the inclusion of the first four bending modes**

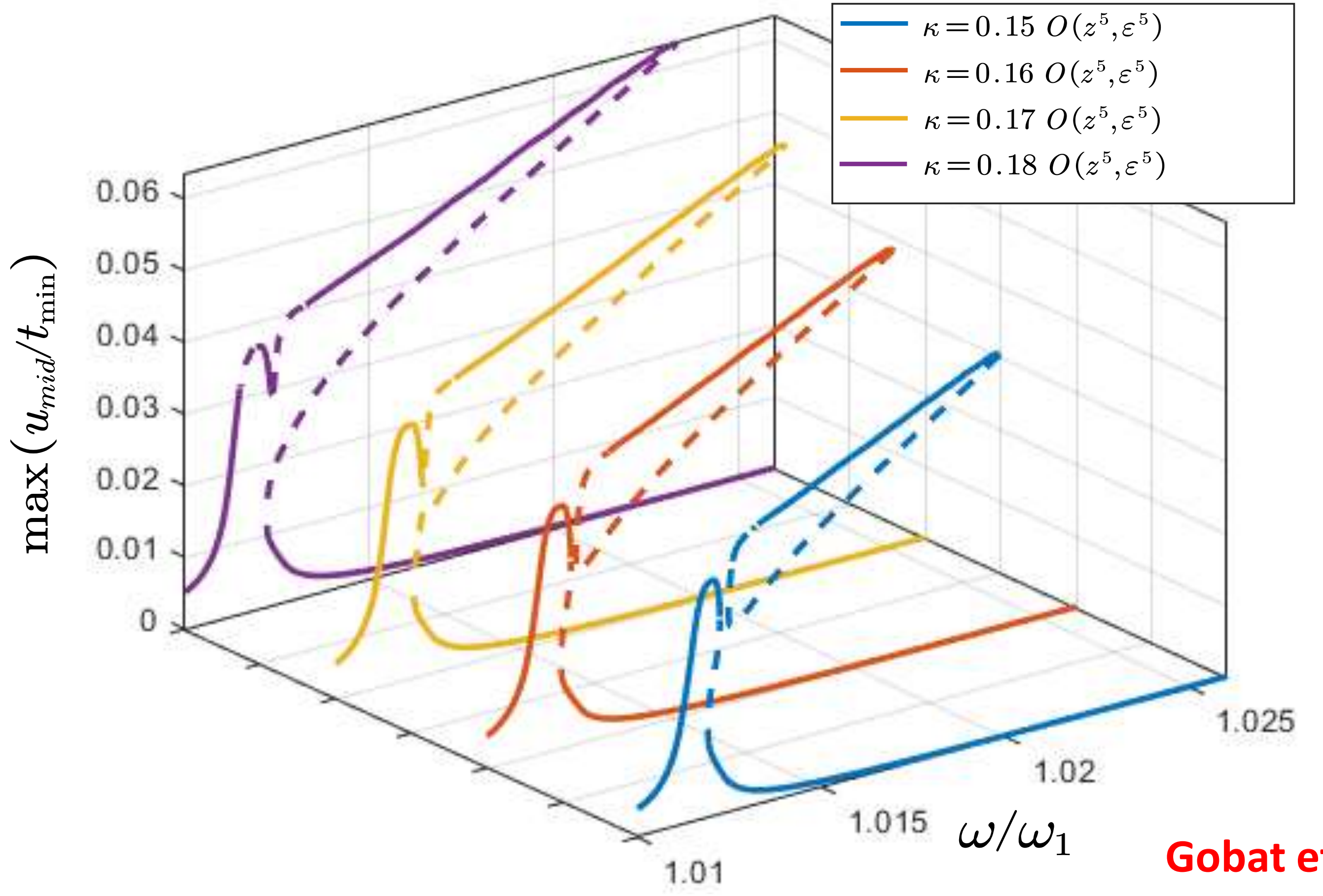


From Uniform to Variable-thickness

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} + \mathbf{G}(\mathbf{U},\ \mathbf{U}) + \mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U}) = 0.18\mathbf{M}\boldsymbol{\phi}_{\mathrm{B1}}\cos(\Omega \mathrm{t})$$





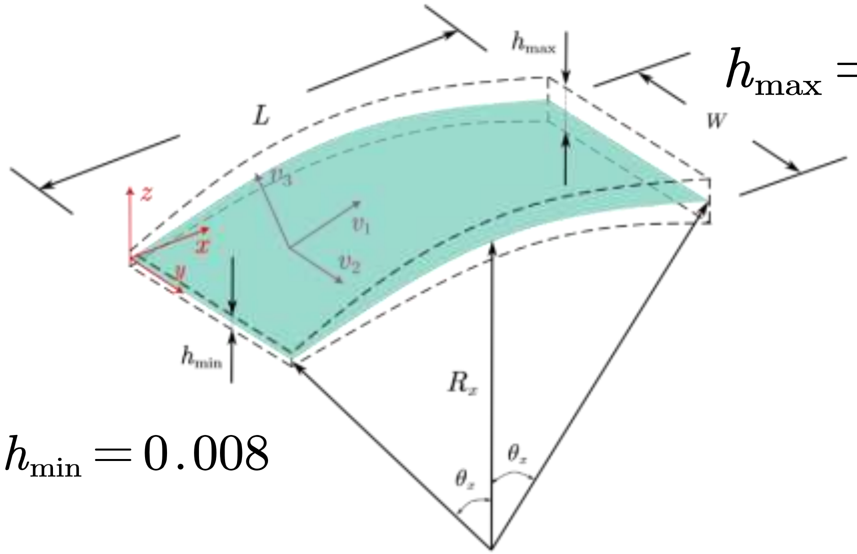


Gobat et al.

From Uniform to Variable-thickness

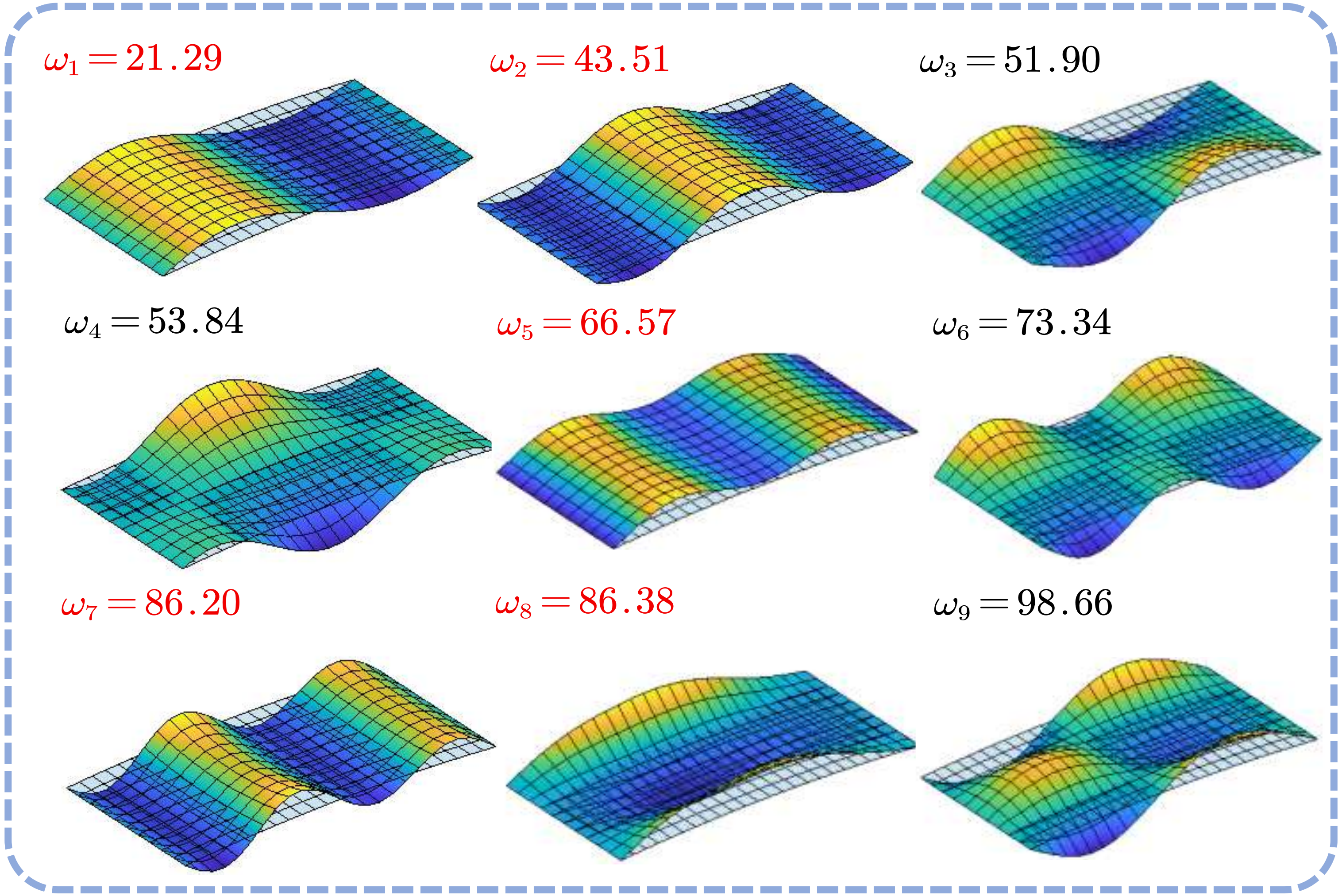
$$h_{\max} = 0.01$$

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} + \mathbf{G}(\mathbf{U},\ \mathbf{U}) + \mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U}) = \kappa \mathbf{M} \boldsymbol{\phi}_{\mathrm{B1}} \cos(\Omega t)$$



$$h_{\min} = 0.008$$

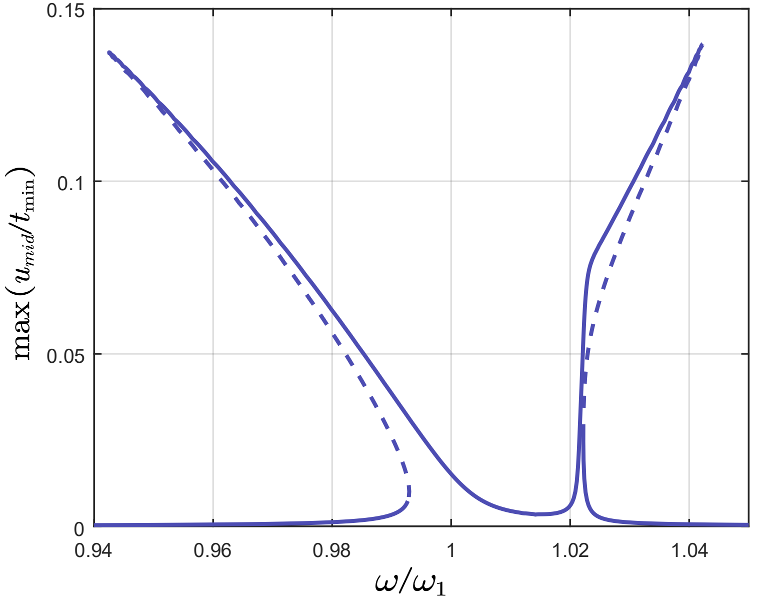
$$h(x) = h_{\min} + \frac{h_{\max} - h_{\min}}{x_{\max} - x_{\min}}(x - x_{\min})$$

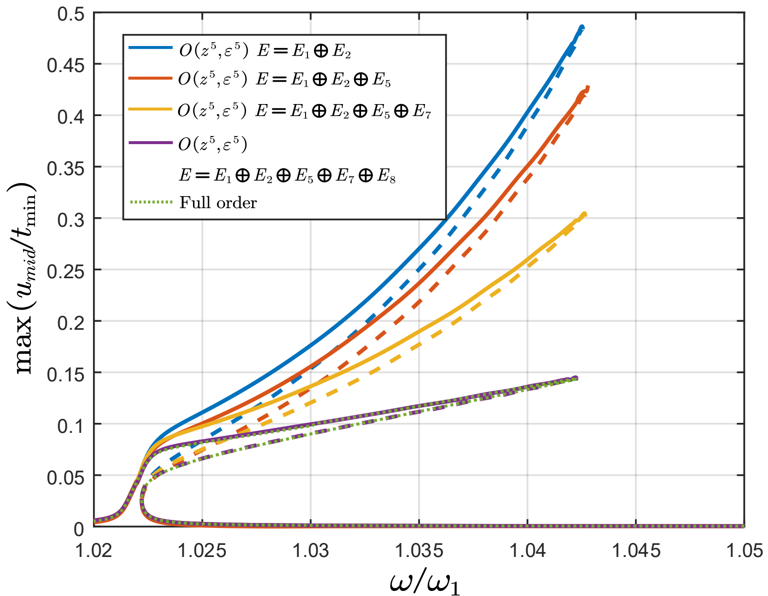


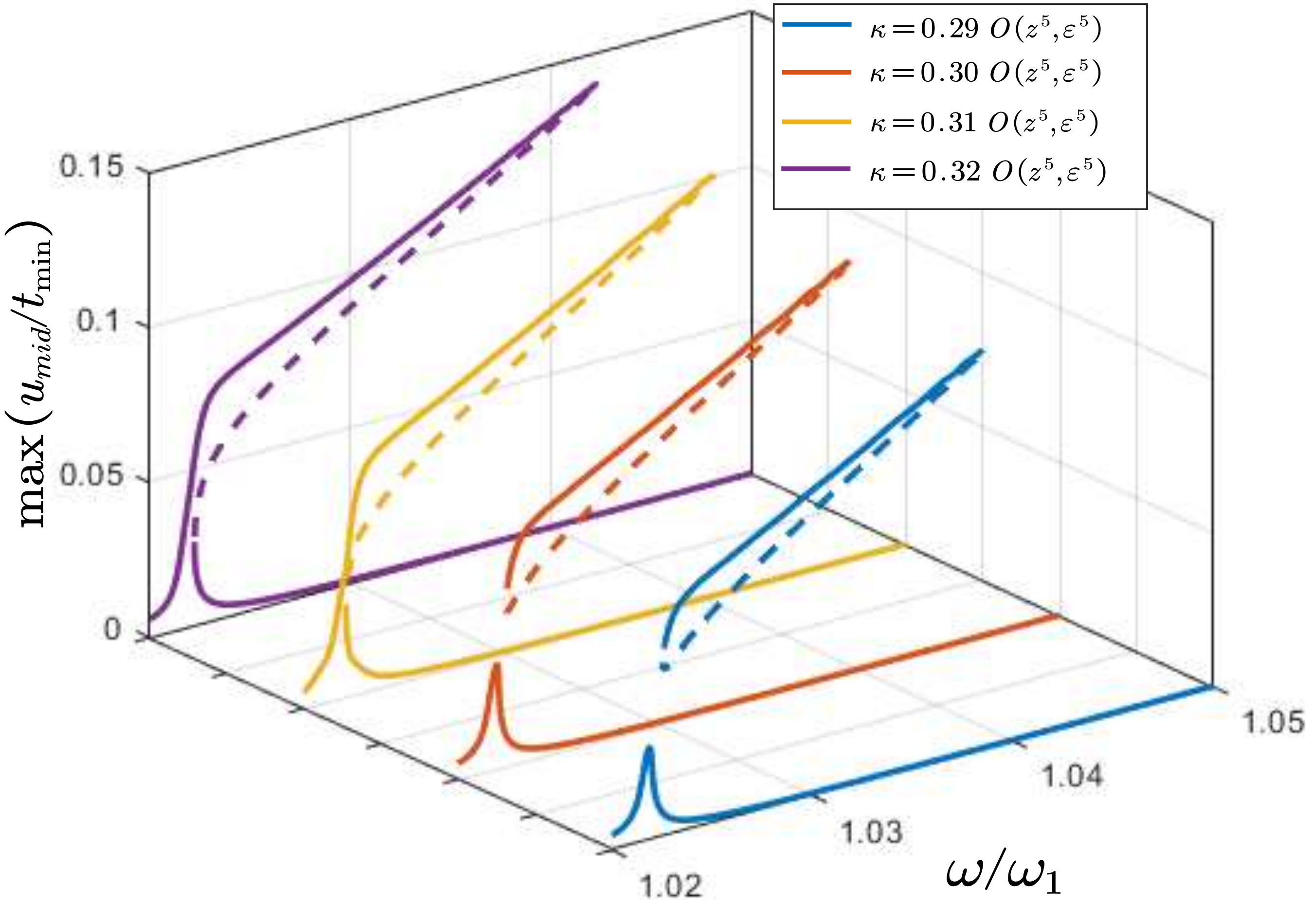
- **With a further reduction in the thickness of the thinner region, the frequency ratio between the first and second modes increases to 2.04**

From Uniform to Variable-thickness

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} + \mathbf{G}(\mathbf{U},\ \mathbf{U}) + \mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U}) = 0.32\mathbf{M}\boldsymbol{\phi}_{\mathrm{B1}}\cos(\Omega \mathrm{t})$$







Master mode selection: a user guideline

Master modes	Order of DPIM	CPU time (s) ROM	CPU time (s) FOM	Time radio
$E = E_1$	$\mathcal{O}(3)$	6.42	92788	14 452.95
	$\mathcal{O}(5)$	49.39		1878.70
$E = E_1 \oplus E_2$	$\mathcal{O}(3)$	17.07	129639	7594.55
	$\mathcal{O}(5)$	376.68		344.16
$E = E_1 \oplus E_2 \oplus E_5 \oplus E_8$	$\mathcal{O}(3)$	72.52	172950	2384.86
	$\mathcal{O}(5)$	3685.24		46.93
$E = E_1 \oplus E_2 \oplus E_5 \oplus E_7 \oplus E_8$	$\mathcal{O}(3)$	122.89	152196	1238.47
	$\mathcal{O}(5)$	6507.12		23.39

The choice of master modes is a key determinant of the performance and efficiency of a ROM !

The guiding principle is to pre-identify the key resonant terms !

Master mode selection: a user guideline

For the conventional CNF method $\lambda_k = \sum_{i=1}^{n} m_i \lambda_i,\ \lambda_i = \omega_i \text{ or } \bar{\omega}_i,\ \text{with } m_i \geqslant 0 \text{ and } \sum_{i=1}^{n} m_i = p$

For $h_{\min} = 0.009 \quad h_{\max} = 0.01$

Order 2 of DPIM

$$\boxed{2\omega_1 \approx \omega_2,} \omega_1 + \omega_2 \approx \omega_5, \omega_1 + \omega_3 \approx \omega_6, \omega_1 + \omega_5 \approx \omega_7, \omega_1 + \omega_5 \approx \omega_8,$$

$$\omega_1 + \bar{\omega}_5 \approx \bar{\omega}_2, \omega_1 + \bar{\omega}_6 \approx \bar{\omega}_3, \omega_1 + \bar{\omega}_7 \approx \bar{\omega}_5, \omega_1 + \bar{\omega}_8 \approx \bar{\omega}_5$$

$$\boxed{2\omega_2 \approx \omega_8,} \omega_2 + \omega_4 \approx \omega_9, \omega_2 + \bar{\omega}_8 \approx \bar{\omega}_2, \omega_3 + \bar{\omega}_6 \approx \bar{\omega}_1$$

Order 3 of DPIM

$$\boxed{3\omega_1 \approx \omega_5,} 2\omega_1 + \omega_2 \approx \omega_8, 2\omega_1 + \omega_4 \approx \omega_9, 2\omega_1 + \bar{\omega}_5 \approx \bar{\omega}_1, 2\omega_1 + \bar{\omega}_8 \approx \bar{\omega}_2, \omega_1 + \omega_6 + \bar{\omega}_2 \approx \omega_3$$

$$\omega_1 + \omega_6 + \bar{\omega}_3 \approx \omega_2, \omega_1 + \omega_7 + \bar{\omega}_4 \approx \omega_4, \omega_1 + \omega_8 + \bar{\omega}_2 \approx \omega_5, \omega_1 + \omega_8 + \bar{\omega}_5 \approx \omega_2, \omega_1 + 2\bar{\omega}_2 \approx \bar{\omega}_5$$

$$\omega_1 + \bar{\omega}_2 + \bar{\omega}_3 \approx \bar{\omega}_6, \omega_1 + \bar{\omega}_2 + \bar{\omega}_5 \approx \bar{\omega}_7, \omega_1 + \bar{\omega}_2 + \bar{\omega}_5 \approx \bar{\omega}_8, \omega_1 + 2\bar{\omega}_4 \approx \bar{\omega}_7 \cdots$$

Identifying the invariance-breaking terms is an efficient way to find the master modes !

Master modes selected as $E = E_1 \oplus E_2 \oplus E_5 \oplus E_8$

Master mode selection: a user guideline

For $h_{\min} = 0.008 \quad h_{\max} = 0.01$

Order 2 of DPIM

$$\boxed{2\omega_1 \approx \omega_2}, \omega_1 + \omega_2 \approx \omega_5, \omega_1 + \omega_3 \approx \omega_6, \omega_1 + \omega_4 \approx \omega_6, \omega_1 + \omega_5 \approx \omega_7$$

$$\omega_1 + \omega_5 \approx \omega_8, \omega_1 + \omega_6 \approx \omega_9, \omega_1 + \bar{\omega}_2 \approx \bar{\omega}_1, \omega_1 + \bar{\omega}_5 \approx \bar{\omega}_2, \omega_1 + \bar{\omega}_6 \approx \bar{\omega}_3$$

$$\omega_1 + \bar{\omega}_6 \approx \bar{\omega}_4, \omega_1 + \bar{\omega}_7 \approx \bar{\omega}_5, \omega_1 + \bar{\omega}_8 \approx \bar{\omega}_5$$

$$\boxed{2\omega_2 \approx \omega_7}, \boxed{2\omega_2 \approx \omega_8} \cdots$$

Order 3 of DPIM

$$\boxed{3\omega_1 \approx \omega_5}, 2\omega_1 + \omega_2 \approx \omega_7, 2\omega_1 + \omega_2 \approx \omega_8, 2\omega_1 + \omega_3 \approx \omega_9$$

$$2\omega_1 + \omega_4 \approx \omega_9, 2\omega_1 + \bar{\omega}_7 \approx \bar{\omega}_2, 2\omega_1 + \bar{\omega}_8 \approx \bar{\omega}_2$$

$$2\omega_1 + \bar{\omega}_9 \approx \bar{\omega}_4, \omega_1 + \omega_2 + \bar{\omega}_7 \approx \bar{\omega}_1, \omega_1 + \omega_2 + \bar{\omega}_8 \approx \bar{\omega}_1 \cdots$$

Master modes selected as $E = E_1 \oplus E_2 \oplus E_5 \oplus E_7 \oplus E_8$

We have previously validated that these selected master modes are sufficient to reconstruct the FOM results

Conclusion & Future works

Highlights

1. **Nonlinear shell dynamics framework** based on invariant manifold ROMs

2. Addresses strategies tackling multiple, **complex nonlinearities: 1:2 resonance, hardening/softening, isolas**

3. Provides **practical guidelines** for master mode selection in nonlinear ROM construction

4. Demonstrates accuracy and efficiency on **shells with curvature and imperfections**

Future works

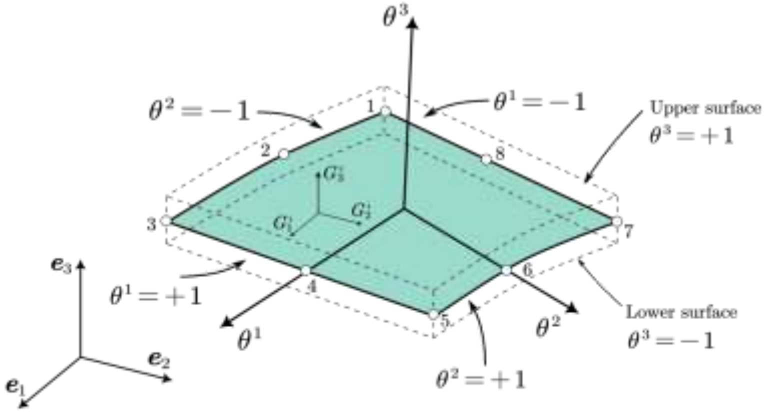
1. Based on the validation results of this work, we will proceed to study more **practical engineering problems** concerning complex nonlinear geometrically thin structures, which are not convenient to analyze using solid elements

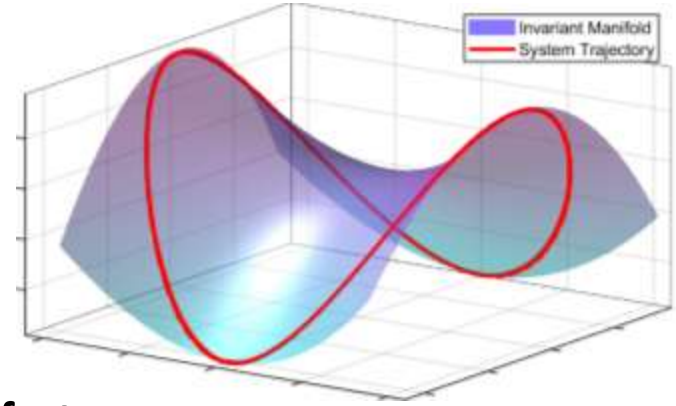
2. On this basis, we will also consider more complex working conditions, including **fluid-structure interaction** problems and **material nonlinearity** issues

Open-Source Code & Resources

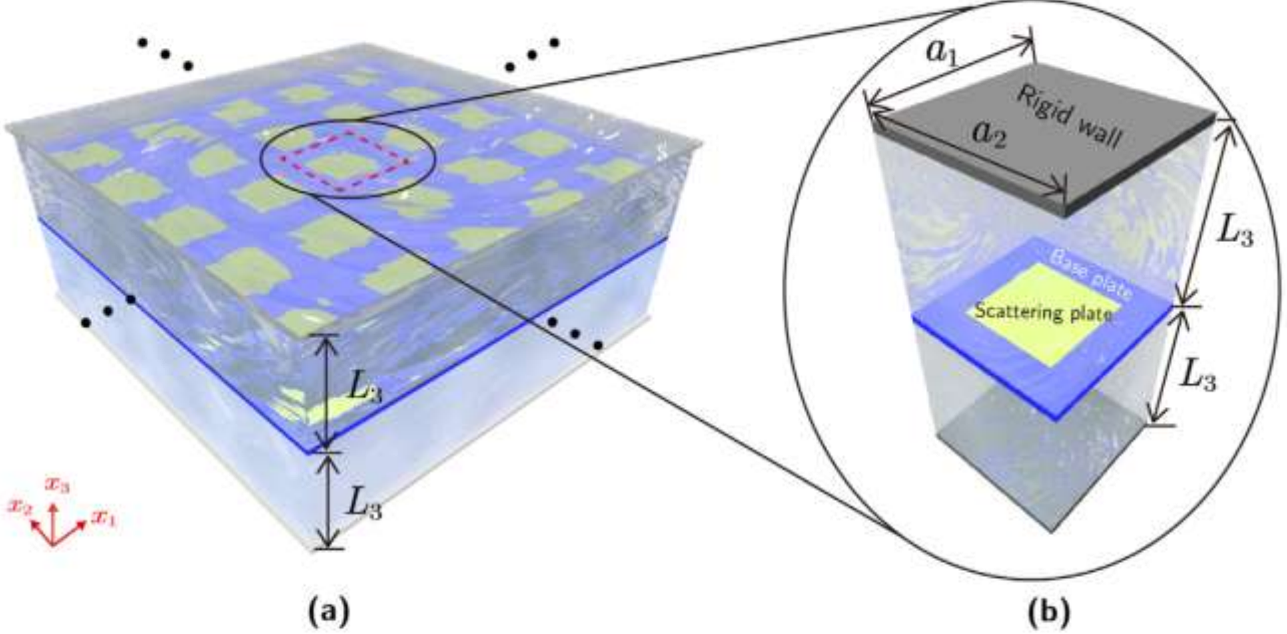
evry-dynamics (Structure dynamics research group)

DPIM ROM based on solid-shell element





Metamaterial with hydro-elastic effect



ÉCOLE DOCTORALE
Sciences mécaniques et énergétiques, matériaux et géosciences (SMEMAG)

Thanks for your listening

16-17 octobre 2025
Les journées annuelles du GDR EX-MODELI à Lille dans les locaux de l'ENSAM

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