LaMCoS
Laboratoire de Mécanique
des Contacts et des Structures
UMR 5259

EX-MODELI
EXPLOITATION & MODÉLISATION des DYNAMIQUES NON LINÉAIRES
GdR CNRS 2134

A frequency-domain framework for Lyapunov stability and bifurcation analysis of quasi-periodic solutions via Koopman-Hill projections

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LaMCoS / UMR CNRS 5259 / INSA Lyon

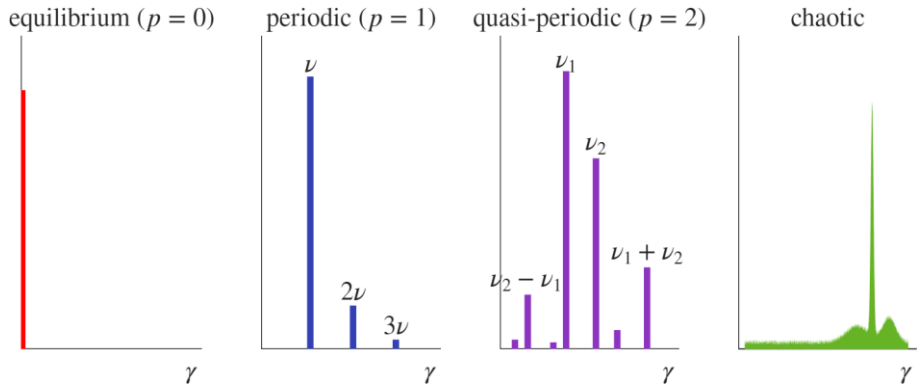
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Quasi-periodic motion

Motions and their spectra:

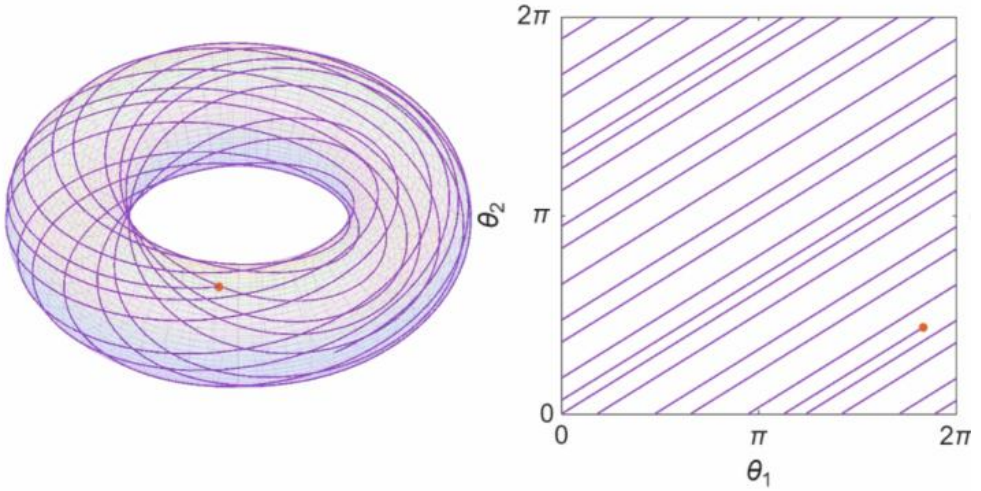
- Equilibrium ($p = 0$): constant, a single line at zero frequency
- Periodic ($p = 1$): lines at $\nu, 2\nu, 3\nu, \ldots$, and the motion repeats after $T = \frac{2\pi}{\nu}$
- Quasi-periodic ($p \geq 2$): lines at $k_1\nu_1 + k_2\nu_2 + \ldots$, and it never repeats
- Chaotic: continuous spectrum, with no finite set of base frequencies

Frequency spectra



Quasi-periodic motion:

- Two or more incommensurate frequencies
  $\nu_1, \nu_2, \ldots, \nu_p$ e.g. for $p = 2$, $\frac{\nu_1}{\nu_2} \notin \mathbb{Q}$
- The trajectory never exactly repeats
- Trajectory winds densely on an invariant torus $\mathbb{T}^p$
- FFT shows combination harmonics
  $k_1\nu_1 + k_2\nu_2 + \ldots + k_p\nu_p$ with $k_i \in \mathbb{Z}$



Harmonic Balance Method (HBM) for quasi-periodic solutions

Dynamical system $\dot{\boldsymbol{q}}(t) = f(\boldsymbol{q}(t), t), \qquad \boldsymbol{q} \in R^n$

Approximate the torus solution with a truncated Fourier series

$$\boldsymbol{q}(t) = \sum_{k \in \mathcal{K}^p} \widehat{\boldsymbol{Q}_k} e^{i\boldsymbol{k}\cdot\boldsymbol{\nu}t} = (\boldsymbol{\Gamma}(\boldsymbol{\nu}) \otimes \boldsymbol{I_n})\widehat{\boldsymbol{Q}}, \qquad \text{with } \boldsymbol{\nu} = (\nu_1, \dots, \nu_p) \text{: the base frequencies}$$

Substitute the Fourier expansion into the invariance equation

$$\mathbf{R}(\mathbf{Q}, \boldsymbol{\nu}) = (\nabla(\nu) \otimes I_n)\, \boldsymbol{Q} - \boldsymbol{F}(\boldsymbol{Q}) = \mathbf{0}$$

Size grows as $\mathrm{n}(2N+1)^p$**:** exponential in the number of frequencies.

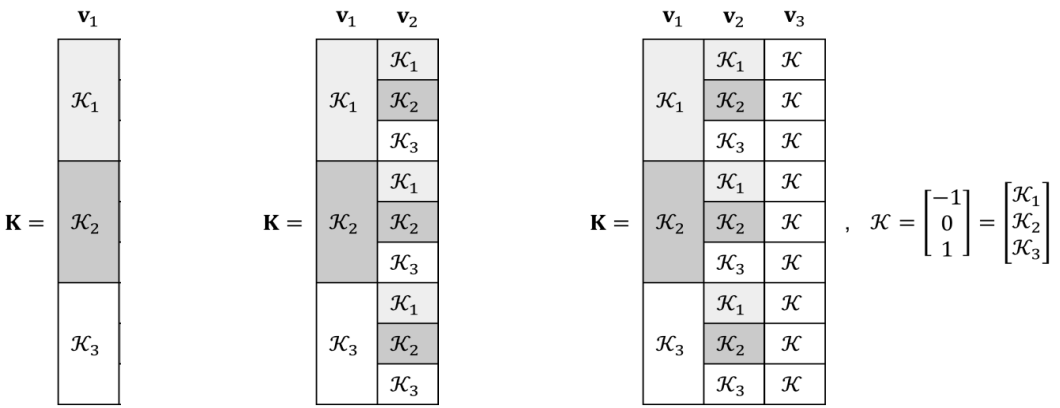


Illustration of matrix K, for N = 1

- n: Number of degrees of freedom
- N: Harmonic truncation order of HBM

Stability analysis

Dynamical system

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{f}(\boldsymbol{q}(t), t), \qquad q \in \mathbb{R}^n$$

- $\boldsymbol{f}$ is assumed to be Lipschitz continues and continuously differentiable with respect to $\mathbf{q}$
- $\boldsymbol{q}^p(t_0)$is reference solution
- $\boldsymbol{q}(t_0)$ is neighboring trajectory
- $\Delta\boldsymbol{q}(t_0) = \boldsymbol{q}(t_0) - \boldsymbol{q}^p(t_0)$: The perturbed solution

Variational equation around $\boldsymbol{q}^p(t_0)$

$$\Delta\dot{\boldsymbol{q}}(t) = \left.\frac{\partial f(\boldsymbol{q},t)}{\partial \boldsymbol{q}}\right|_{q^p(t)} \Delta\boldsymbol{q}(t) = \boldsymbol{J}(t)\,\Delta\boldsymbol{q}(t)$$

A set of fundamental solutions:

$$\Delta\boldsymbol{q}(t) = \boldsymbol{\Psi}(t, t_0)\Delta\boldsymbol{q}(t_0)$$

Stability reduces to how $\boldsymbol{\Psi}(t, 0)$ behaves as $t \to \infty$

For periodic solutions:

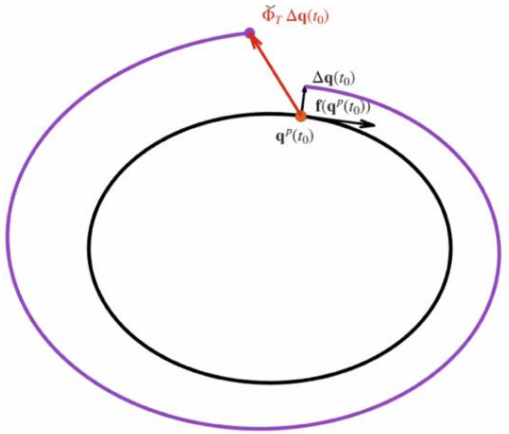
$$f(\boldsymbol{q}(t), t) = f(\boldsymbol{q}(t+T), t+T)$$

$$\Delta\boldsymbol{q}(T) = \boldsymbol{M}_T\Delta\boldsymbol{q}(0)$$

Monodromy matrix

$$\boldsymbol{M}_T = \boldsymbol{\Psi}(T, 0) \rightarrow \text{self-map}$$

Eigenvalues of Monodromy matrix are the Floquet multipliers μ_i.



Stable if all $|\boldsymbol{\mu_i}| < \mathbf{1}$

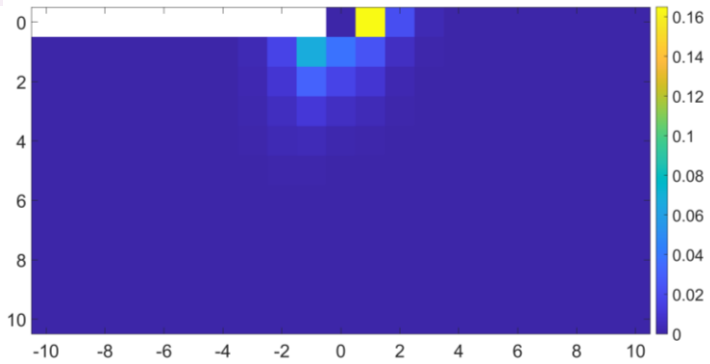
Objectives

- **Stability analysis of quasi-periodic solutions in the frequency domain:** bypassing long time integrations required by existing methods
- **Bifurcation classification of quasi-periodic solutions**: leveraging the same frequency-domain framework

Koopman–Hill framework for frequency-domain stability and bifurcation analysis of quasi-periodic solutions

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Harmonic selection procedure



Problem:

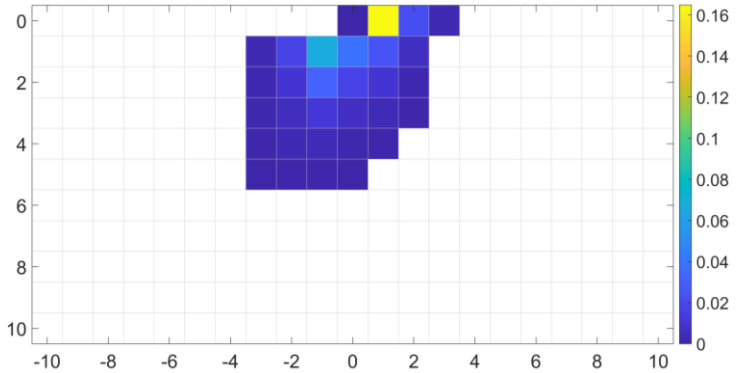
- Full basis HBM: $n(2N+1)^p$
- Most harmonics contribute negligibly to the solution

Harmonic selection procedure:

- Operate on a converged full-basis solution $Q \in R^{n(2N+\mathbb{1})^p}$
- For each DOF d: retain only harmonics whose content $A_{k,d}$ exceeds a relative threshold σ_d

- Global set: $\mathcal{H}_d = \{k \in [-N, N]^p \setminus \{0\} : A_{k,d} > \sigma_d\}$

$$\mathcal{H} = \bigcup_{d=1}^{n} \mathcal{H}_d$$



Acceptance of retained set:

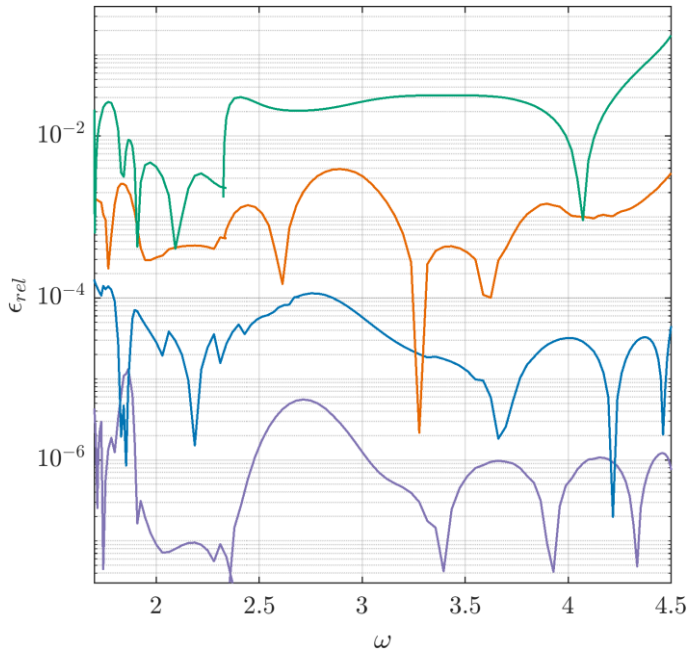
$$\varepsilon_{\text{rel}} < \varepsilon$$

Continuation runs entirely on reduced system: $|\mathcal{H}| \ll (2N+1)^p$

Harmonic selection for Duffing oscillator

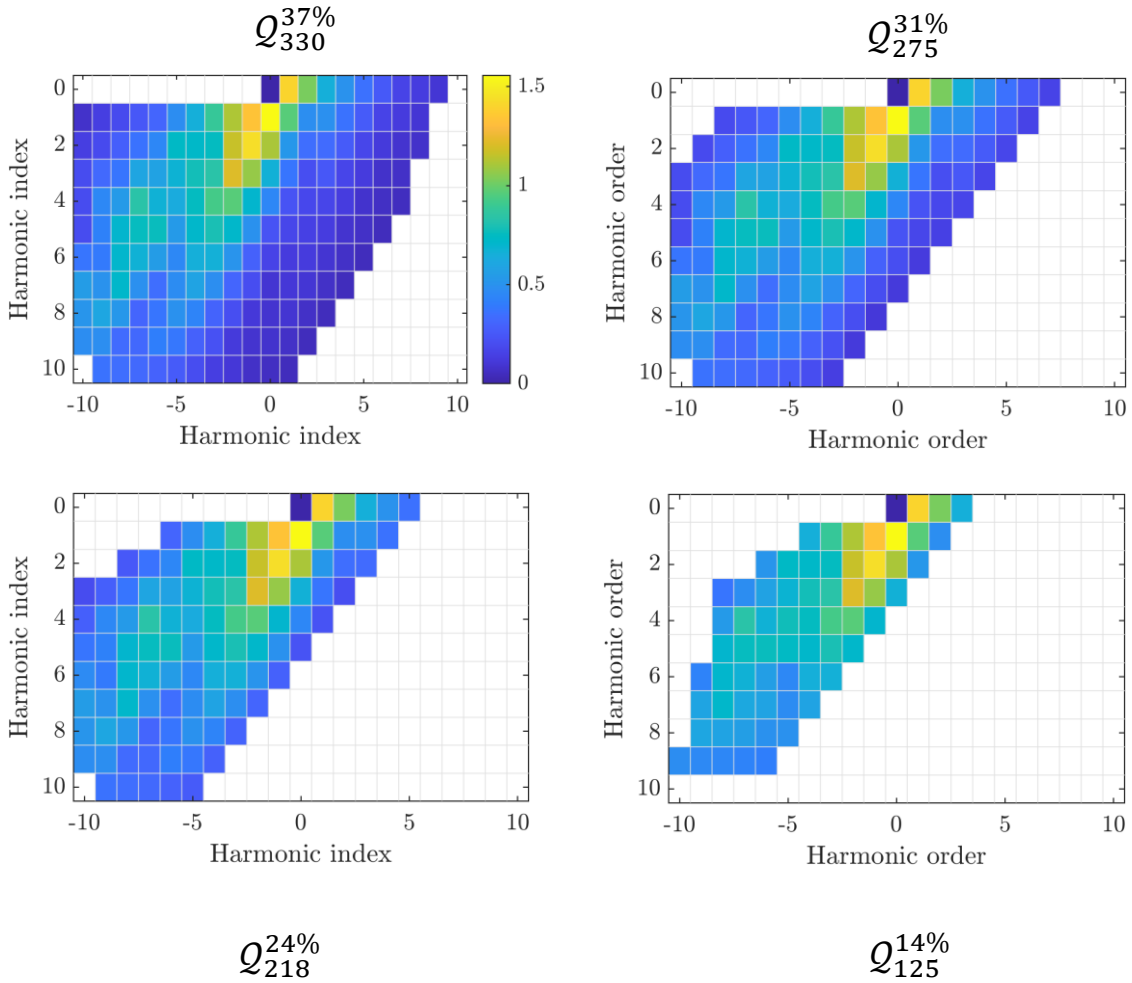
Quasi-periodically forced Duffing oscillator:

$$\ddot{x} + 2\zeta\dot{x} + x + \xi x^3 = f_1 \cos(\omega_1 t) + f_2 \cos(\omega_2 t)$$



The relative error between the amplitude of harmonically selection and full basis solutions $Q_{882}^{100\%}$.

N=20, n=2, n(2*N+1)=882 → full basis solutions $Q_{882}^{100\%}$



The computational time reduces roughly proportionate to the retained harmonic percentage.

Koopman–Hill framework for frequency-domain stability and bifurcation analysis of quasi-periodic solutions

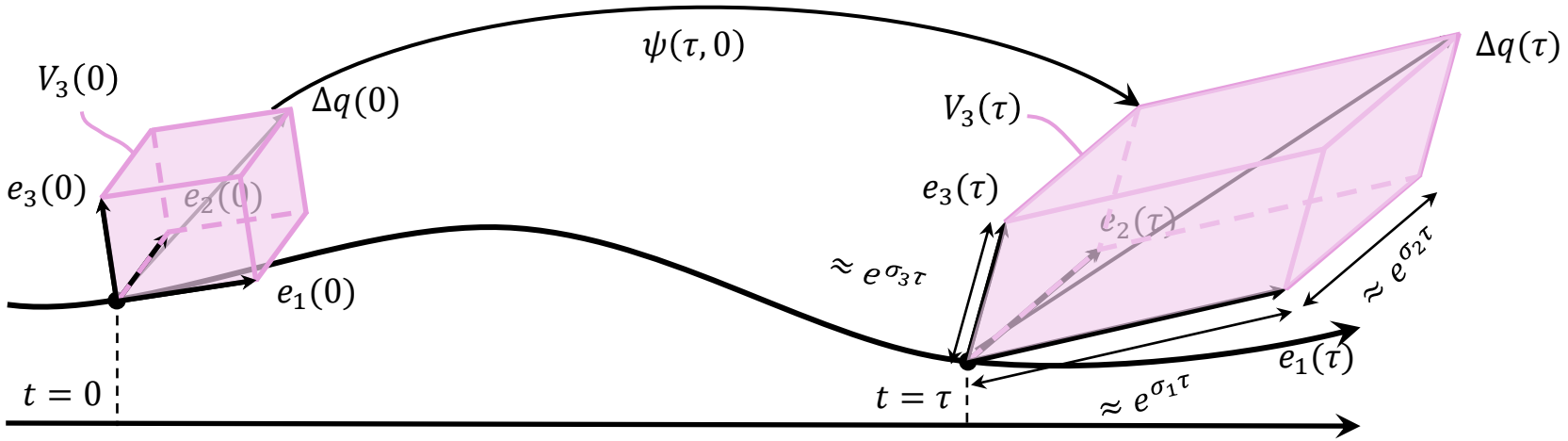
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Lyapunov exponents

Lyapunov (characteristic) exponents definition [1]

$$\chi(\Delta q_0) = lim\ sup_{t\to\infty} \frac{1}{t} \ln|\Psi(t,0)\Delta q_0|$$

The long-term average exponential growth rate of each perturbations direction defines Lyapunov exponents



[1] G. Benettin, et. al., (1980), "Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems: A method for computing all of them," Meccanica.

Frequency-domain Lyapunov exponents via Koopman-Hill framework

Hill matrix as a byproduct of HBM

$$\mathbf{R}_{\mathrm{Q}} = \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{Q}}$$

Koopman-Hill framework (Bayer & Leine 2023) [2]

$$M_T \approx C e^{(\boldsymbol{R_Q} T)} W$$

$$C = (1, 0, \ldots, 0) \otimes \boldsymbol{I}_{\mathrm{n}}$$

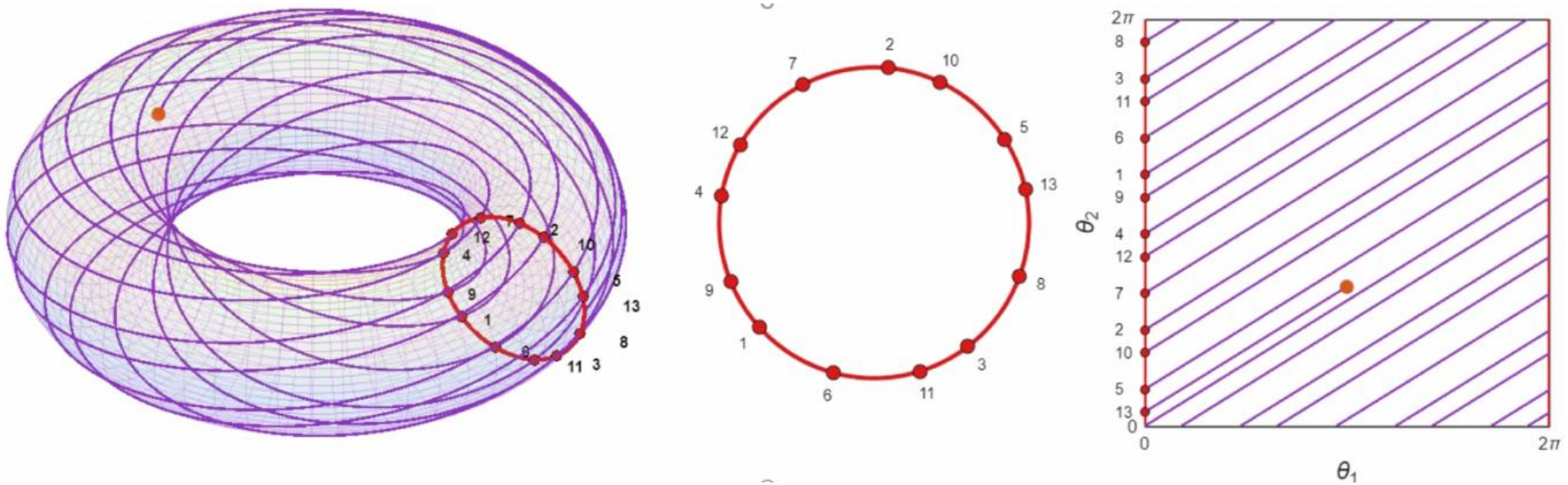
$$\mathrm{W} = (1,\ 2, 0,\ \ldots, 2,\ 0)^{\mathsf{T}} \otimes \boldsymbol{I}_{\mathrm{n}}$$

Discrete fundamental matrix at boundary point ρ:

$$\psi(\boldsymbol{\rho}_{\mathrm{k}}) \approx \boldsymbol{C}\, e^{(\boldsymbol{R_Q}\tau)} \boldsymbol{W}(\boldsymbol{\rho}_{\mathrm{k}})$$

$W(\rho)$ is evaluated at N_{Ch} points

$$\boldsymbol{W}(\boldsymbol{\rho}_{\mathrm{k}}) = (1,\ 2\cos(n_2\boldsymbol{\rho}_{\mathrm{k}}),\ 2\sin(n_2\boldsymbol{\rho}_{\mathrm{k}}),\ \ldots,\ 2\cos(n_N\boldsymbol{\rho}_{\mathrm{k}}),\ 2\sin(n_N\rho_{\mathrm{k}}))^{\mathsf{T}} \otimes \boldsymbol{I}_{\mathrm{n}}$$

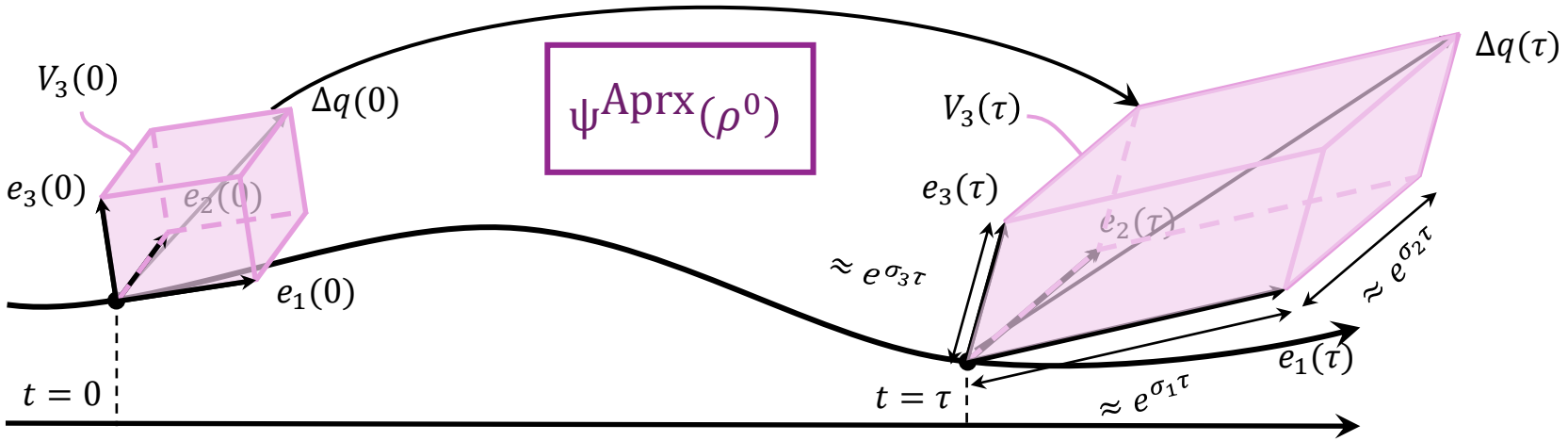


[2] Bayer, F., & Leine, R. I. (2023). "Sorting-free Hill-based stability analysis of periodic solutions through Koopman analysis." Nonlinear Dynamics.

Lyapunov exponent computation via the Koopman-Hill framework

At each step $k = 1, \dots, N_{map}$, we propagate

$$\psi^{(k)} = \psi^{\mathrm{Aprx}}\left(\rho^{(k-1)}\right) \cdot E^{(k-1)}$$



We completely bypass the time integration to perform all the computation in frequency domain

Spectrum recovery

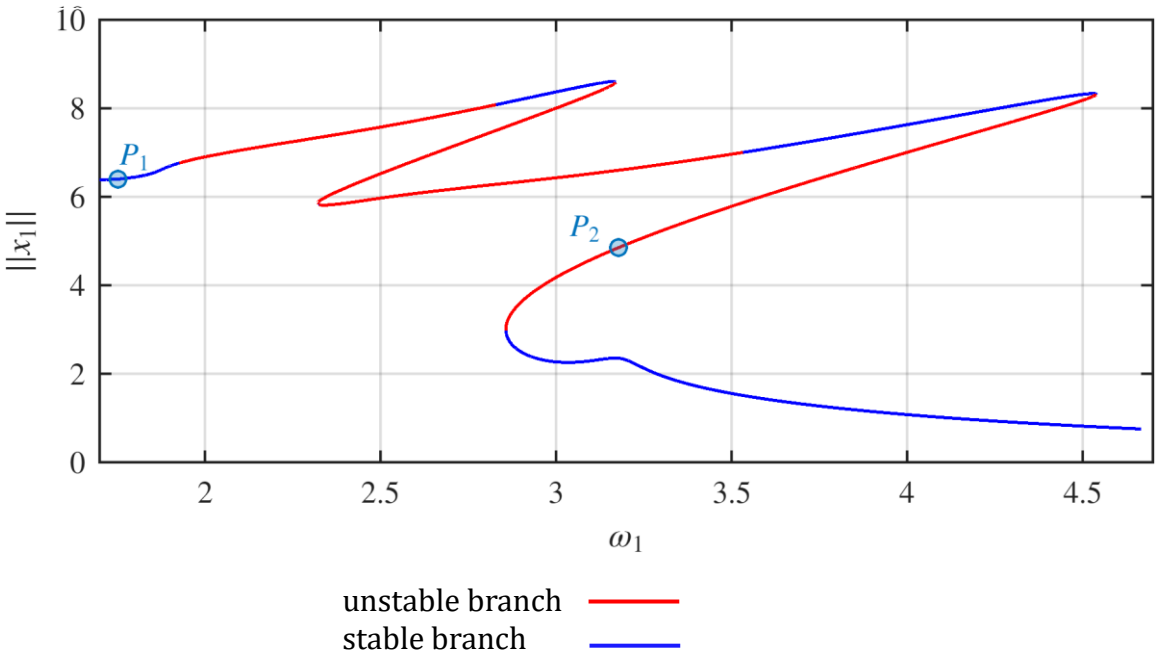
$$\sigma^{(\mathrm{m})} = \lim_{\{k \to \infty\}} \frac{V_m^k}{k\tau} = \sigma^1 + \cdots + \sigma_{\mathrm{m}}$$
$$\sigma^1 = \sigma^{(1)}, \sigma_{\mathrm{m}} = \sigma^{(\mathrm{m})} - \sigma^{(\mathrm{m}-1)}$$

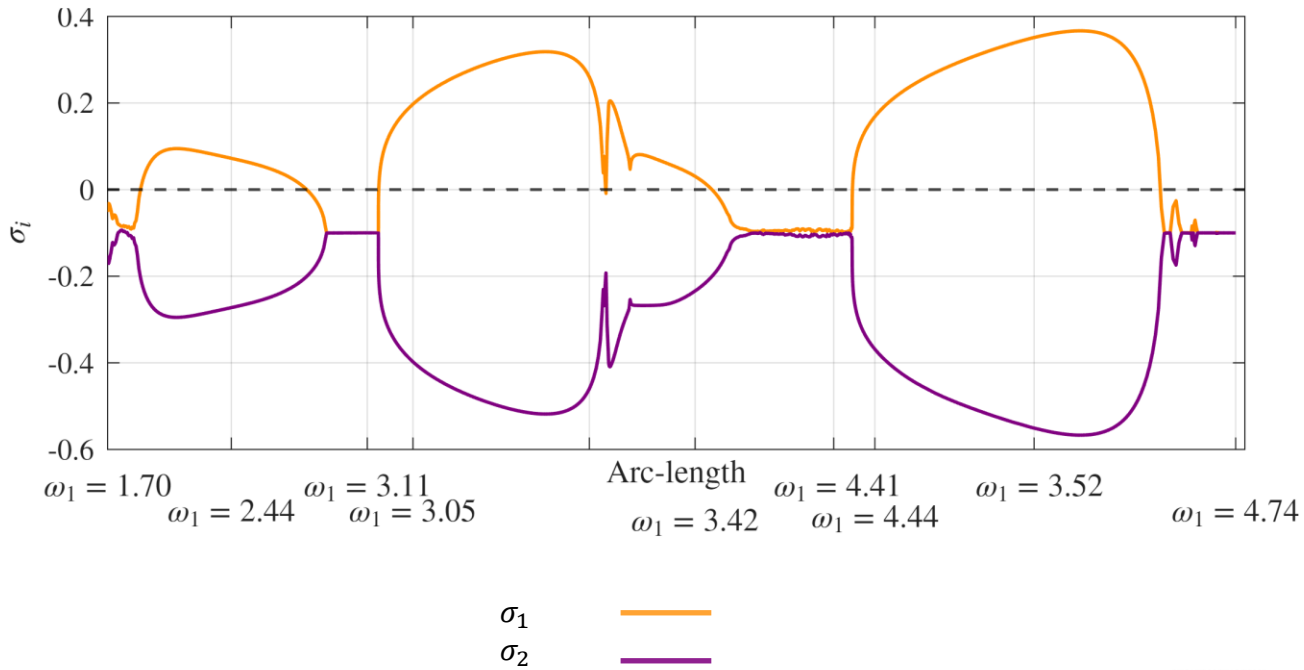
Stability analysis and Lyapunov spectrum

Quasi periodically forced Duffing oscillator

$$\ddot{x} + 2\zeta\dot{x} + x + \xi x^3 = f_1 \cos(\omega_1 t) + f_2 \cos(\omega_2 t)$$

- $\zeta = 0.1, \xi = 0.2,\ f_1 = 5, f_2 = 5$
- $\omega_2 = \frac{\omega_1}{\sqrt{2}}$: two incommensurable frequencies
- $x_1 = x$ displacement
- $x_2 = \dot{x}$ velocity



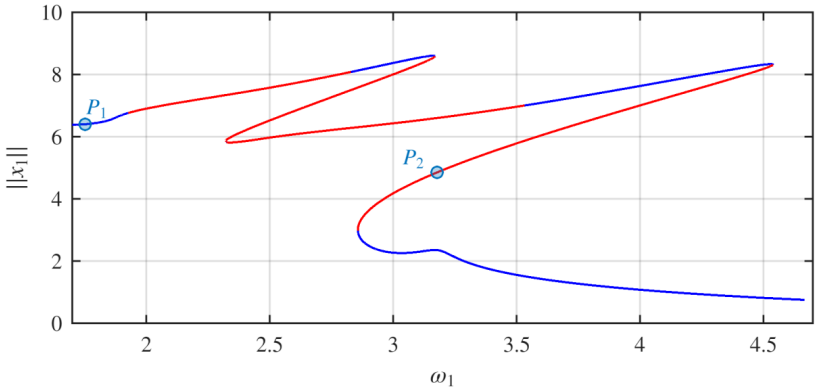


Stability validation: stable point P_1

Quasi periodically forced Duffing oscillator

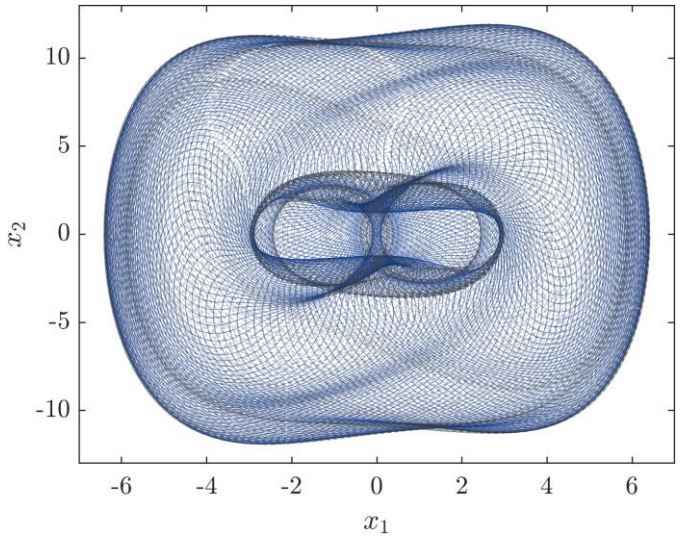
$$\ddot{x} + 2\zeta\dot{x} + x + \xi x^3 = f_1 \cos(\omega_1 t) + f_2 \cos(\omega_2 t)$$

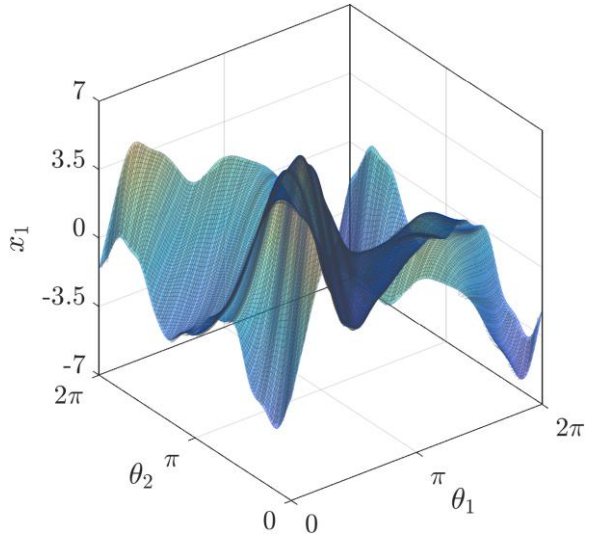
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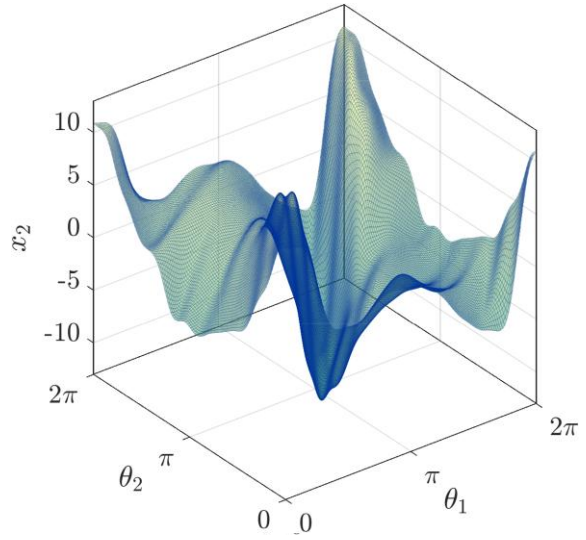


unstable branch
stable branch

Torus function P_1: stable solution at $\omega_1 = 1.75$ ⟶ $[\sigma_1, \sigma_2] = [-0.0563, -0.1293]$







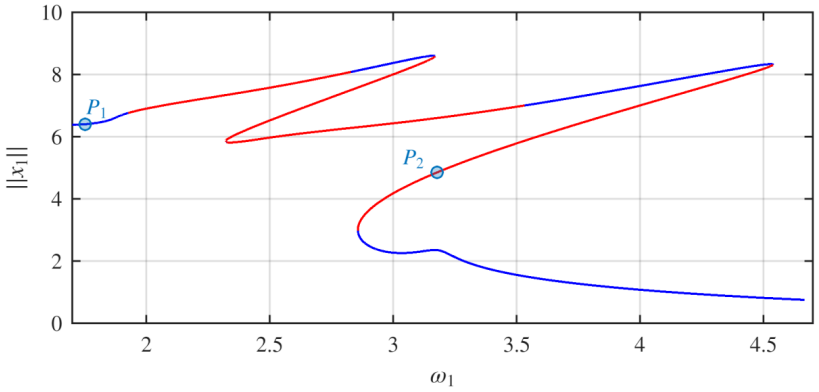
HBM time simulation

Stability validation: unstable point P_2

Quasi periodically forced Duffing oscillator

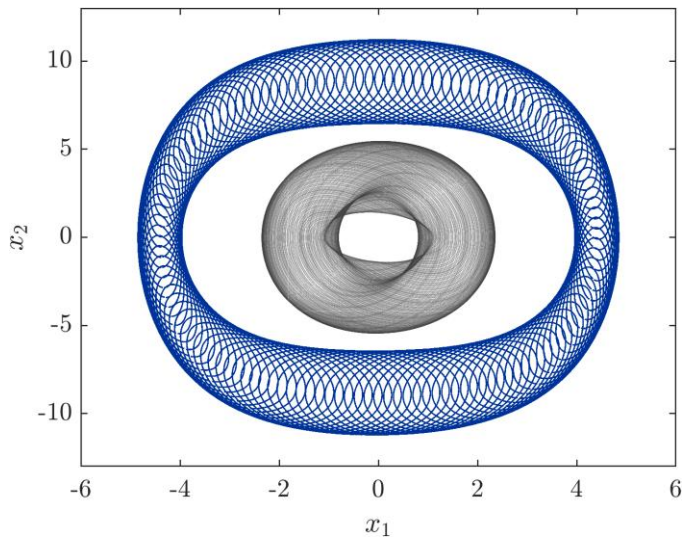
$$\ddot{x} + 2\zeta\dot{x} + x + \xi x^3 = f_1 \cos(\omega_1 t) + f_2 \cos(\omega_2 t)$$

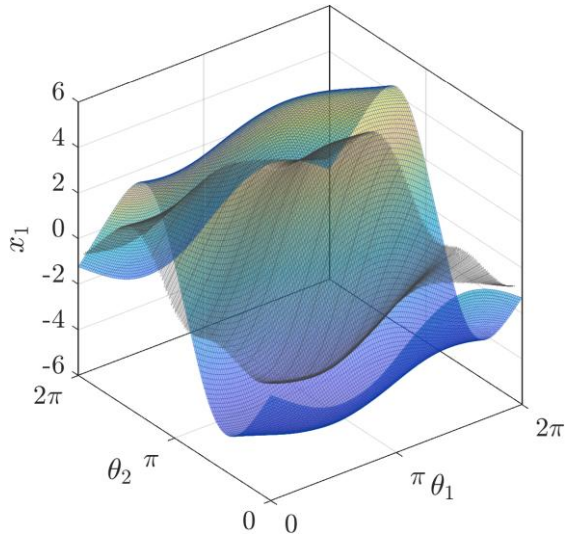
- $x_1 = x$ displacement
- $x_2 = \dot{x}$ velocity

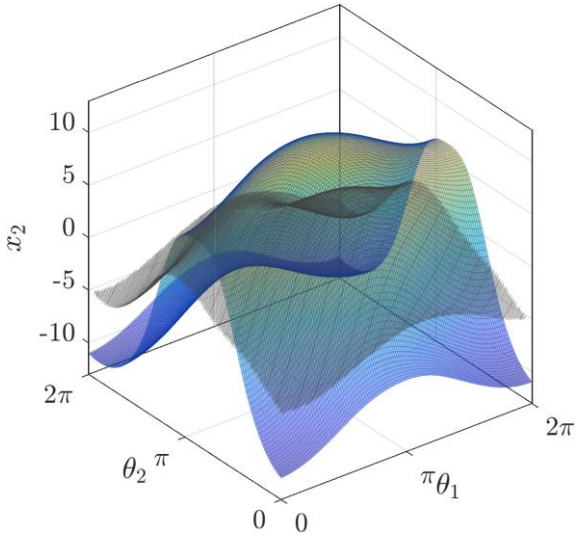


unstable branch
stable branch

Torus function P_2: stable solution at $\omega_1 = 3.17$ ⟶ $[\sigma_1, \sigma_2] = [0.3657, -0.5638]$



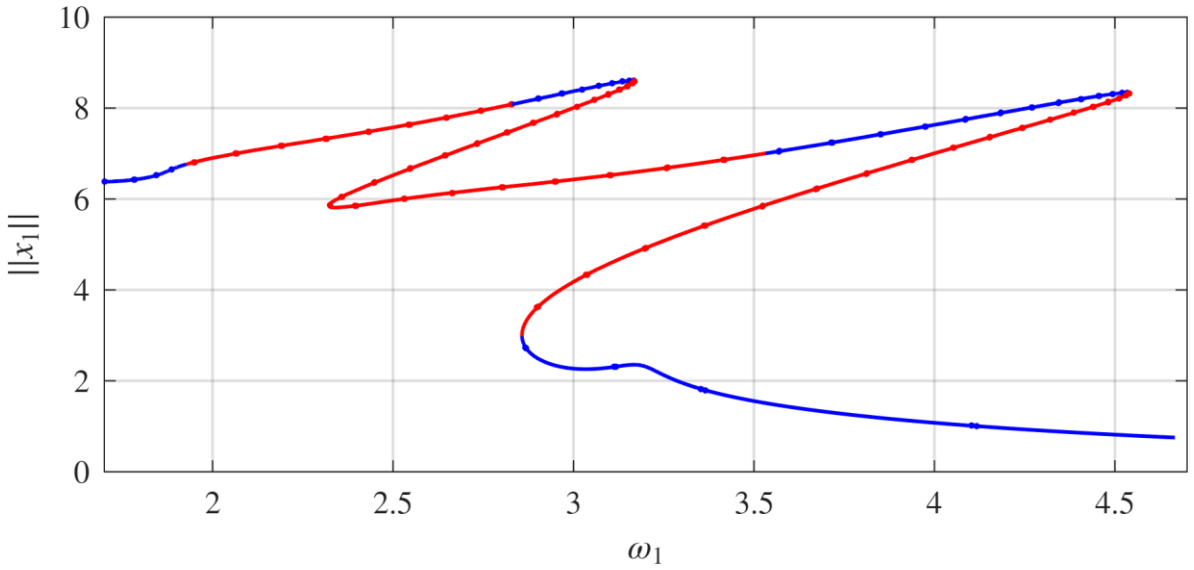




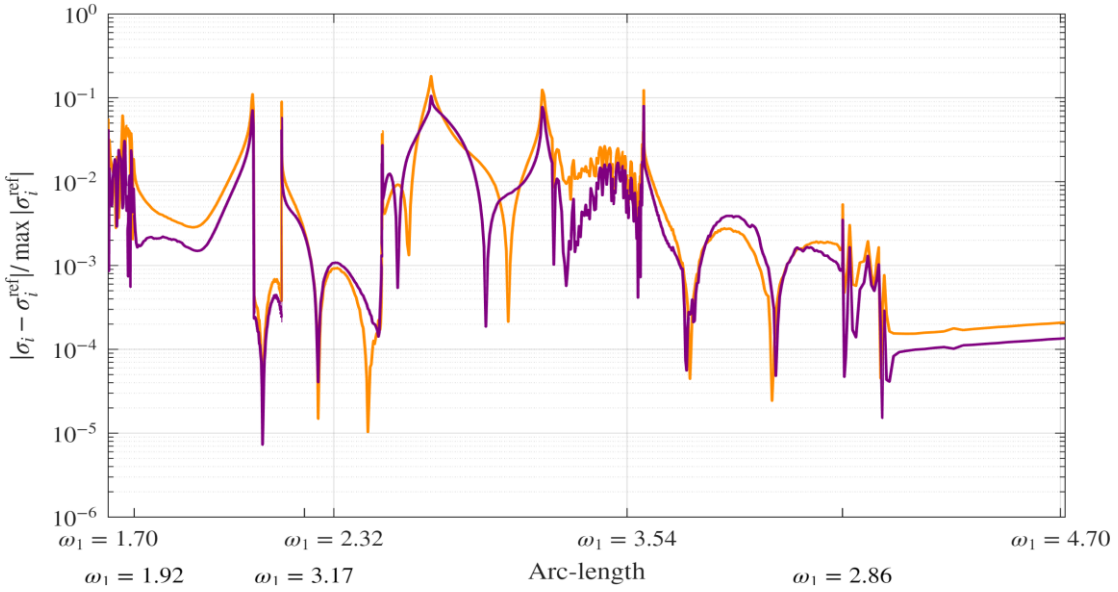
HBM time simulation

Validation: Koopman-Hill Lyapunov vs. time-domain integration

Fiedler et al. [3] compute the Lyapunov exponent spectrum by integrating the variational equation along the torus characteristics



Relative error of the Lyapunov exponent spectrum



Computational time for each point on the continuation branch under same conditions:

- Fiedler's method: each point: 62 s
- Current method: 2 s (31 times faster)

This work: unstable branch — stable branch

Fiedler et al. [1]: unstable branch — stable branch

Relative error in σ_1 — relative error in σ_2

[3] Fiedler, R., Hetzler, H., & Bäuerle, S. (2024). Efficient numerical calculation of Lyapunov-exponents and stability assessment for quasi-periodic motions in nonlinear systems. Nonlinear Dynamics.

Koopman–Hill framework for frequency-domain stability and bifurcation analysis of quasi-periodic solutions

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Bifurcation classification: Dominant Lyapunov Bundle (DLB)

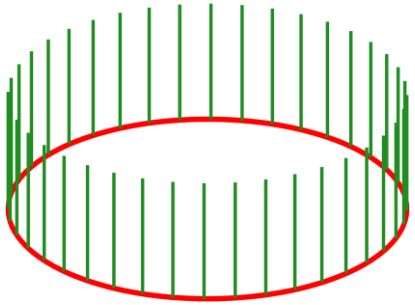
At bifurcation: the Dominant Lyapunov Exponent (DLE) crosses zero, $\sigma_{DLE} \approx 0$

The Lyapunov vectors associated with σ_{DLE} form a bundle over the Poincaré section: the Dominant Lyapunov Bundle (DLB) [4].

Their global topology of DLB identifies the bifurcation type [4].

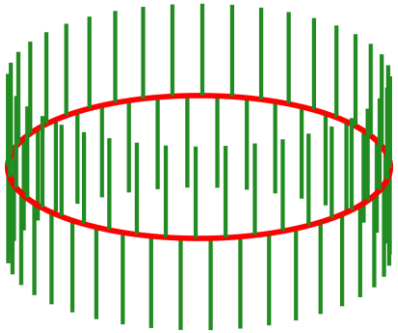
The DLB is recovered in the frequency domain without any additional time integration.

$\mathbf{A^+}$ (orientation-preserving annulus)

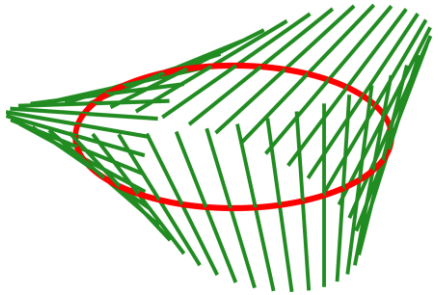


Fold bifurcation

$\boldsymbol{A^-}$ *(orientation-reversing annulus)*

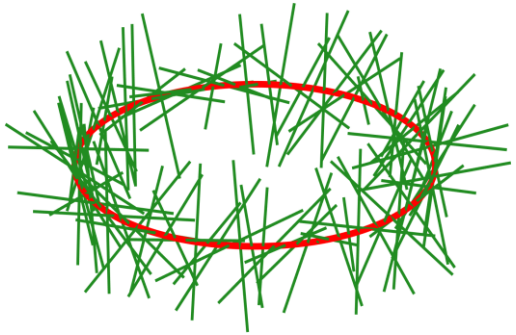


M (Möbius band)



Period-doubling bifurcation

$\boldsymbol{F_{NS}}$ (focus)

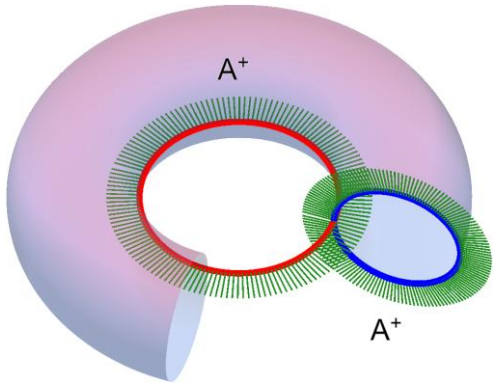


Neimark-Sacker bifurcation

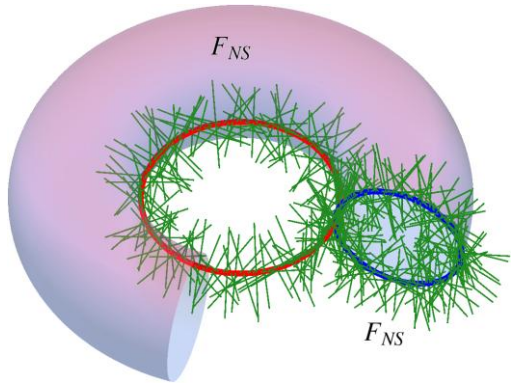
[4] Kamiyama, K., Komuro, M., Endo, T., & Aihara, K. (2014). Classification of bifurcations of quasi-periodic solutions using lyapunov bundles. *International Journal of Bifurcation and Chaos*.

Bifurcation classification: bundles for quasi-periodic solution [4]

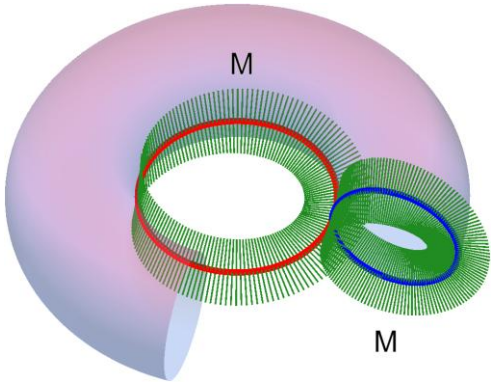
Fold bifurcation: $A^+ \times A^+$
Two quasi-periodic branches collide and disappear.

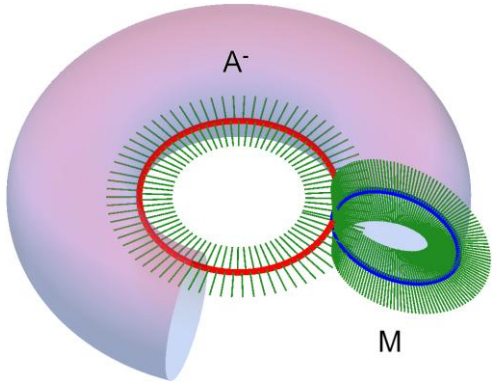


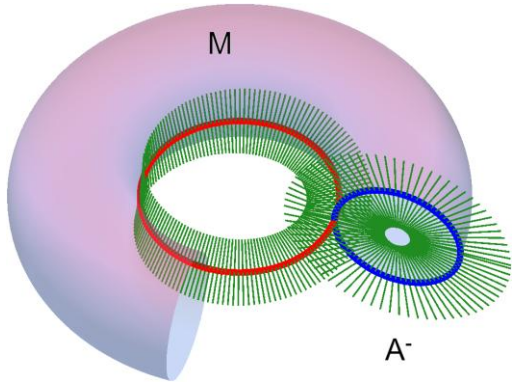
Neimark-Sacker bifurcation: $F_{NS} \times F_{NS}$
The 2-torus T^2 gives birth to a 3-torus T^3



Torus doubling bifurcation: $A^- \times M$, M $\times A^-$, $M \times M$
The quasi-periodic torus is replaced by a doubled torus.



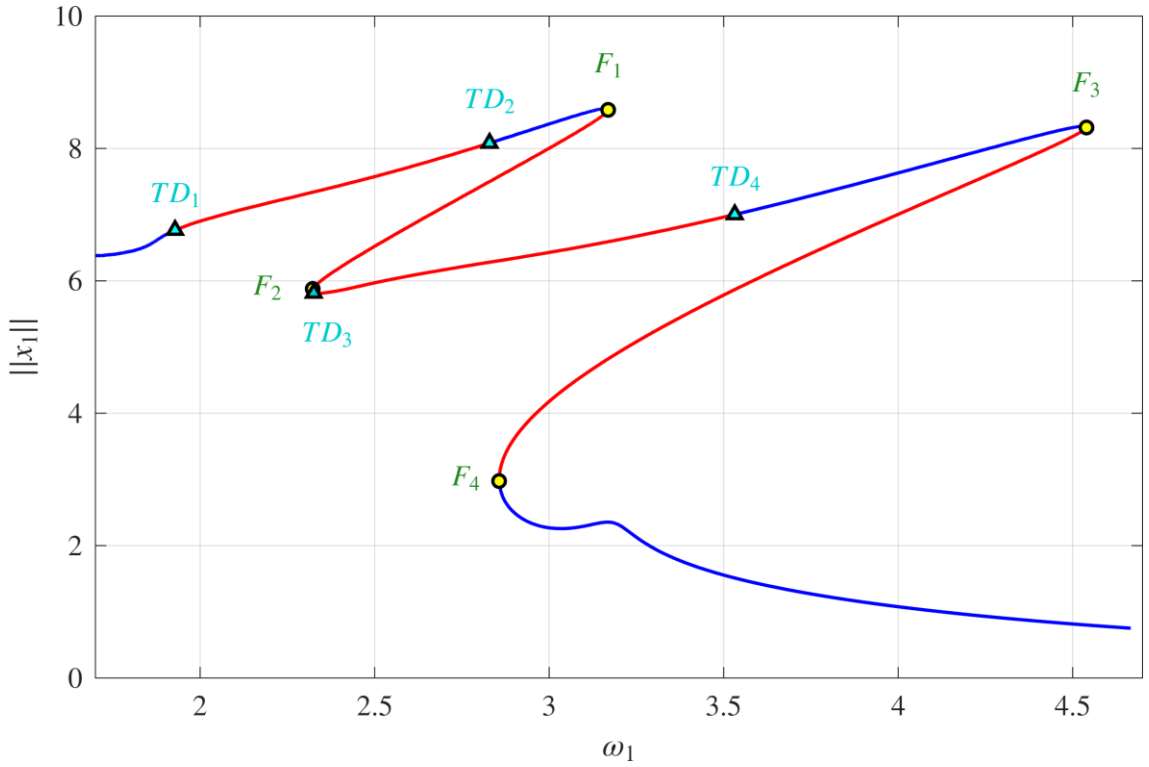




[4] Kamiyama, K., Komuro, M., Endo, T., & Aihara, K. (2014). Classification of bifurcations of quasi-periodic solutions using lyapunov bundles. *International Journal of Bifurcation and Chaos*.

Bifurcation classification

Bifurcation diagram for quasi-periodically forced Duffing oscillator

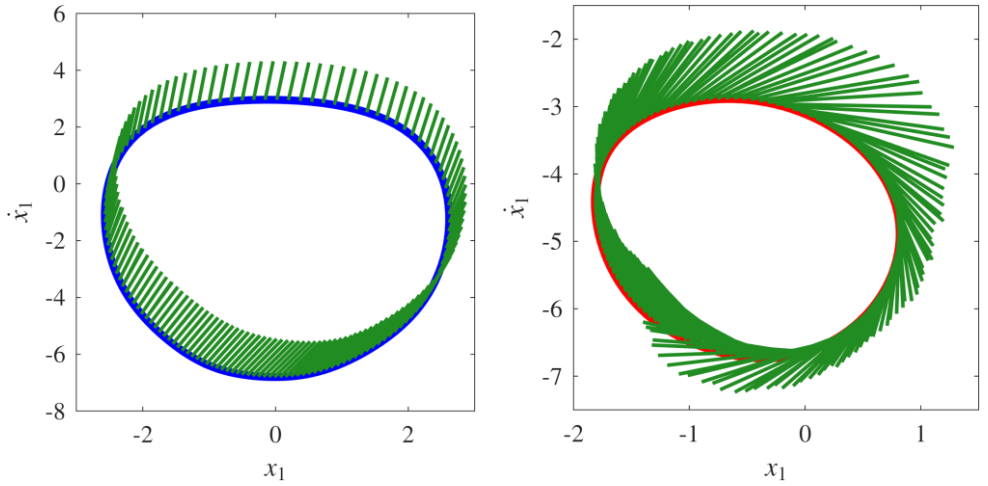


- Four Fold bifurcations (F_1-F_4)
- Four Torus doubling bifurcations ($TD_1 - TD_2$)

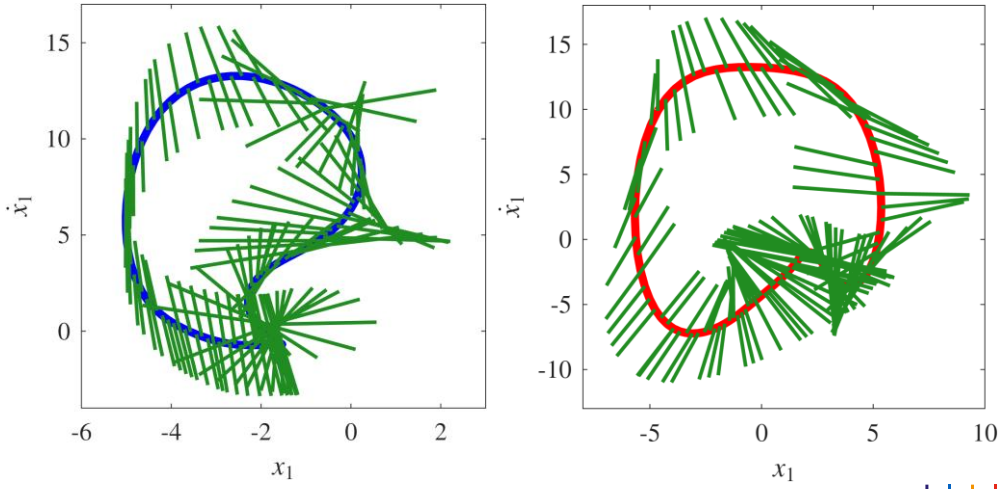
Fold (F) bifurcation Torus doubling (TD) bifurcation

QP stable branch QP unstable branch

Lyapunov bundles for F_1 bifurcation

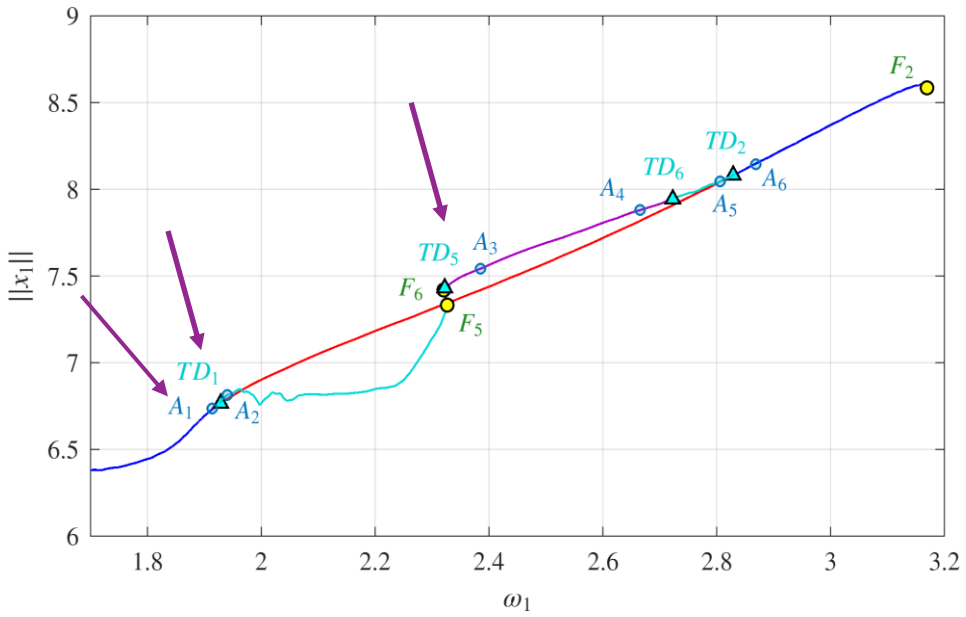


Lyapunov bundles for TD_3 bifurcation

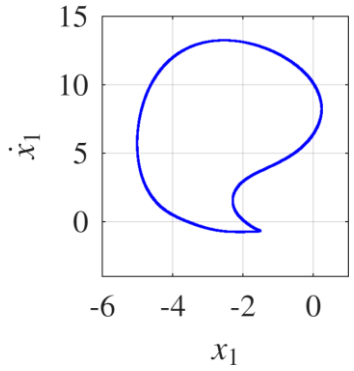


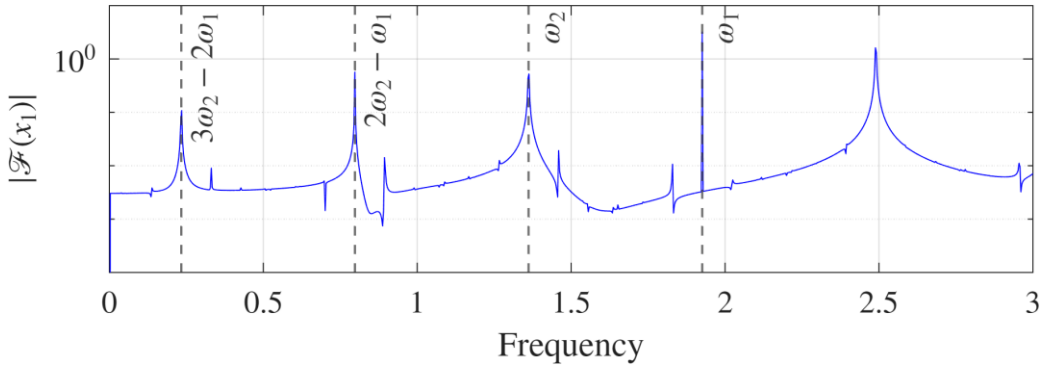
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Bifurcation localization

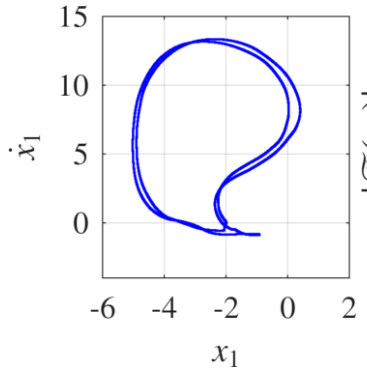


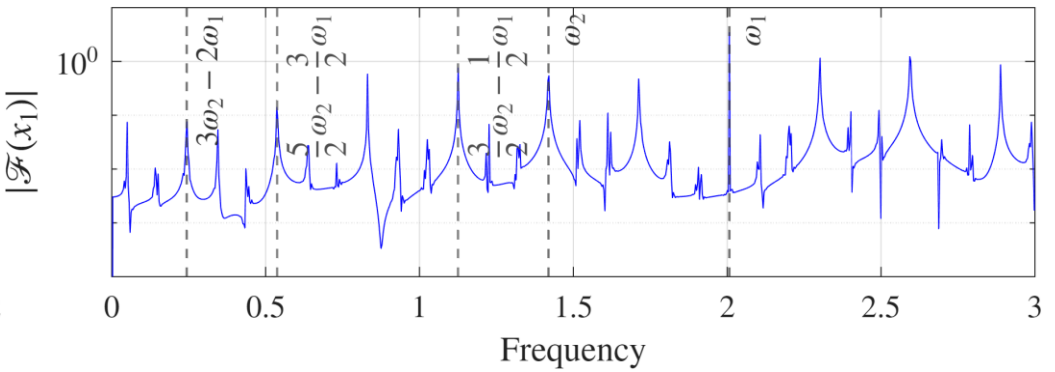
A_1



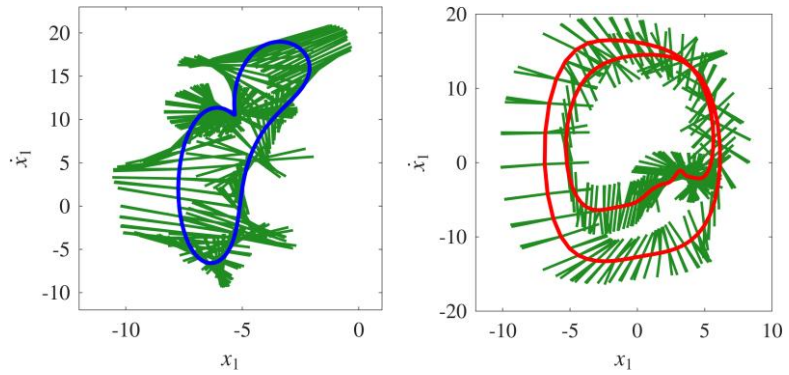


A_2

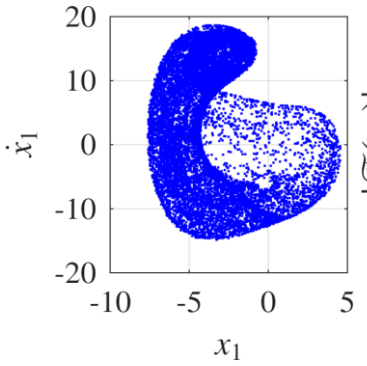


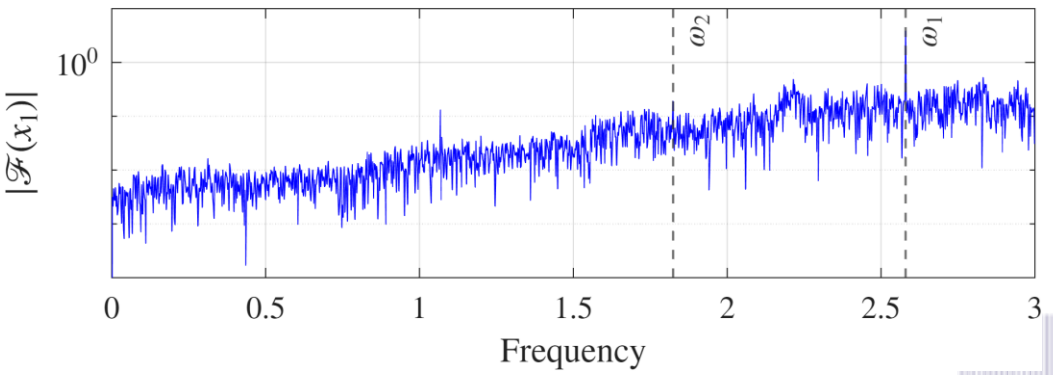


TD_1



A_3



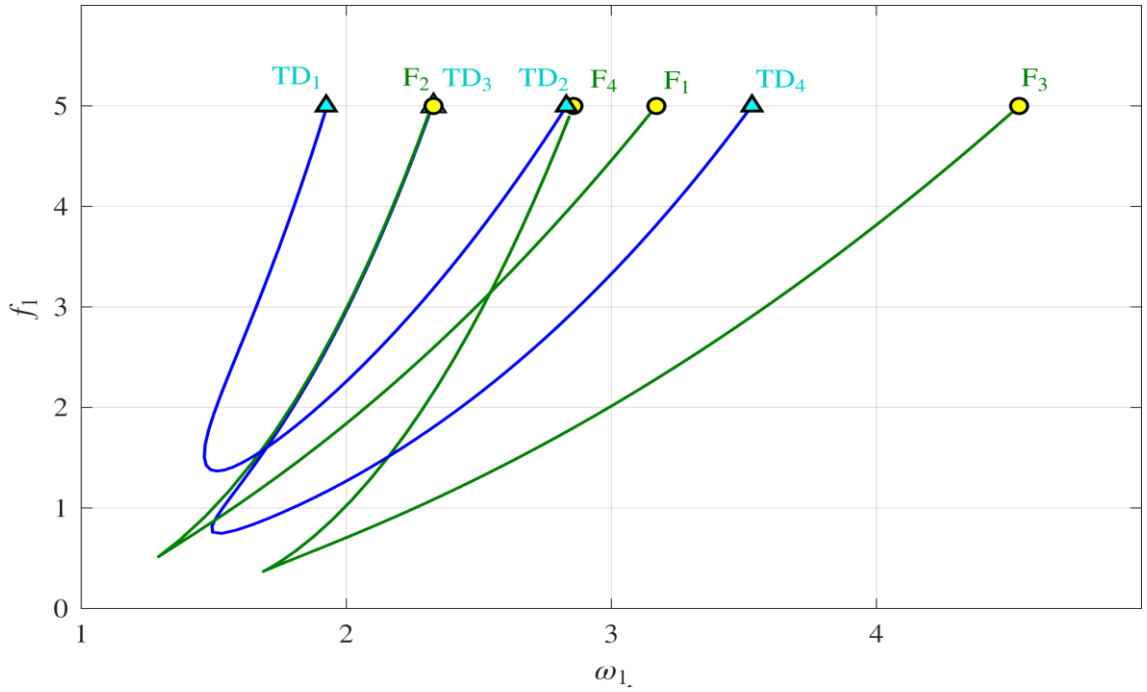


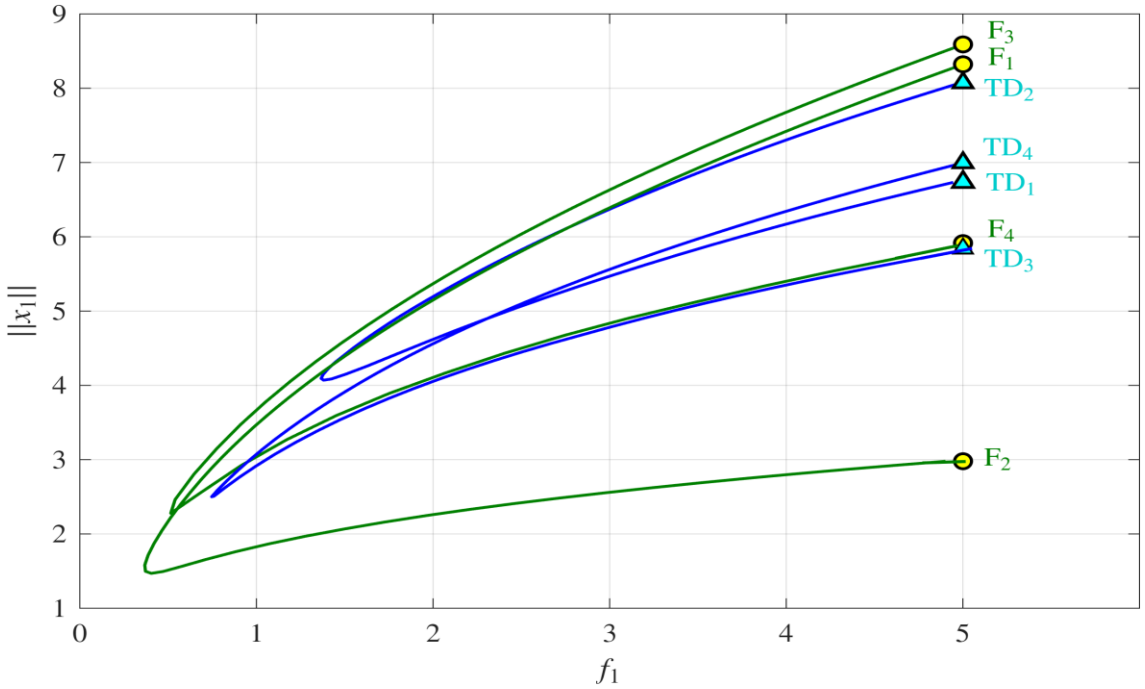
Fold (F) bifurcation ○ Torus doubling (TD) bifurcation △

QP stable branch — QP unstable branch — Torus doubled stable branch — Torus doubled unstable branch —

Bifurcation tracking

Bifurcation tracking for both Folds ($F_1 - F_4$) and torus doubling ($TD_1 - TD_4$) bifurcations with respect to the force





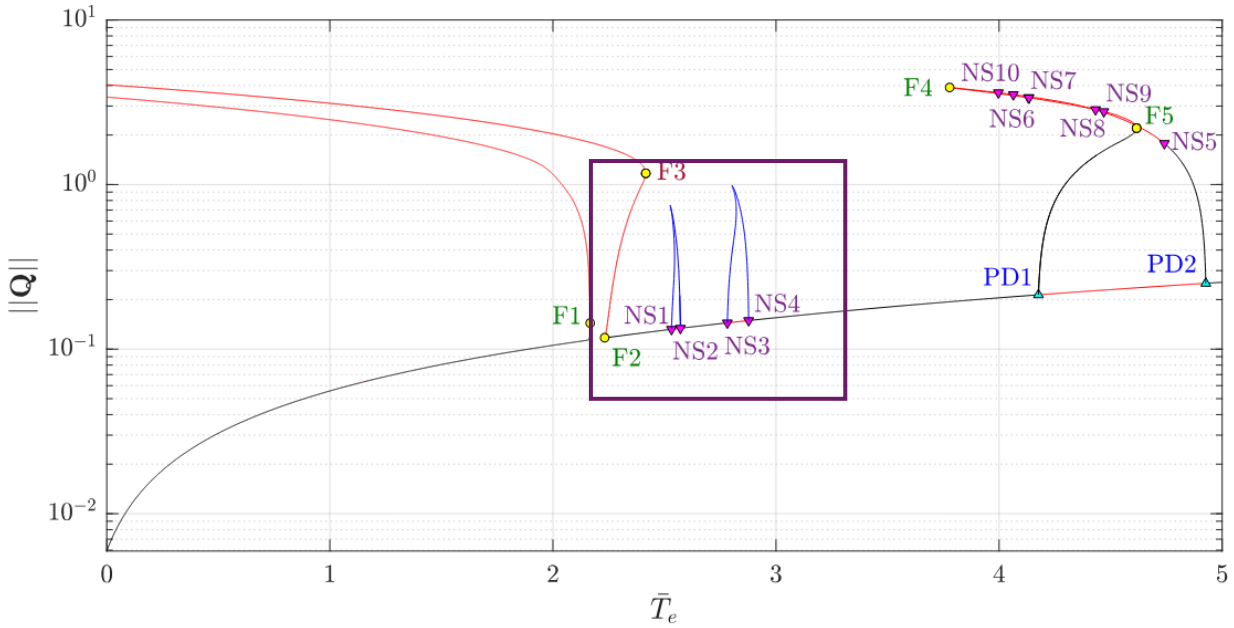
Fold (F) bifurcation Torus doubling (TD) bifurcation

QP stable branch QP unstable branch

TD curve Fold curve

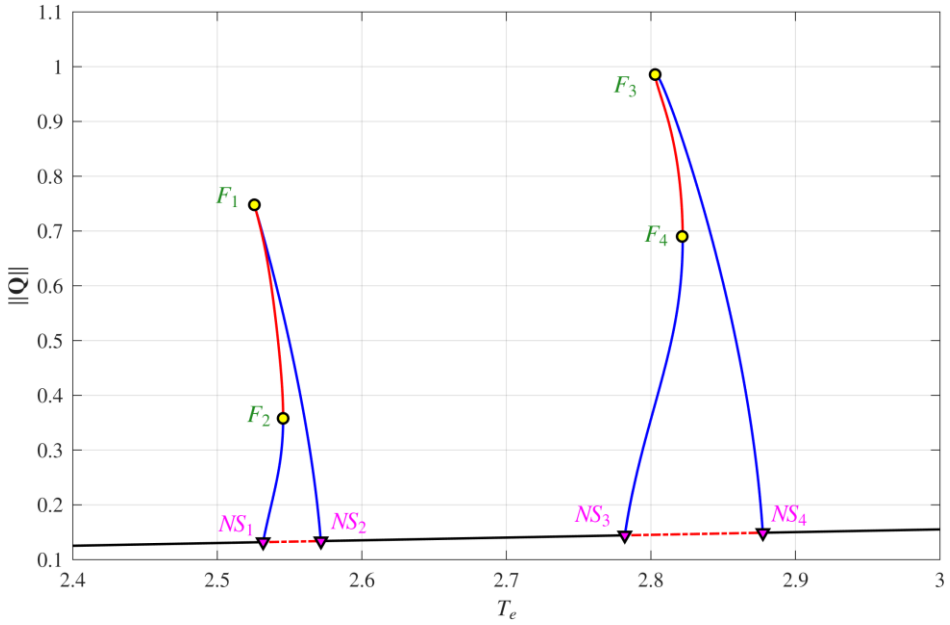
Autonomous system: stability analysis and bifurcation classification

Idealized gear transmission [5]



QP unstable branch
QP stable branch
Fold (F) bifurcation
Neimark Sacker (*NS*) bifurcation

- Two Fold bifurcations on each quasi-periodic branch



- The proposed method works well for autonomous systems

[5] Alcorta, R., *et al.* (2026) Bifurcation analysis of rotating machinery in non-stationary operations: an angular harmonic balance approach. *Nonlinear Dyn*.

Koopman–Hill framework for frequency-domain stability and bifurcation analysis of quasi-periodic solutions

- Introduction
- Harmonic selection
- Stability analysis for quasi-periodic motion
  - Lyapunov stability analysis
  - Stability analysis of a quasi-periodically forced Duffing oscillator
- Bifurcation classification for quasi-periodic motion
  - Bifurcation classification using Lyapunov bundles
  - Quasi-periodically forced Duffing oscillator
  - Idealized gear transmission
- Summary

Summary

Summary of contributions:

1. Stability analysis of quasi-periodic solutions in the frequency domain via Lyapunov exponents and Koopman-Hill projection
2. Bifurcation classification of quasi-periodic solutions
3. Proof of concept: quasi-periodic solution stability & bifurcations robustness under harmonic selection
4. Bifurcation localization and tracking via extended systems
5. Continuation of double-torus branches from torus doubling bifurcations

Perspectives

- Localization for “Hopf-like” bifurcations to higher-dimensional tori, and tracking in parameter space
- Practical implementation of algorithms for higher-dimensional tori: T^p ,$p \geq 3$
- Experimental validation of the current study

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EX-MODELI
EXPLOITATION & MODÉLISATION des DYNAMIQUES NON LINÉAIRES
GdR CNRS 2134

Thank you for your attention

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