Déficultés en analyse modale non-régulière

Mathias Legrand

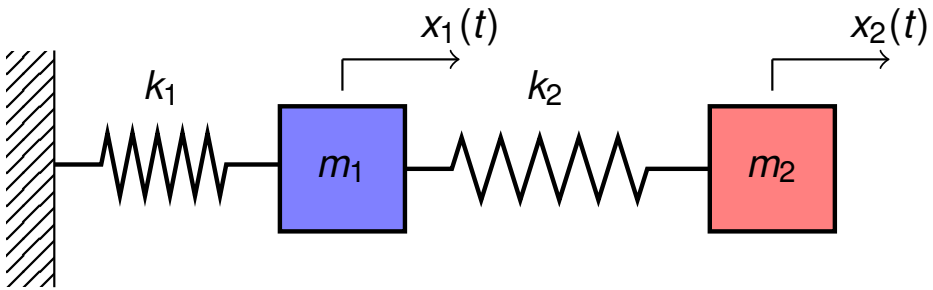
October 15, 2026

1. Modal analysis
2. Unilateral Systems
3. Finite Volume Method
4. Finite Element Method with impacts
5. Frequency Domain
6. Friction
7. Challiculties!

Modal analysis

Linear Modal Analysis: Discrete Systems

Simple mass-spring system



$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}; \qquad \mathbf{x}(t) = [x_1(t) \;\; x_2(t)]^\top \tag{1}$$

Linear modes of vibration

- Assume a periodic solution $\mathbf{x}(t) = \bar{\mathbf{x}} e^{i\omega t}$ then solve the eigenvalue problem[1]

$$(\mathbf{K} - \omega^2 \mathbf{M})\bar{\mathbf{x}} = \mathbf{0} \tag{2}$$

- Natural frequencies can be calculated from the eigenvalues ω^2
- Mode shapes can be calculated from the eigenvectors

[1] Shabana. *Vibration of Discrete and Continuous Systems.* (Springer, 2012).

Nonlinear Modal Analysis: Discrete Systems

Simple nonlinear mass-spring system

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \alpha\mathbf{x}^3 = \mathbf{0} \tag{3}$$

- Strategy to find linear modes no longer apply
- **Definition of nonlinear mode[2]:** Invariant surface of (periodic) orbits in the phase space, tangent to a linear mode at the equilibrium point

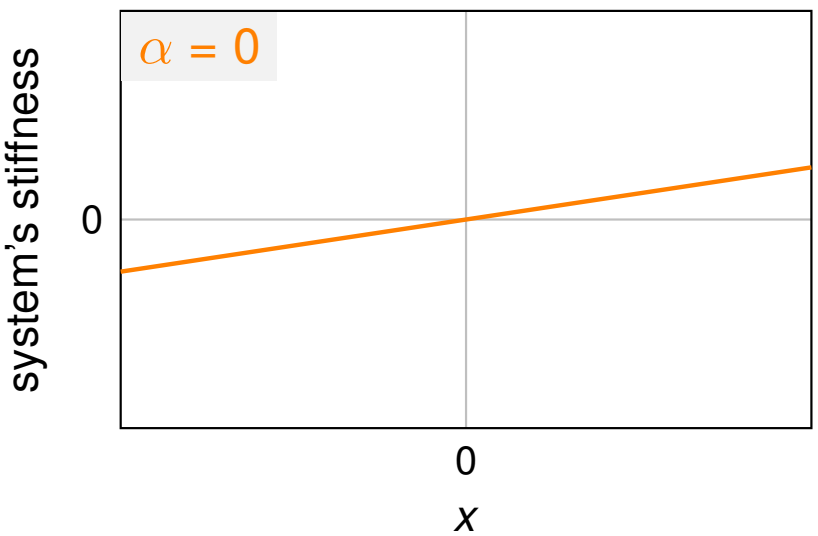


Figure 1: Nonlinear stiffness

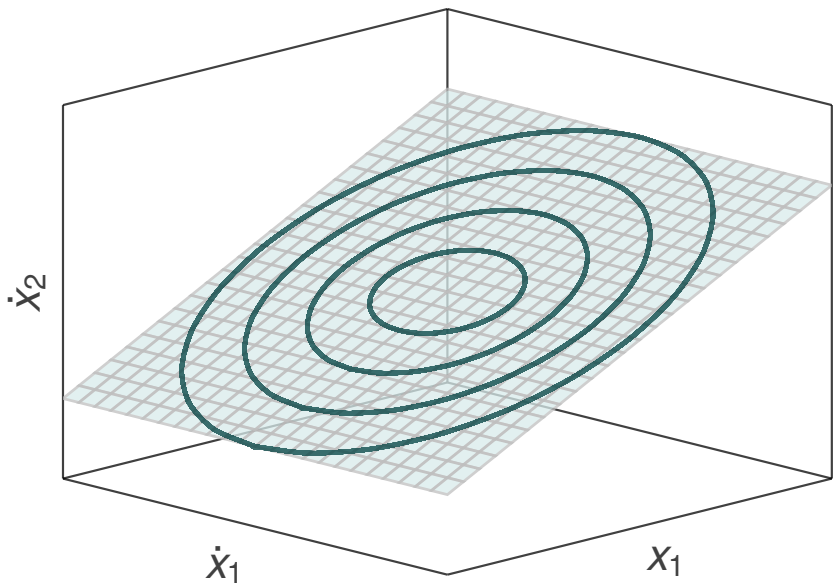


Figure 2: Invariant manifold

[2] Shaw & Pierre. Non-linear normal modes and invariant manifolds. *Journal of sound and Vibration* (1991).

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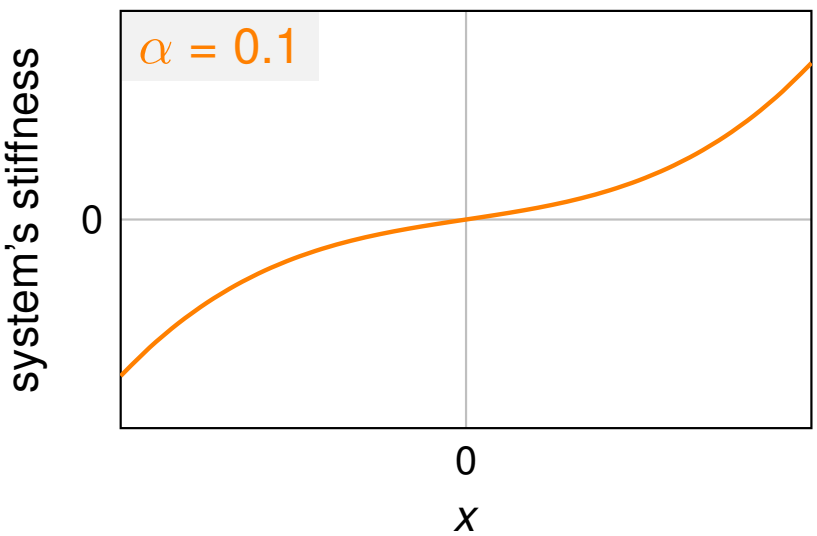


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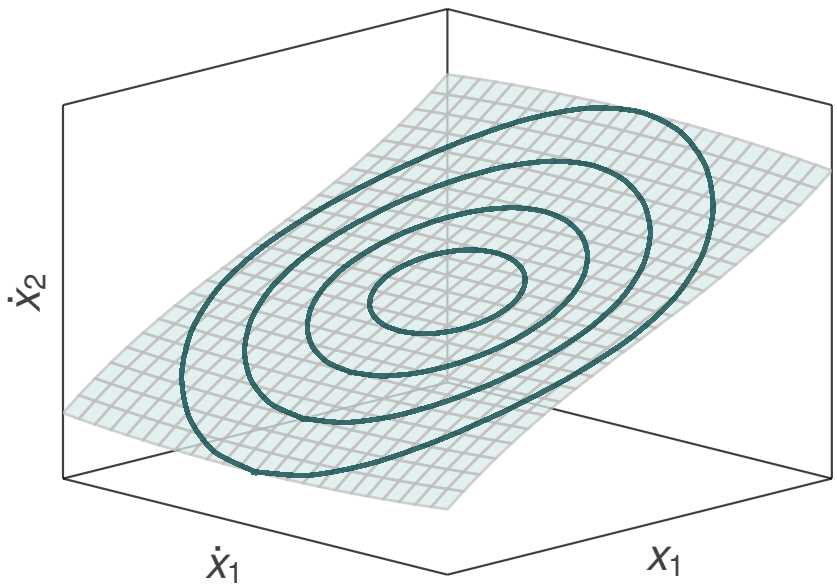


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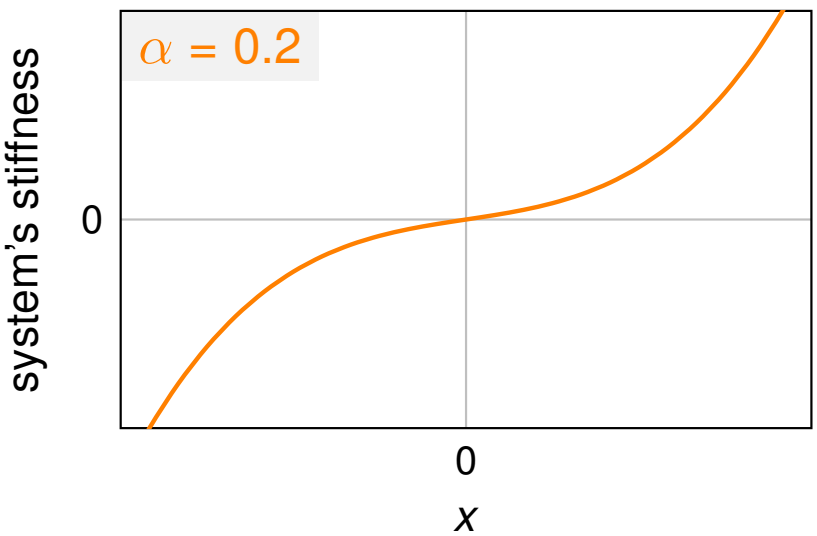


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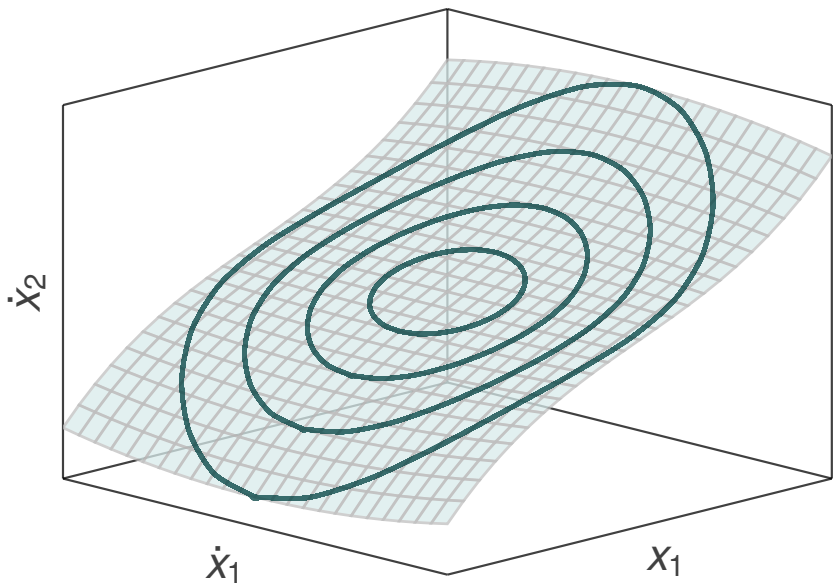


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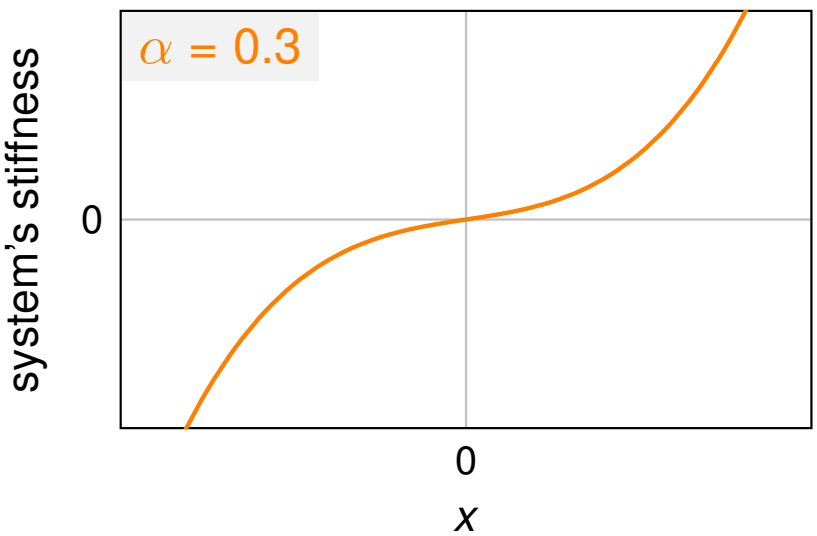


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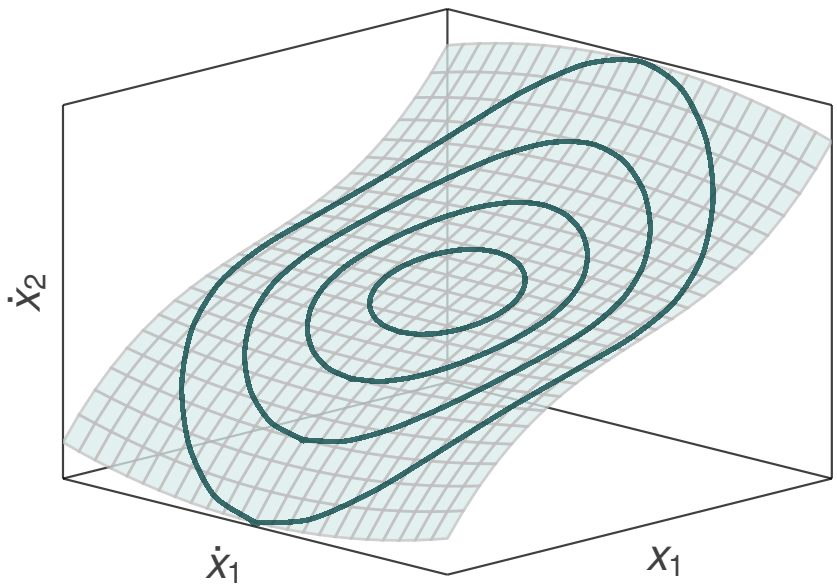


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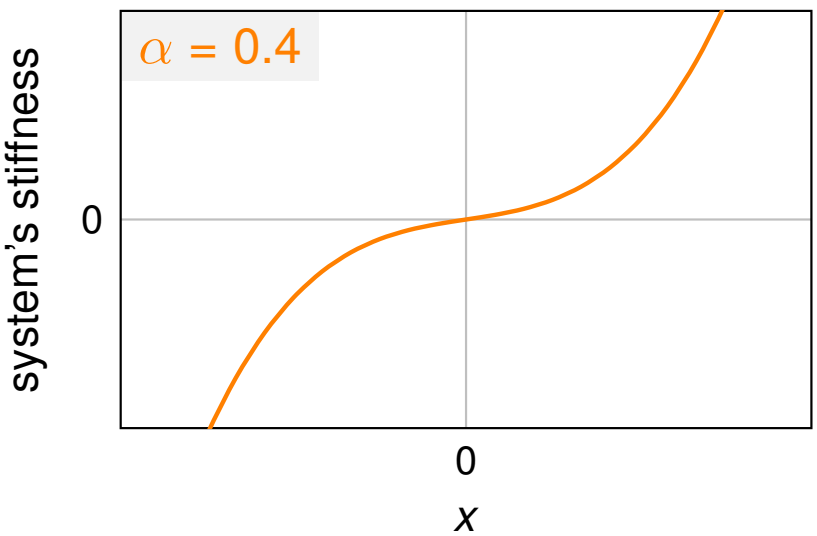


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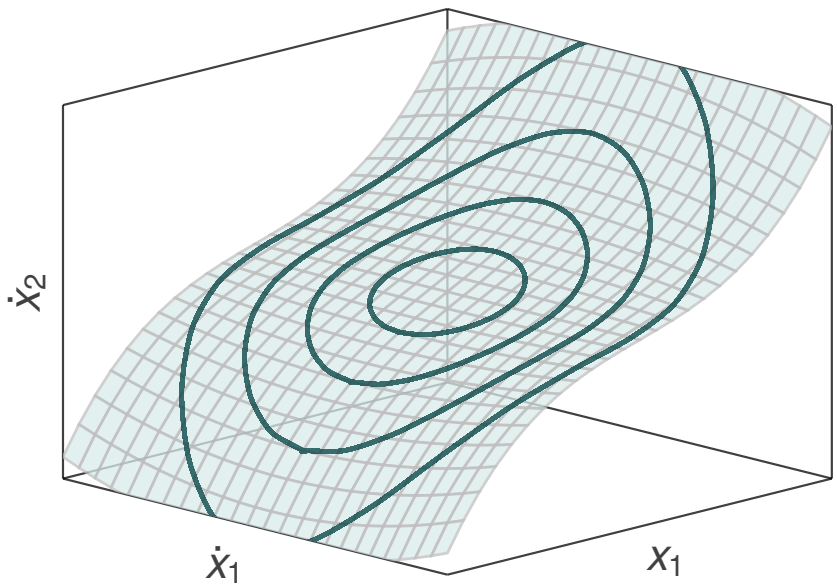


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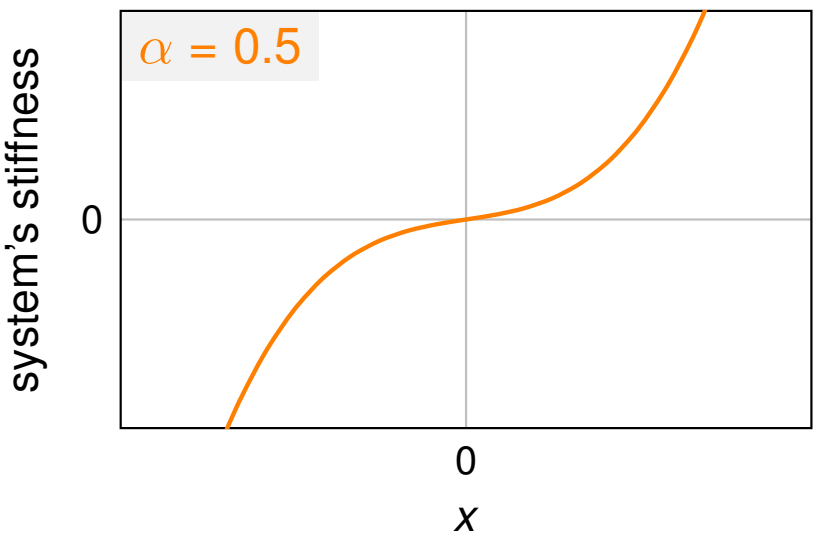


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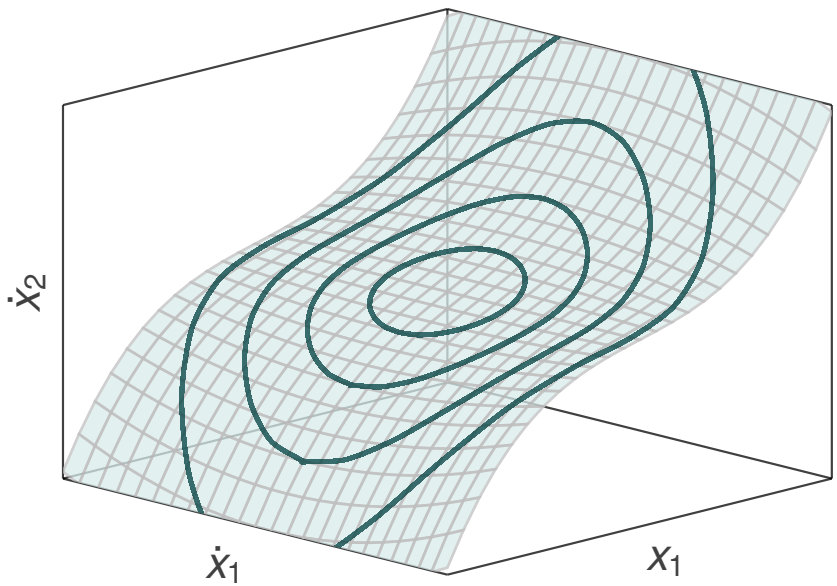


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Nonlinear Modal Analysis

Objective

Predict nonlinear vibratory resonances of periodically-forced systems[3,4]

$$\dot{\mathbf{q}} = \mathbf{f}(t, \mathbf{q}) \;\rightarrow\; \dot{\mathbf{q}} = \mathbf{f}(\mathbf{q}) \tag{4}$$

Frequency-Energy Plot (FEP)

Mode of vibration seen as a backbone curve[5]: every point is a periodic motion

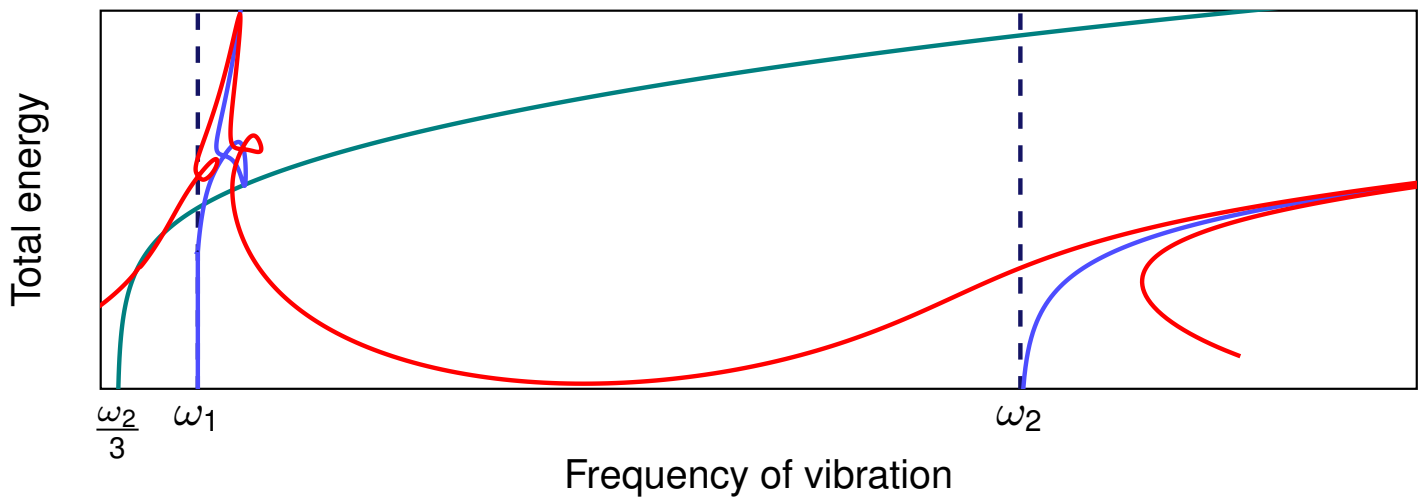


Figure 3: 2-dof nonlinear oscillator FEP

[3] Shaw & Pierre. Non-linear normal modes and invariant manifolds. *Journal of sound and Vibration* (1991).
[4] Haller & Ponsioen. Nonlinear normal modes and spectral submanifolds: existence, uniqueness and use in model reduction. *Nonlinear Dynamics* (2016).
[5] Kerschen, Peeters, Golinval & Vakakis. Nonlinear normal modes, Part I. *Mechanical Systems and Signal Processing* (2009).

Limitation of Nonlinear Modal Analysis

Smoothness assumption

Calculation of nonlinear modes relies on *smoothness* of the nonlinearity **f**: differentiable with respect to unknown **q**

- Center manifold theory
- Normal form theory

Nonsmooth mechanical systems

Systems undergoing impacts or/and friction (nonsmooth nonlinearities): **f** is no longer single-valued and differentiable

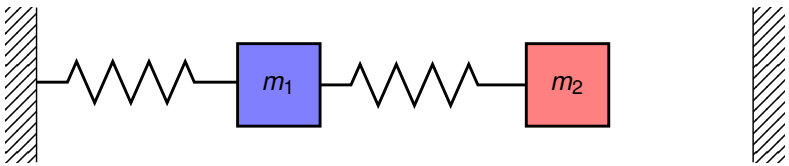


Figure 4: 2-dof impact oscillator

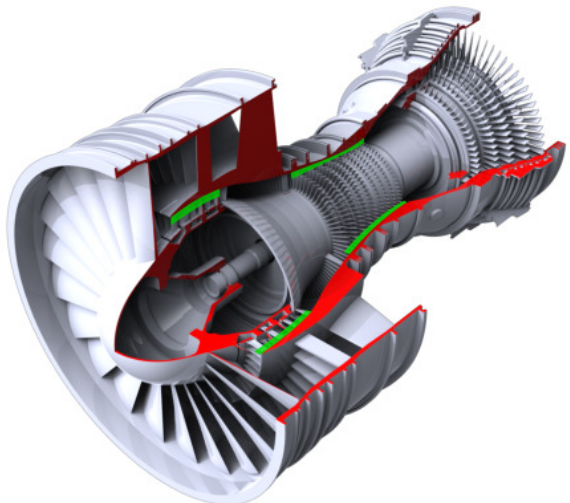


Figure 5: Aircraft engine: blade-casing contact

Unilateral Systems

Initial Value Problem

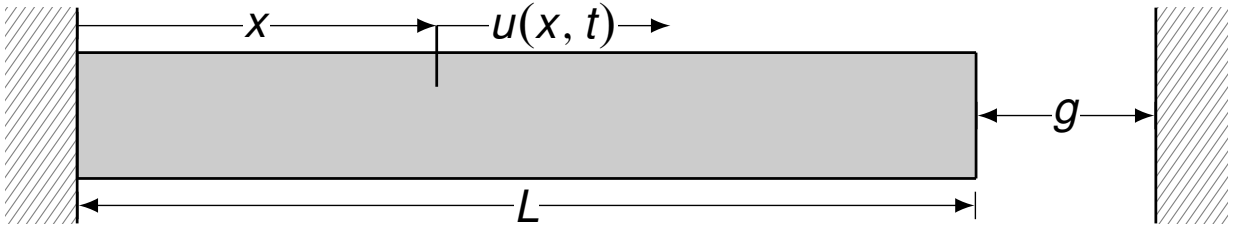


Figure 6: Unilaterally constrained bar

- Local equation (wave equation)

$$\partial_t^2 u(x,t) - c^2 \partial_x^2 u(x,t) = 0 \tag{5}$$

- Dirichlet boundary condition at $x = 0$

$$u(0,t) = 0 \tag{6}$$

- Initial conditions

$$\begin{aligned} u(x,0) &= u_0(x) \\ v(x,0) &= v_0(x) \end{aligned} \tag{7}$$

- Unilateral contact condition at $x = L$

$$g(t) \geq 0,\ r(t) \leq 0,\ r(t)g(t) = 0 \tag{8}$$

  - Gap function

$$g(t) = g_0 - u(L,t) \tag{9}$$

  - Contact force

$$r(t) = \partial_x u(L,t) \tag{10}$$

Periodic solutions → Nonsmooth modes of vibration

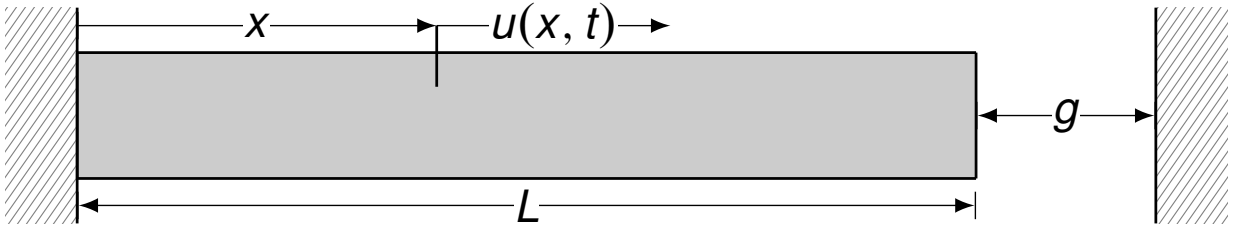


Figure 7: Unilaterally constrained bar

- Local equation (wave equation)

$$\partial_t^2 u(x,t) - c^2 \partial_x^2 u(x,t) = 0 \quad (11)$$

- Dirichlet boundary condition at $x = 0$

$$u(0,t) = 0 \quad (12)$$

- Periodicity (unknown period T)

$$\begin{aligned} u(x,t+T) &= u(x,t) \\ v(x,t+T) &= v(x,t) \end{aligned} \quad (13)$$

- Unilateral contact condition at $x = L$

$$g(t) \geq 0, \; r(t) \leq 0, \; r(t)g(t) = 0 \quad (14)$$

  - Gap function

$$g(t) = g_0 - u(L,t) \quad (15)$$

  - Contact force

$$r(t) = \partial_x u(L,t) \quad (16)$$

Find T-parameterized continuous families of periodic solutions → manifolds where nonsmooth modes of vibration are realized

Finite Volume Method

WFEM features

Fully-discrete scheme

- Basic Finite Volume formulation[6]
- Each volume $\mathcal{C}_i$ embeds constant state $\mathbf{Q}_i^{(n)} = \left[\sigma_i^{(n)} \quad v_i^{(n)}\right]^\top$

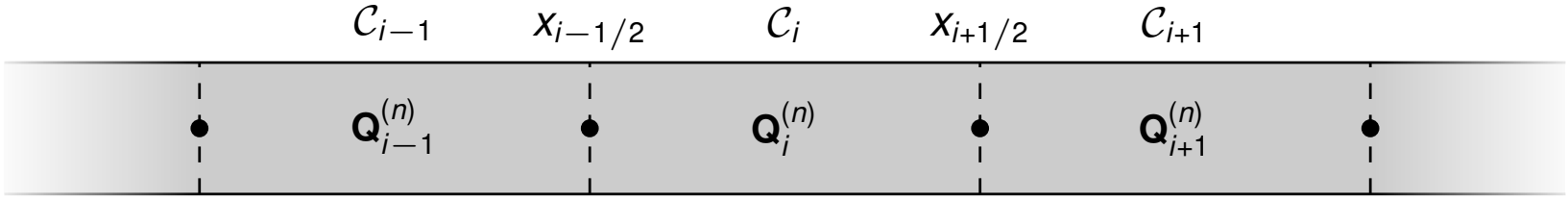


Figure 8: WFEM discretization

- Momentum conservation is applied to each finite volume
- Propagation of information along characteristic lines

$$\mathbf{Q}_i^{(n+1)} = \frac{\Delta t}{\Delta x}\mathbf{B}\left(\mathbf{D}_1\mathbf{Q}_{i-1}^{(n)} - \mathbf{D}_2\mathbf{Q}_{i+1}^{(n)}\right), \quad \mathbf{B} = \begin{bmatrix} 0 & -E \\ -1/\rho & 0 \end{bmatrix}, \qquad \Delta t = \Delta x / c \tag{17}$$

Advantages

- Simultaneous space-time discretization compliant with travelling waves
- No impact law is needed $\rightarrow$ (1) Conservation of energy and (2) No chattering

[6] Shorr. *The Wave Finite Element Method.* (Springer, 2004).

Compact Matrix Form

Time-domain iterations

Matrix **A** embeds *inertial* and *stiffness* terms as well as *boundary conditions*

$$\mathbf{Q}^{(n+1)} = \mathbf{A}\mathbf{Q}^{(n)} \iff \mathbf{Q}^{(n)} = \mathbf{A}^n\mathbf{Q}^{(0)}, \quad \mathbf{Q}^{(n)} = \left[\mathbf{Q}_1^{(n)\top} \dots \mathbf{Q}_N^{(n)\top}\right]^\top \tag{18}$$

Treatment of contact: conditional switch at $x = L$

- Free Phase Matrix $\mathbf{A}_\mathrm{f}$ where $\sigma(L,t) = 0$



Figure 9: Open gap

- Contact Phase Matrix $\mathbf{A}_\mathrm{c}$ where $u(L,t) = g_0$

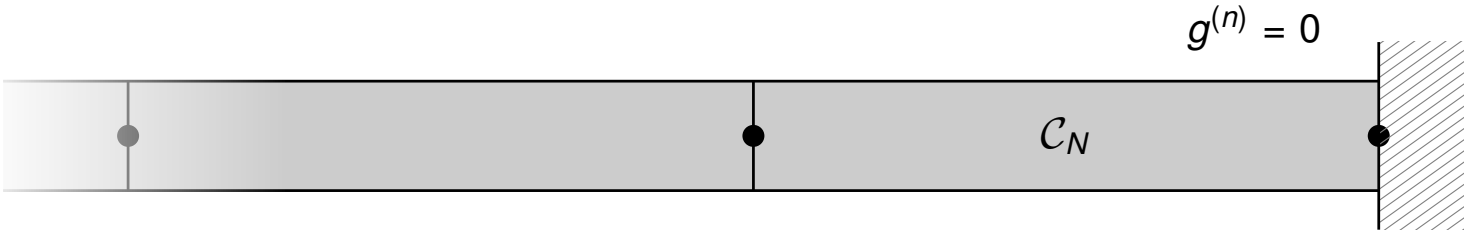


Figure 10: Closed gap

Solution Method

Shooting technique with unknown period T

- Assumption: only one contact phase and one free phase per period
- Periodicity condition: $\mathbf{Q}^{(n_T)} = \mathbf{Q}^{(0)}$, where $n_T \Delta t = T$
- Solution with m—steps in free phase and p—steps in contact phase

$$\mathbf{Q}^{(n_T)} = \mathbf{A}_c^p \mathbf{A}_f^m \mathbf{Q}^{(0)}, \quad n_T = m + p \tag{19}$$

Problem formulation

Given $\mathbf{A}_c$ and $\mathbf{A}_f$, find $m \in \mathbb{N}^+$, $p \in \mathbb{N}^+$ and $\mathbf{Q}^{(0)} \in \mathbb{R}^{2N\times 1,*}$ such that[7]

$$(\mathbf{A}_c^p \mathbf{A}_f^m - \mathbf{I})\mathbf{Q}^{(0)} = \mathbf{0} \tag{20}$$

together with the following conditions

Condition 1 Point of contact do not penetrate the obstacle
condition 2 Point of contact must be stuck to the obstacle during contact
condition 3 Contact force must be negative

→ Something like inequality-constrained kernel finding (eigenvalue) problem

[7] Yoong, Thorin & Legrand. Nonsmooth modal analysis of an elastic bar subject to a unilateral contact constraint. *Nonlinear Dynamics* (2018).

Backbone Curves

Types of nonsmooth modes

- Three categories: *Main*, *subharmonic* and *internally resonant* NSMs

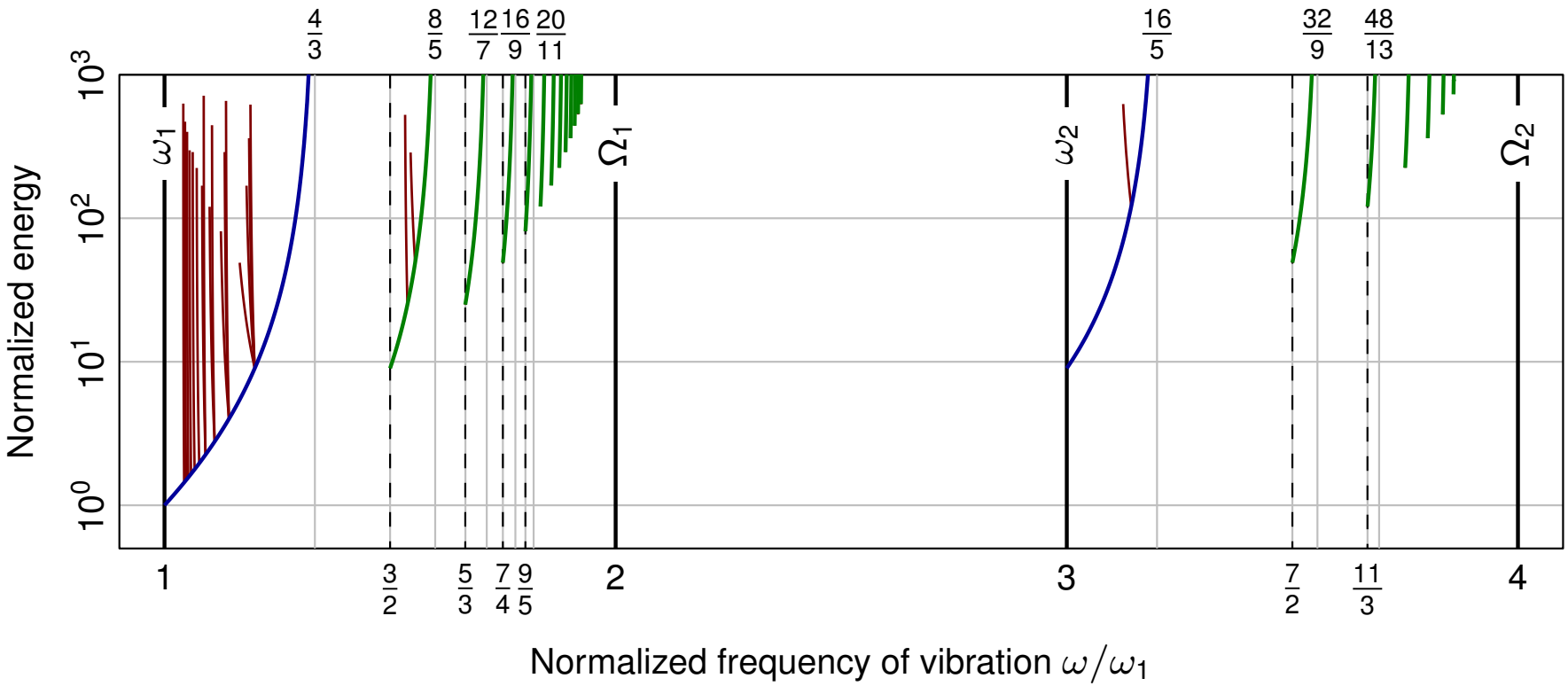


Figure 11: FEP: main [——], subharmonic [——] and internally resonant [——]

Main NSM

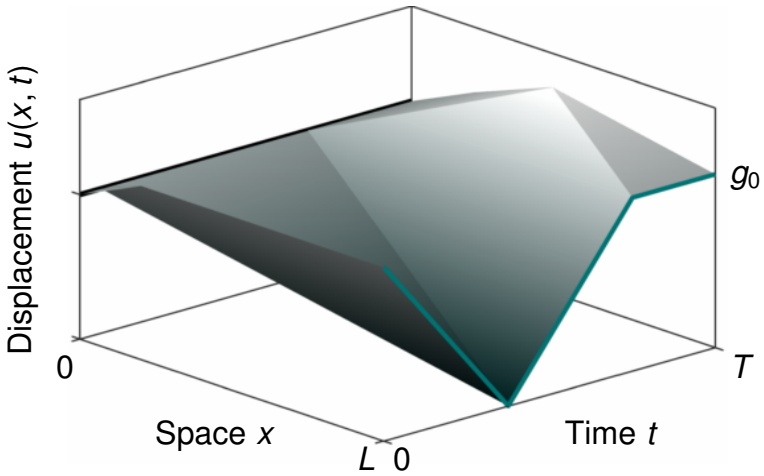


Figure 12: First main nonsmooth mode

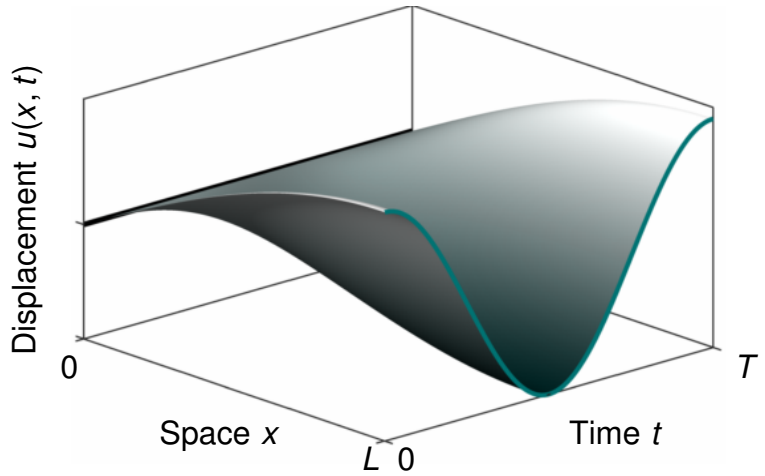


Figure 13: First linear mode

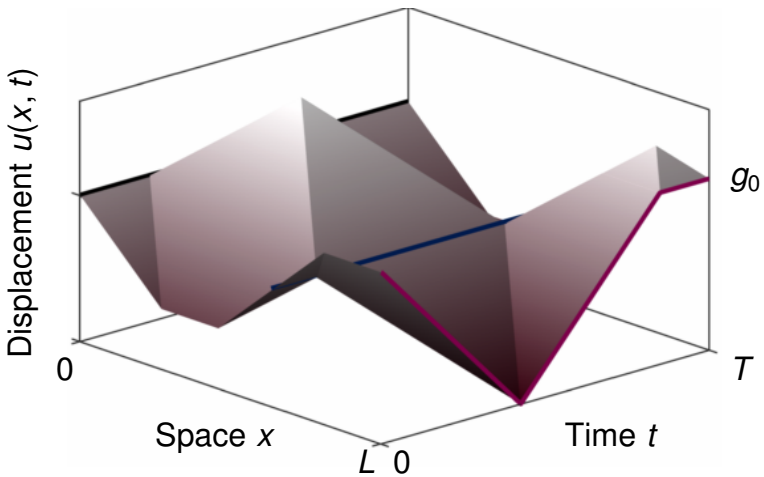


Figure 14: Second main nonsmooth mode

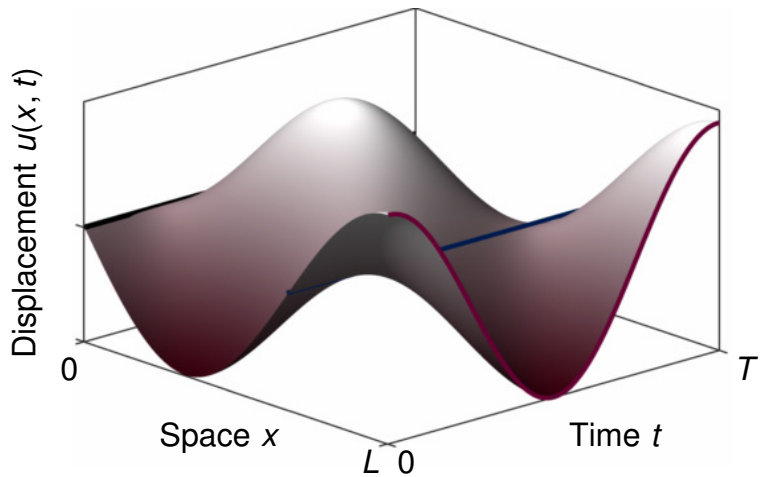


Figure 15: Second linear mode

Other NSM

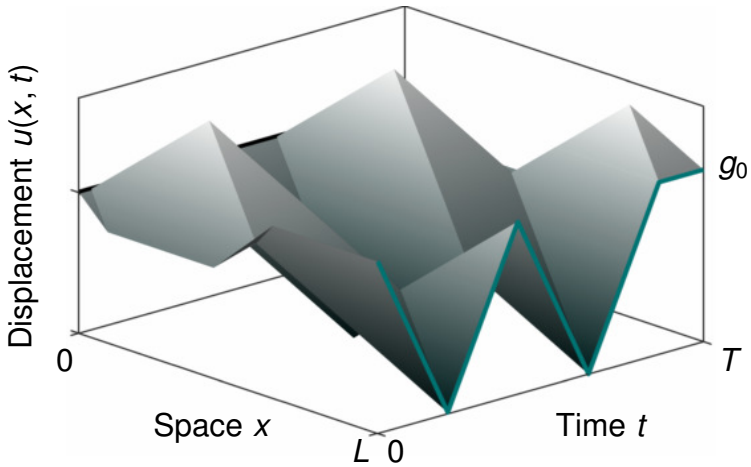


Figure 16: Subharmonic NSM at $\omega_2/2$

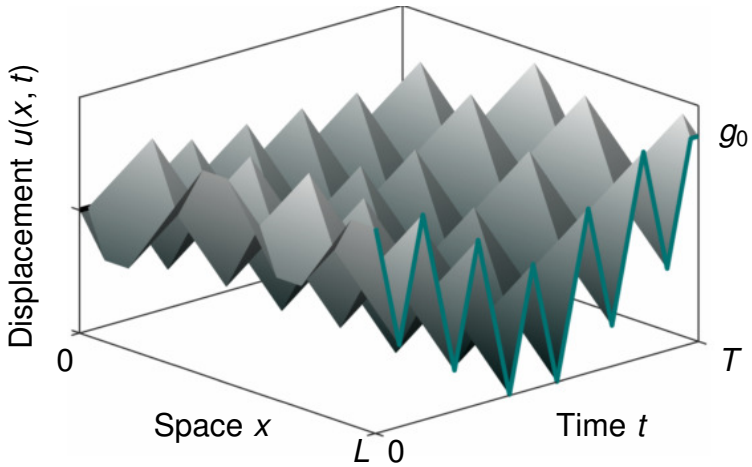


Figure 17: Internally resonant NSM at $\omega_4/6$

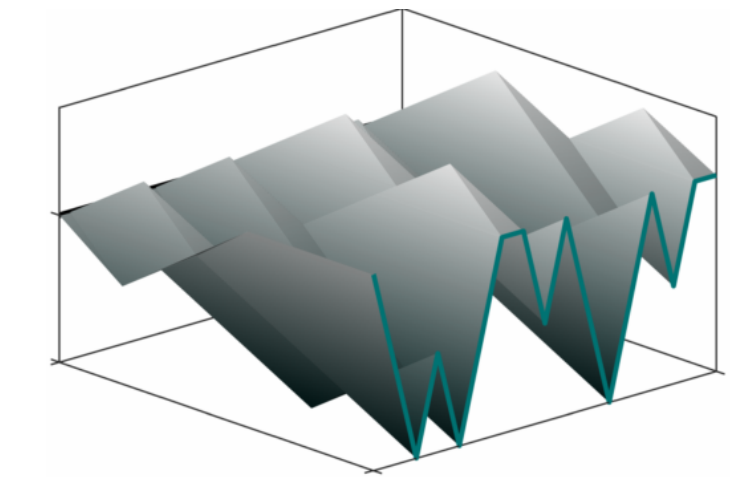


Figure 18: Two-contact per period NSM

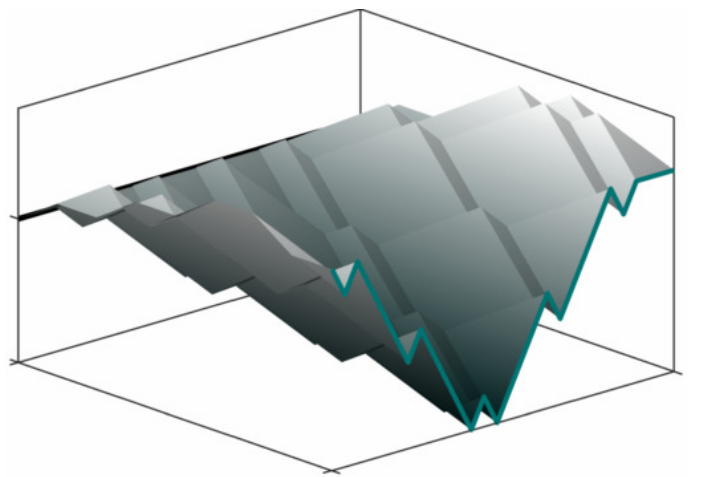
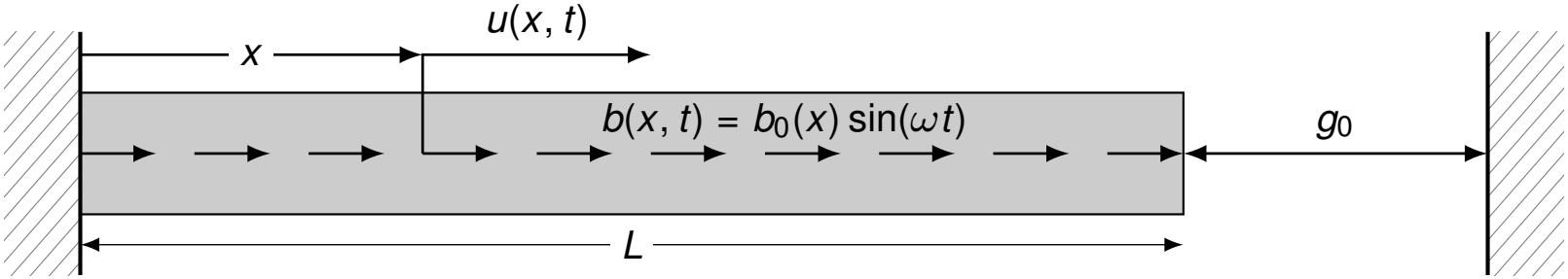


Figure 19: Internally resonant NSM at $\omega_4/6$

Forced Response under Periodic Excitation

Slightly-damped fixed–free bar



Forced response plots

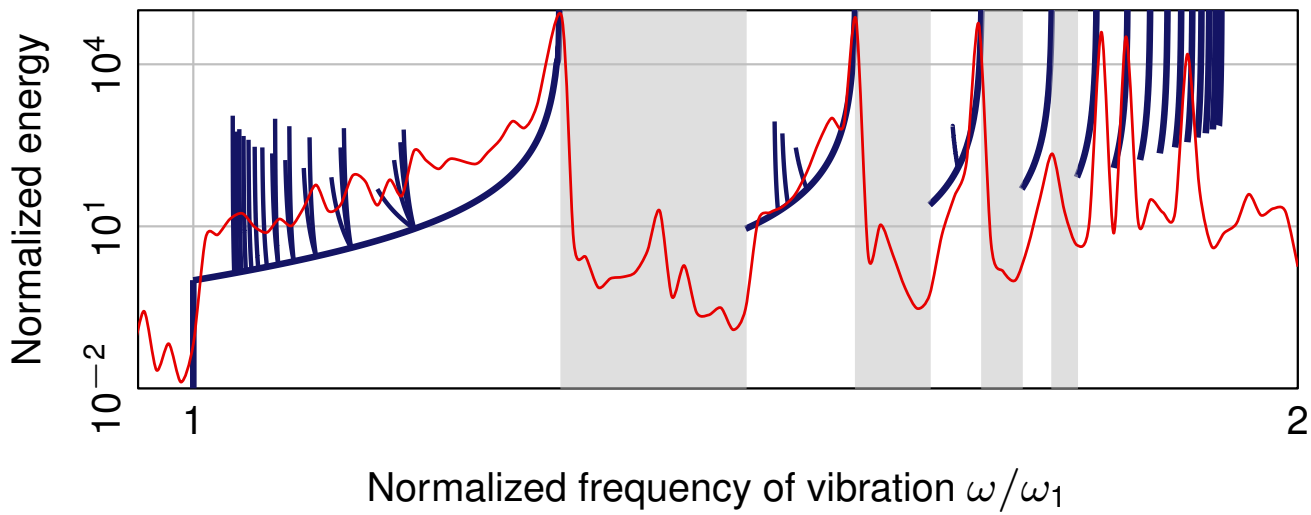
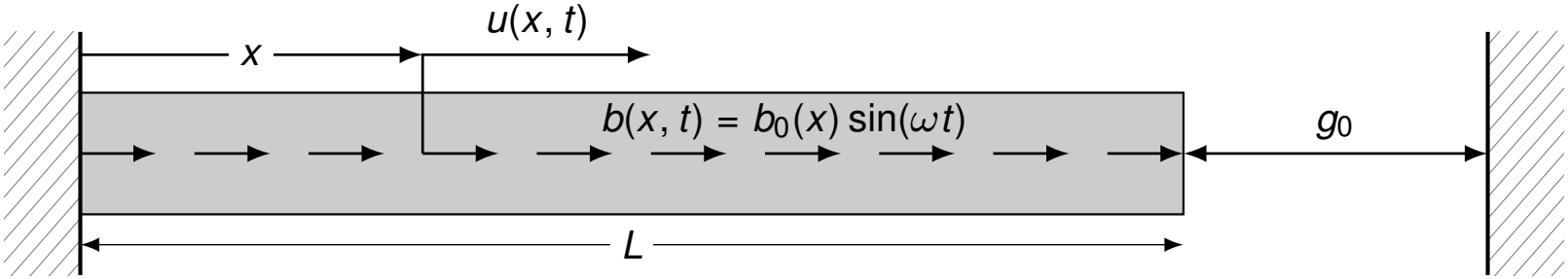


Figure 20: FEP for $g_0 > 0$: nonsmooth modes [——] and forced response [——]

Forced Response under Periodic Excitation

Slightly-damped fixed–free bar



Forced response plots

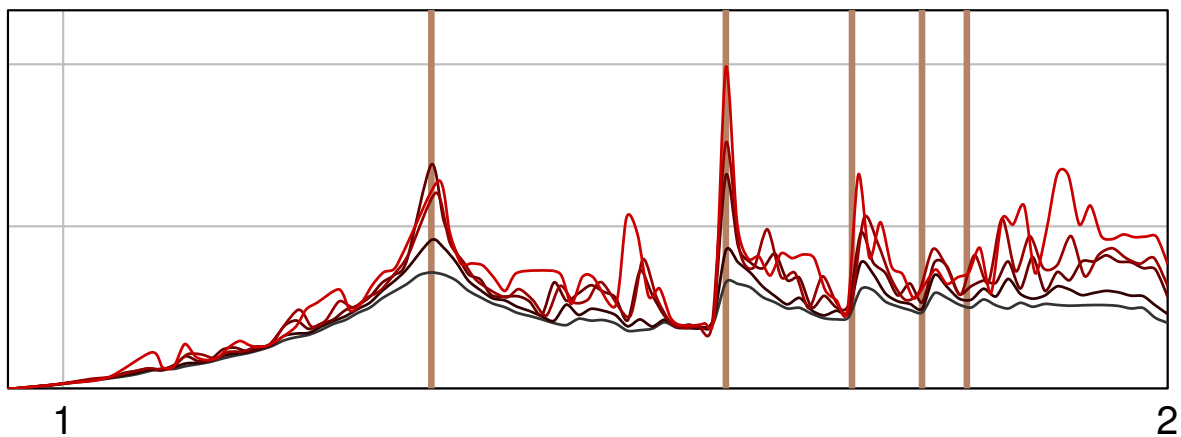
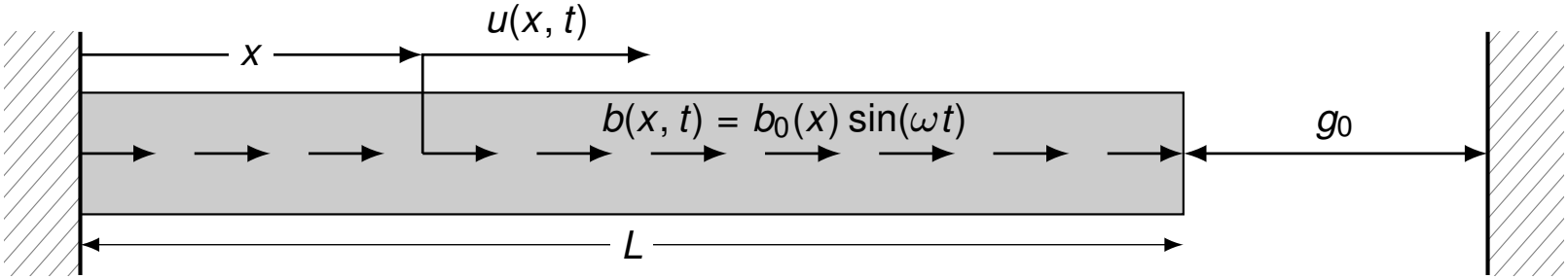


Figure 20: FEP for $g_0 = 0$

Forced Response under Periodic Excitation

Slightly-damped fixed–free bar



Forced response plots

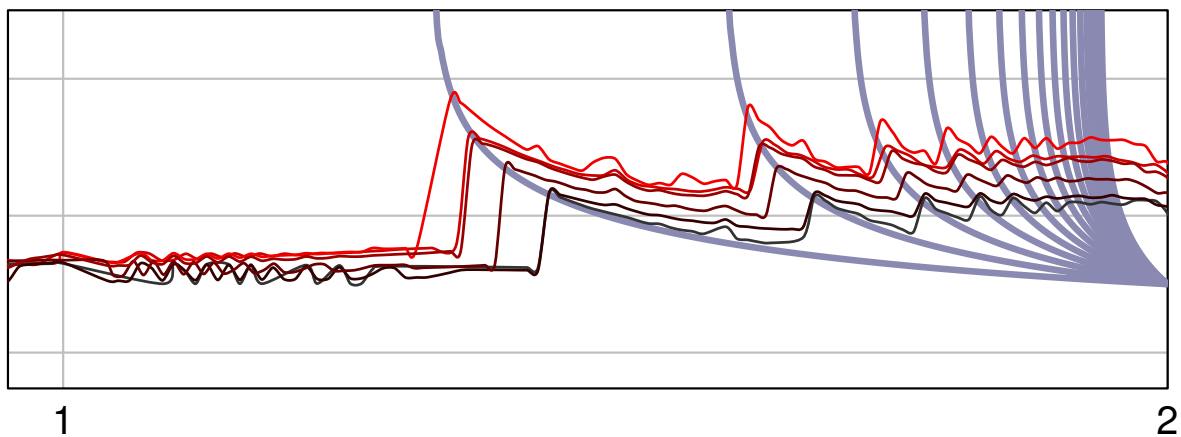


Figure 20: FEP for $g_0 < 0$

Comparison

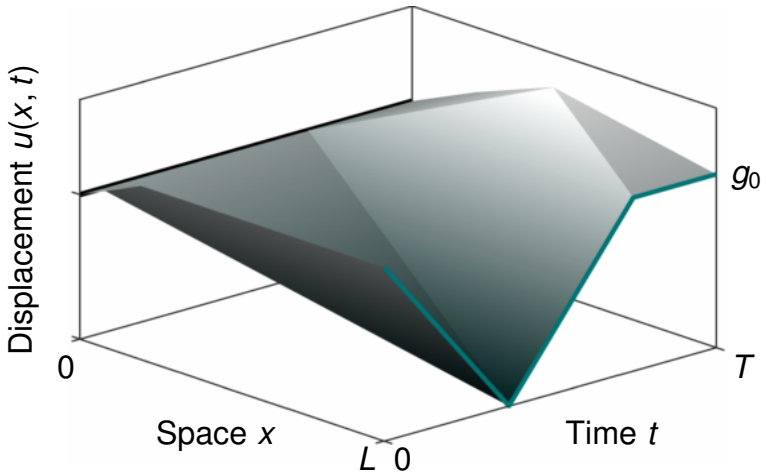


Figure 21: First main nonsmooth mode

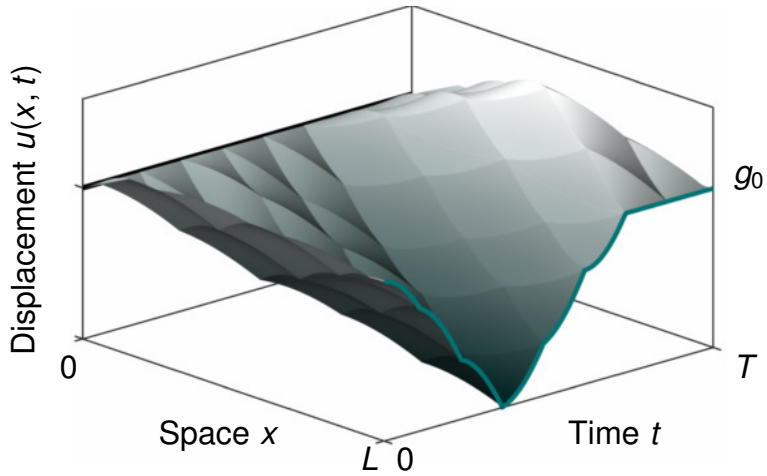


Figure 22: Forced steady-state around ω_1

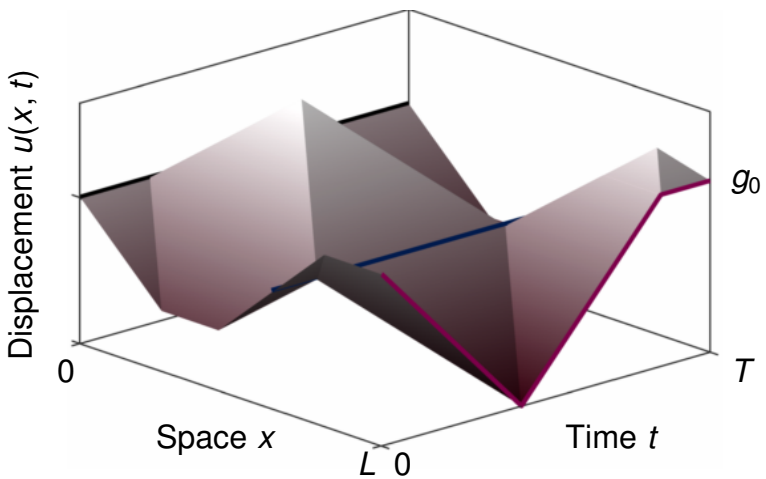


Figure 23: Second main nonsmooth mode

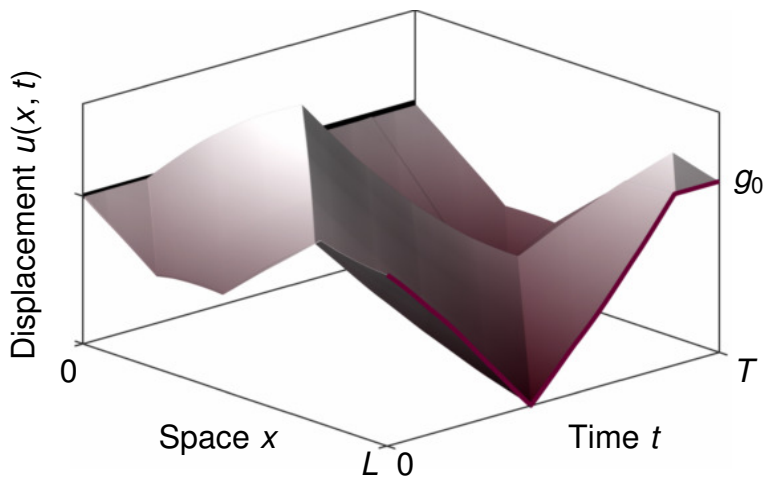


Figure 24: Forced steady-state around ω_2

Finite Element Method with impacts

Governing equations for N dof

Full version

- local equation: $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{A}^\top \lambda$
- Signorini's conditions: $g_0 - u_N(t) \geq 0$, $\lambda(t) \leq 0$, $(g_0 - u_N(t)) \cdot \lambda(t) = 0$
- constant energy $\dot{\mathbf{u}}^\top \mathbf{M}\dot{\mathbf{u}} + \mathbf{u}^\top \mathbf{K}\mathbf{u} = E(t) = E(0)$

Simplified version: one impact per period

- periodic motion of period T
- governing equations (Shooting-like formulation/Poincaré map)

$$\begin{aligned} & u_N(0) = g_0 \\ & \mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \qquad \forall t \in]0, T[\\ & \mathbf{u}(T) = \mathbf{u}(0) \qquad \text{[displacement continuity]} \\ & \dot{\mathbf{u}}^-(T) = \mathbf{S}\dot{\mathbf{u}}^+(0) \qquad \text{[velocity jump]} \end{aligned} \tag{21}$$

where

$$\mathbf{S} = \begin{bmatrix} \mathbf{I}_{N-1,N-1} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix} \tag{22}$$

- ▷ nonlinear problem in T (period)
- ▷ lost information: $\forall t \in]0, T[$, $g_0 - u_N(t) \geq 0$

Explicit solution

- Projection onto the linear modal space: $\mathbf{u}(t) = \mathbf{P}\mathbf{q}(t)$

$$\begin{bmatrix} \cos(\mathbf{\Omega}\, T) - \mathbf{I} & \sin(\mathbf{\Omega}\, T) \\ -\mathbf{\Omega}\, \sin(\mathbf{\Omega}\, T) & \mathbf{P}^{-1}\mathbf{S}\mathbf{P}\mathbf{\Omega} + \mathbf{\Omega}\, \cos(\mathbf{\Omega}\, T) \end{bmatrix} \begin{pmatrix} \mathbf{q}(0) \\ \dot{\mathbf{q}}^{+}(0) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{23}$$

  where $\mathbf{\Omega} = \mathbf{diag}(\omega_i)_{i=1,\dots,N}$ [matrix of linear eigenvalues]

- Displacement at $t = 0$

$$\mathbf{u}(0) = \alpha\left(\mathbf{P}\mathbf{\Omega}^{-1}(\mathbf{I} - \cos(\mathbf{\Omega}\, T))^{-1} \sin(\mathbf{\Omega}\, T)\mathbf{P}^{-1}\right)\mathbf{e}_N, \quad \forall\, T \tag{24}$$

  - α is a normalisation factor
  - above solutions not always admissible: some should be eliminated

Theorem[8]

- N distinct periods for linear modes: $i \neq j \Rightarrow T_i/T_j \notin \mathbb{N}$
- Admissible linear grazing modes exist

Then:

1. For all $T > 0$, there is a unique solution
2. Initial data is a smooth manifold $\mathcal{V}_0 = \{(\mathbf{u}(0;T), \dot{\mathbf{u}}^{+}(0;T))\}$
3. For all j, if $T < T_j$ and $T \simeq T_j$ then $\mathbf{u}(\cdot\,;T)$ is admissible
4. For all j, if $T > T_j$ and $T \simeq T_j$, $\mathbf{u}(\cdot\,;T)$ is not admissible

[8] Legrand, Junca & Heng. Nonsmooth modal analysis of a N-degree-of-freedom system undergoing a purely elastic impact law. *Communications in Nonlinear Science and Numerical Simulation* (2017).

Spectrum of vibration

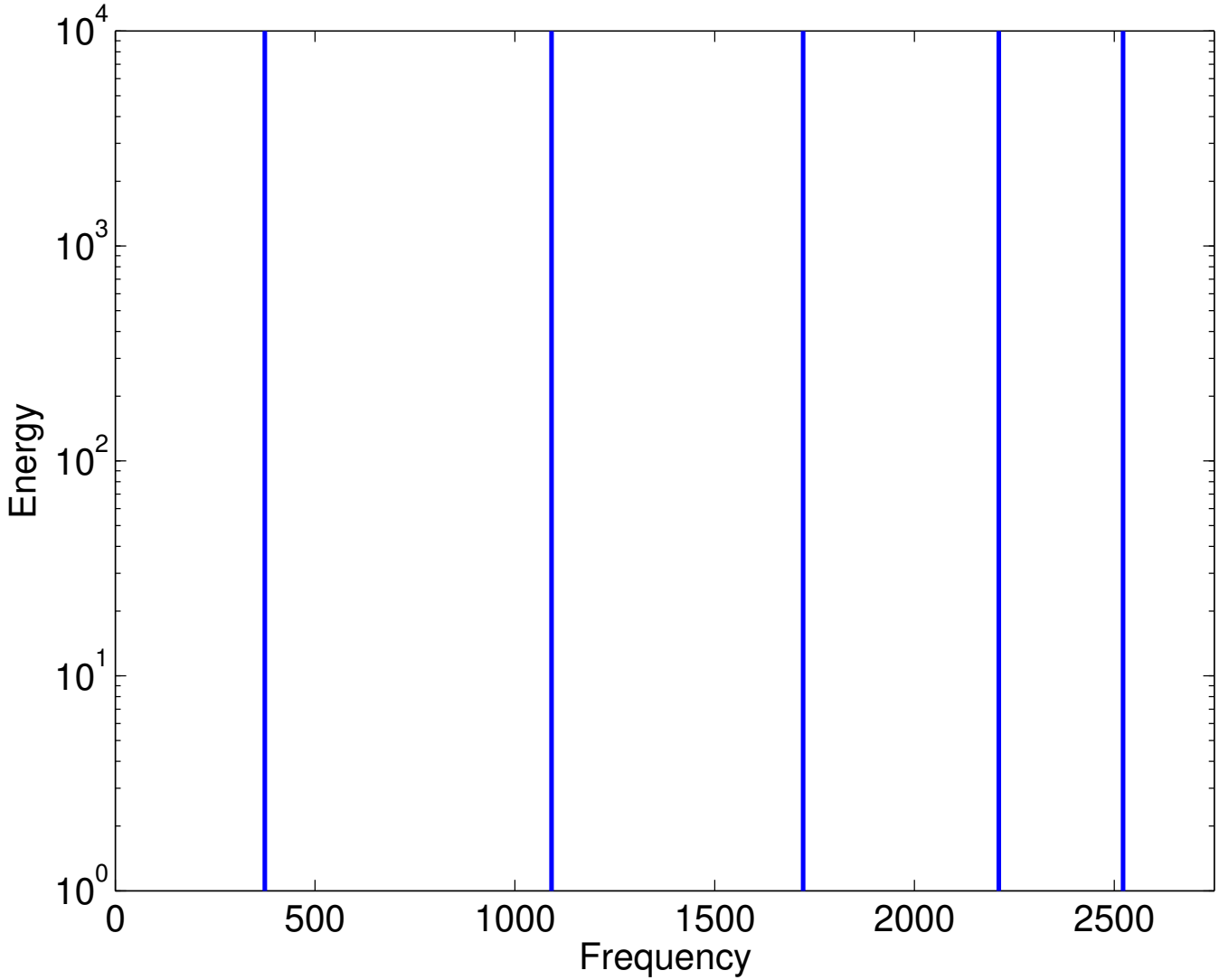


Figure 25: Frequency Energy Plot for a 5-dof system

Spectrum of vibration

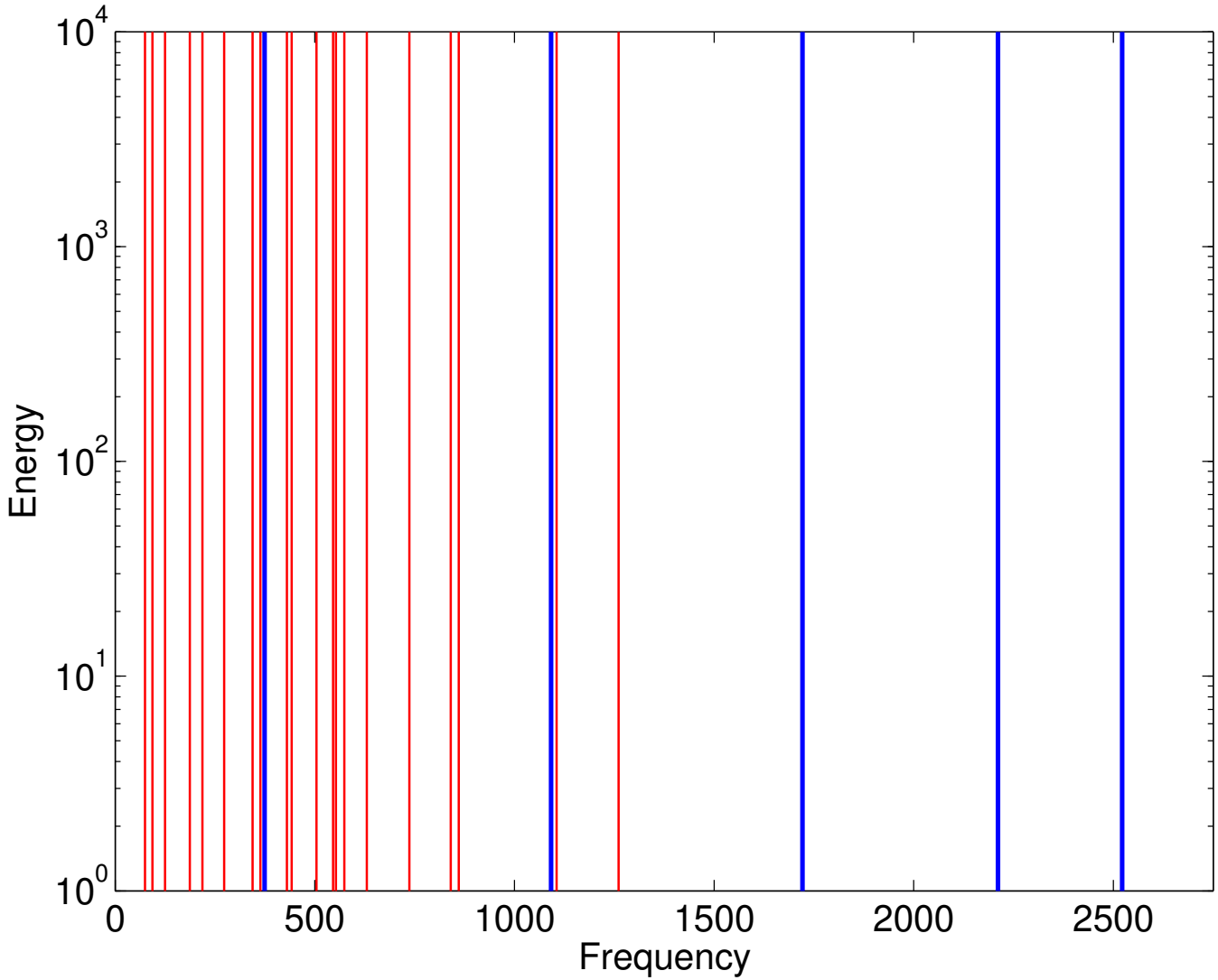


Figure 25: Frequency Energy Plot for a 5-dof system

Spectrum of vibration

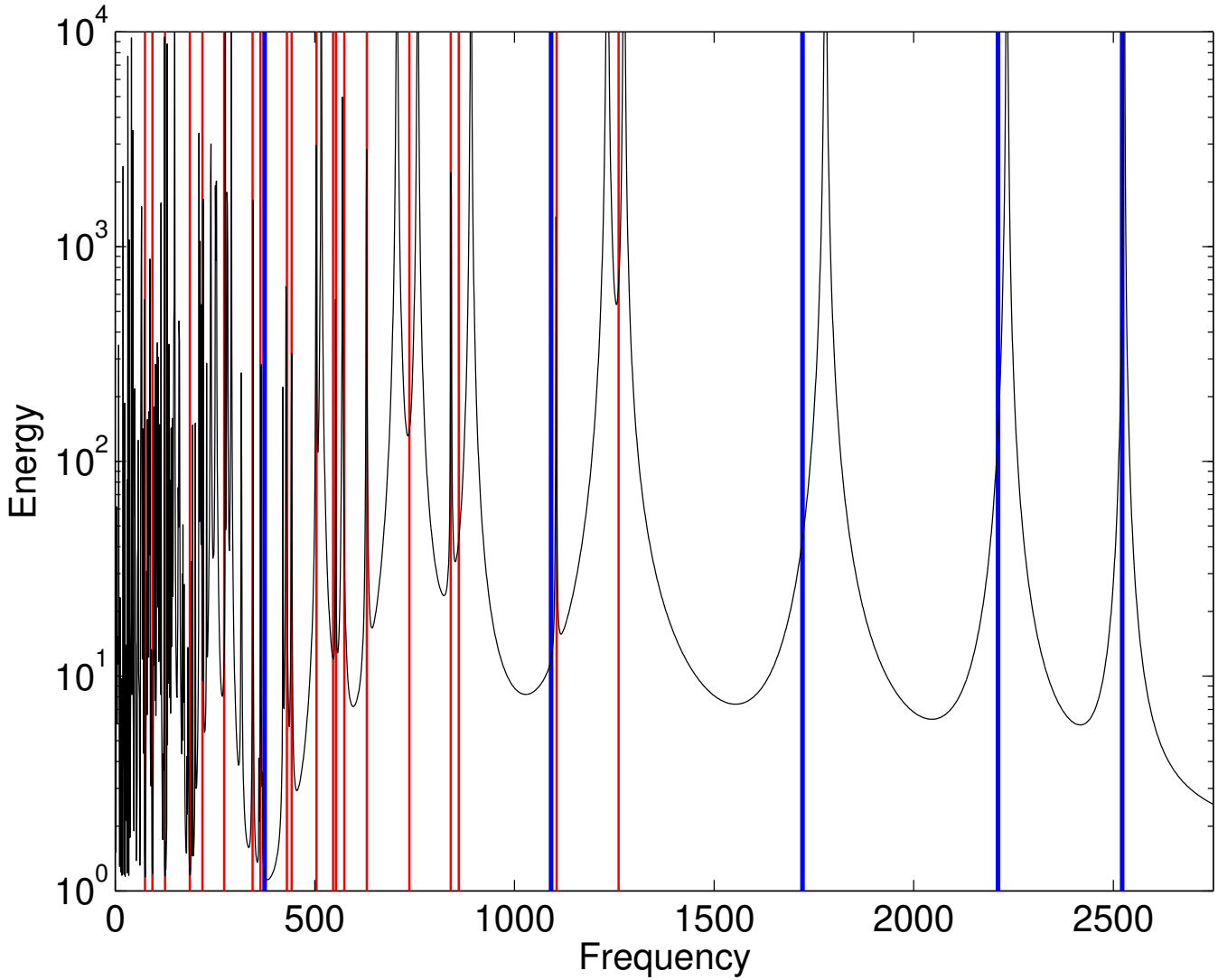


Figure 25: Frequency Energy Plot for a 5-dof system

Spectrum of vibration

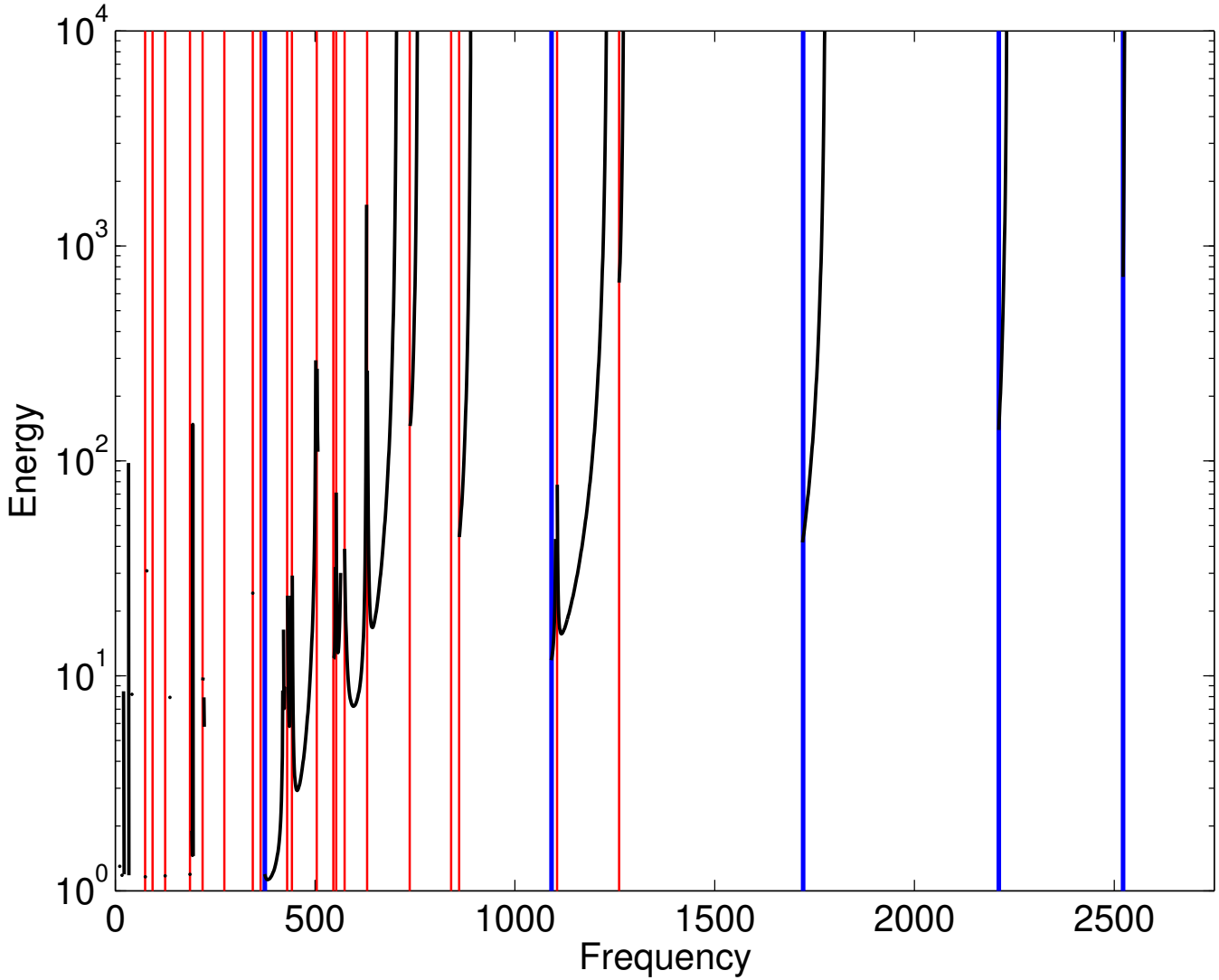


Figure 25: Frequency Energy Plot for a 5-dof system

Spectrum of vibration

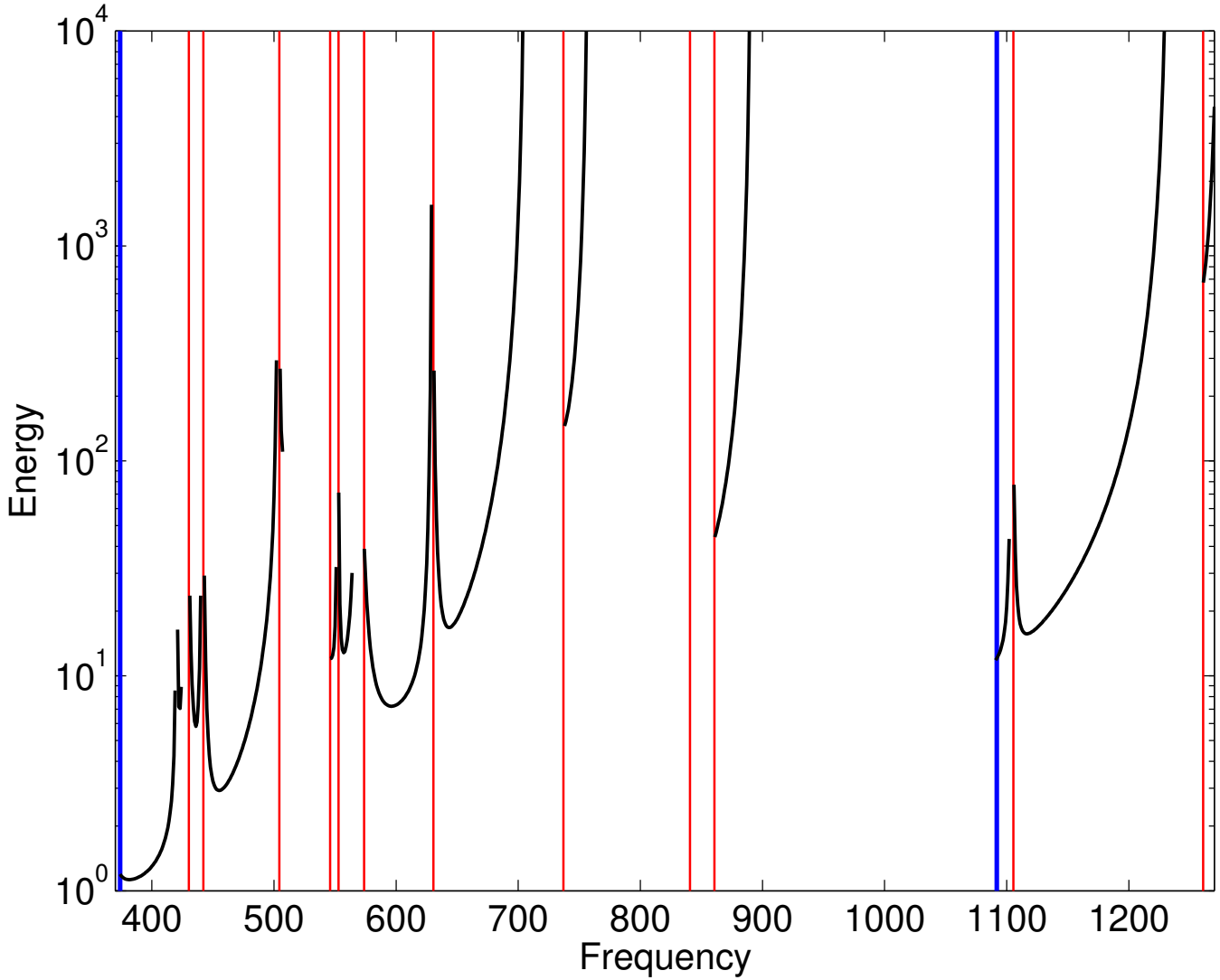


Figure 25: Frequency Energy Plot for a 5-dof system

Spectrum of vibration

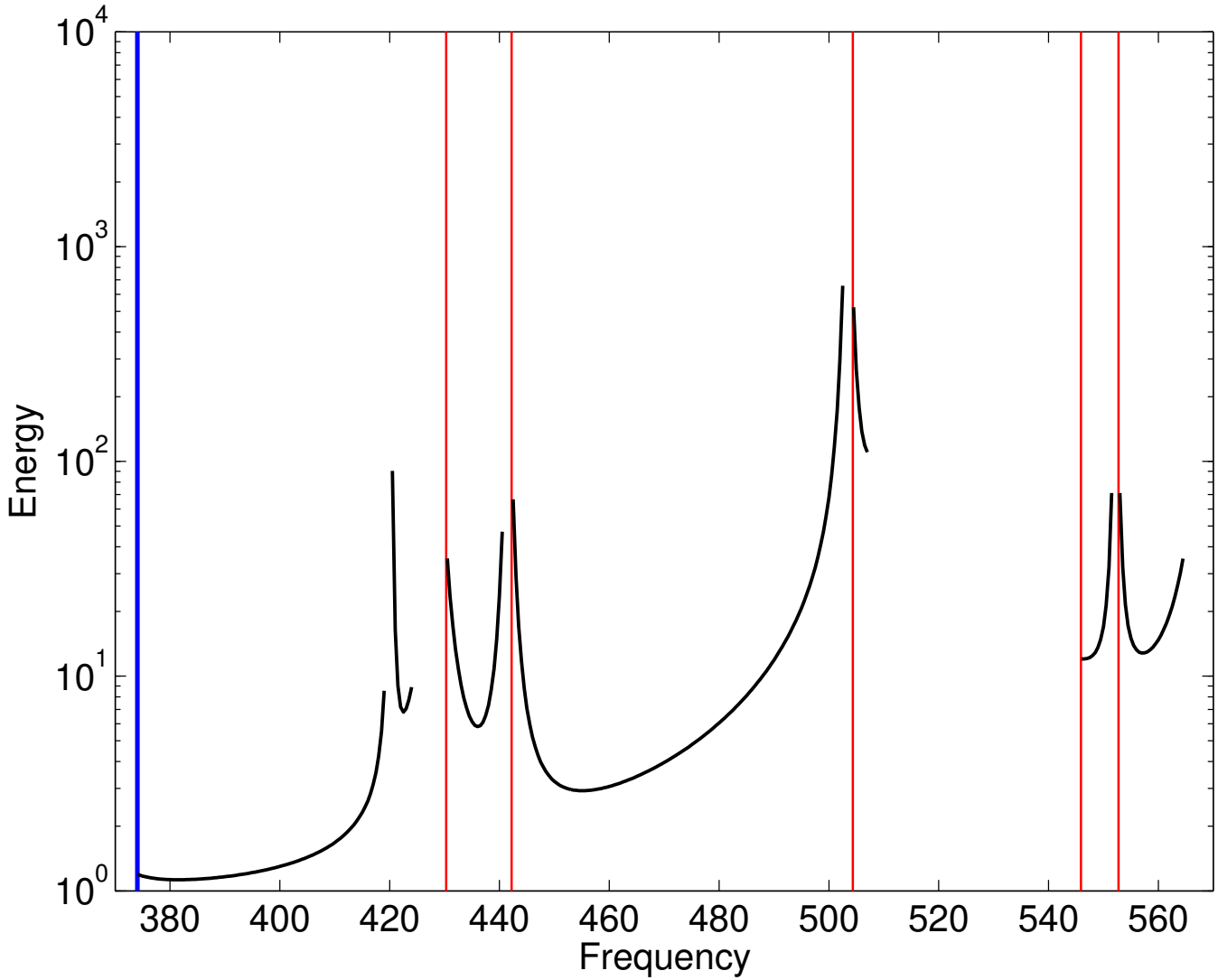


Figure 25: Frequency Energy Plot for a 5-dof system

Manifolds

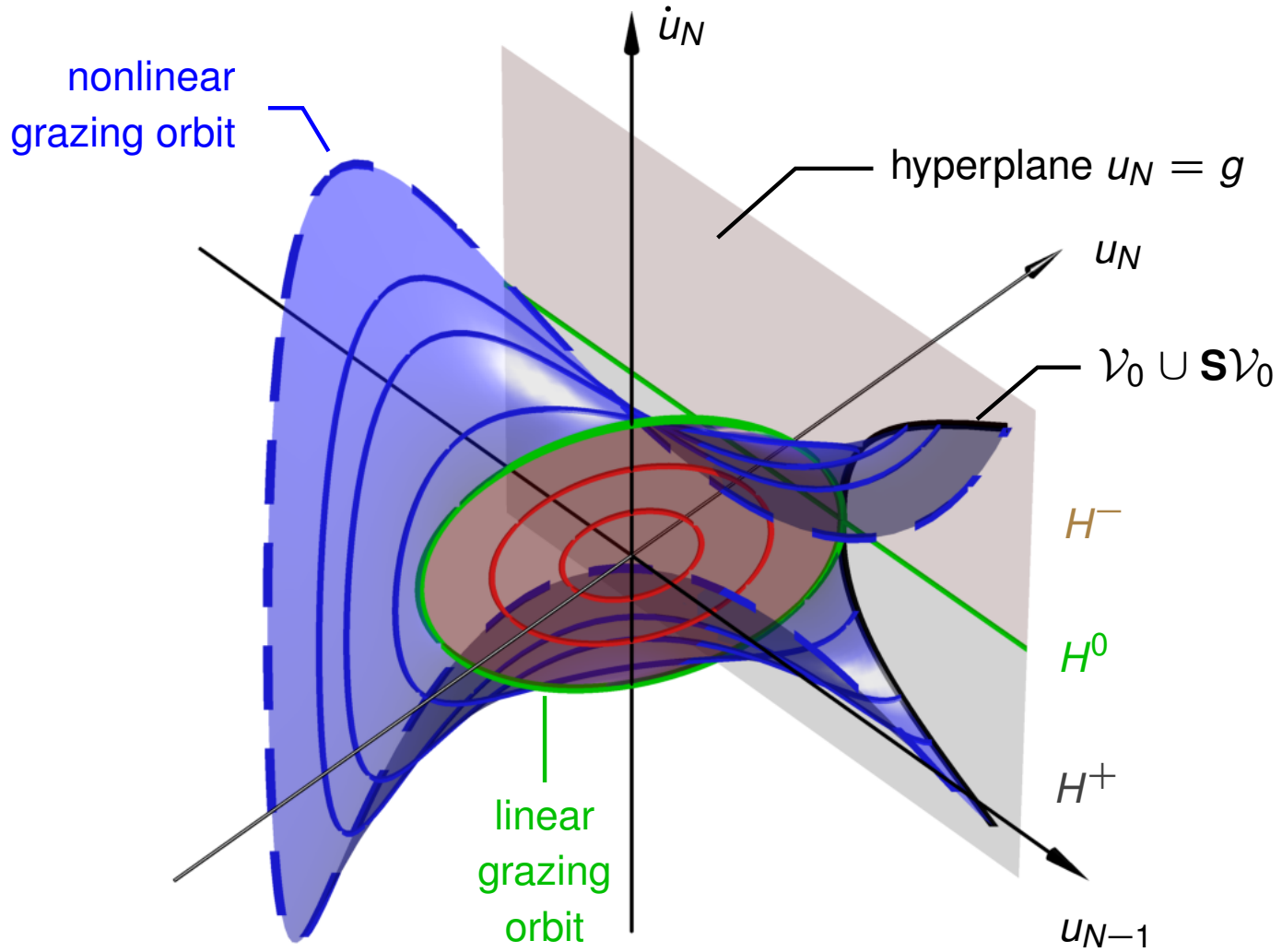


Figure 26: Generic manifold

Invariant manifolds for $g_0 > 0$

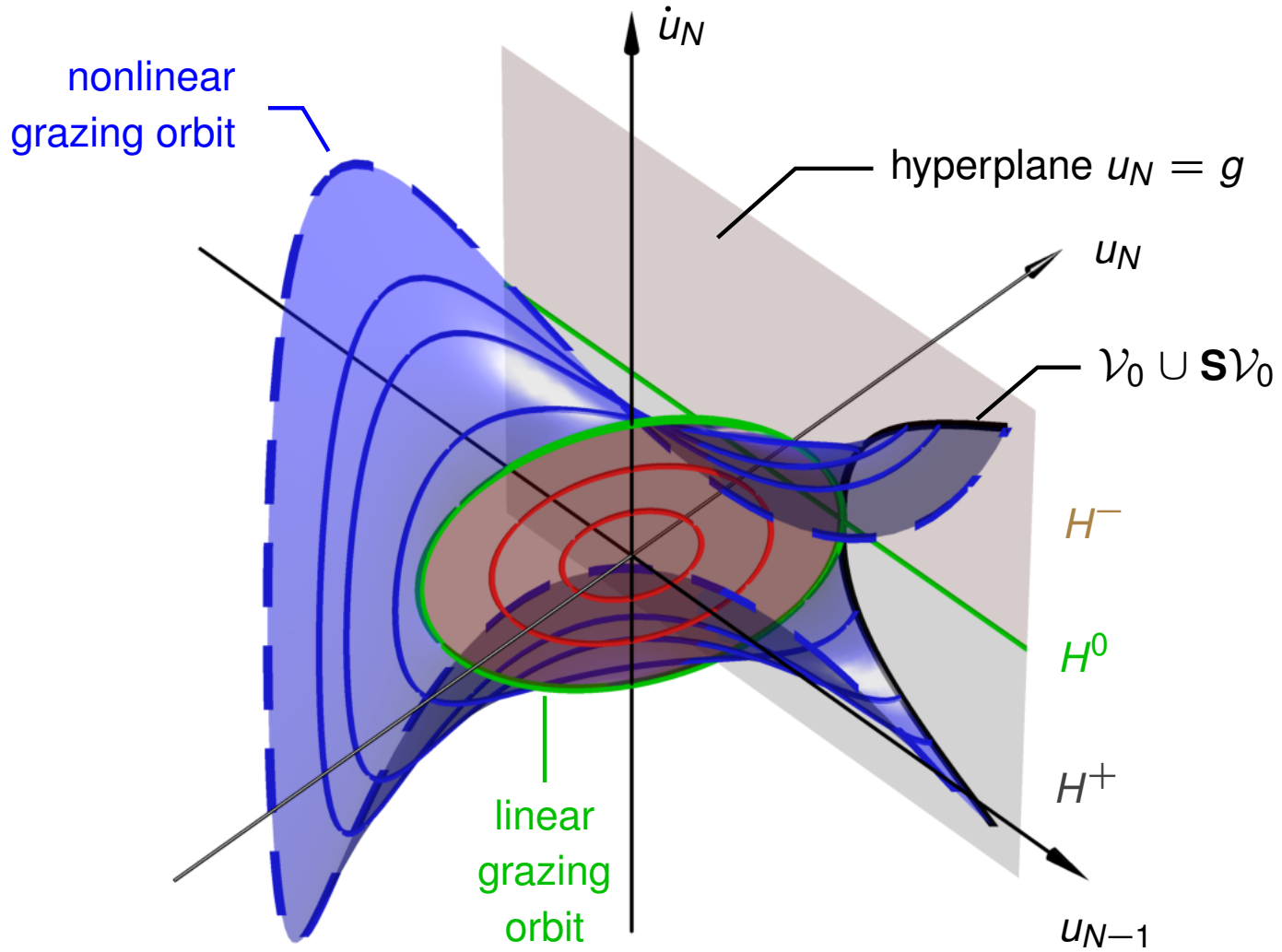


Figure 27: Generic manifold

Invariant manifolds for $g_0 > 0$

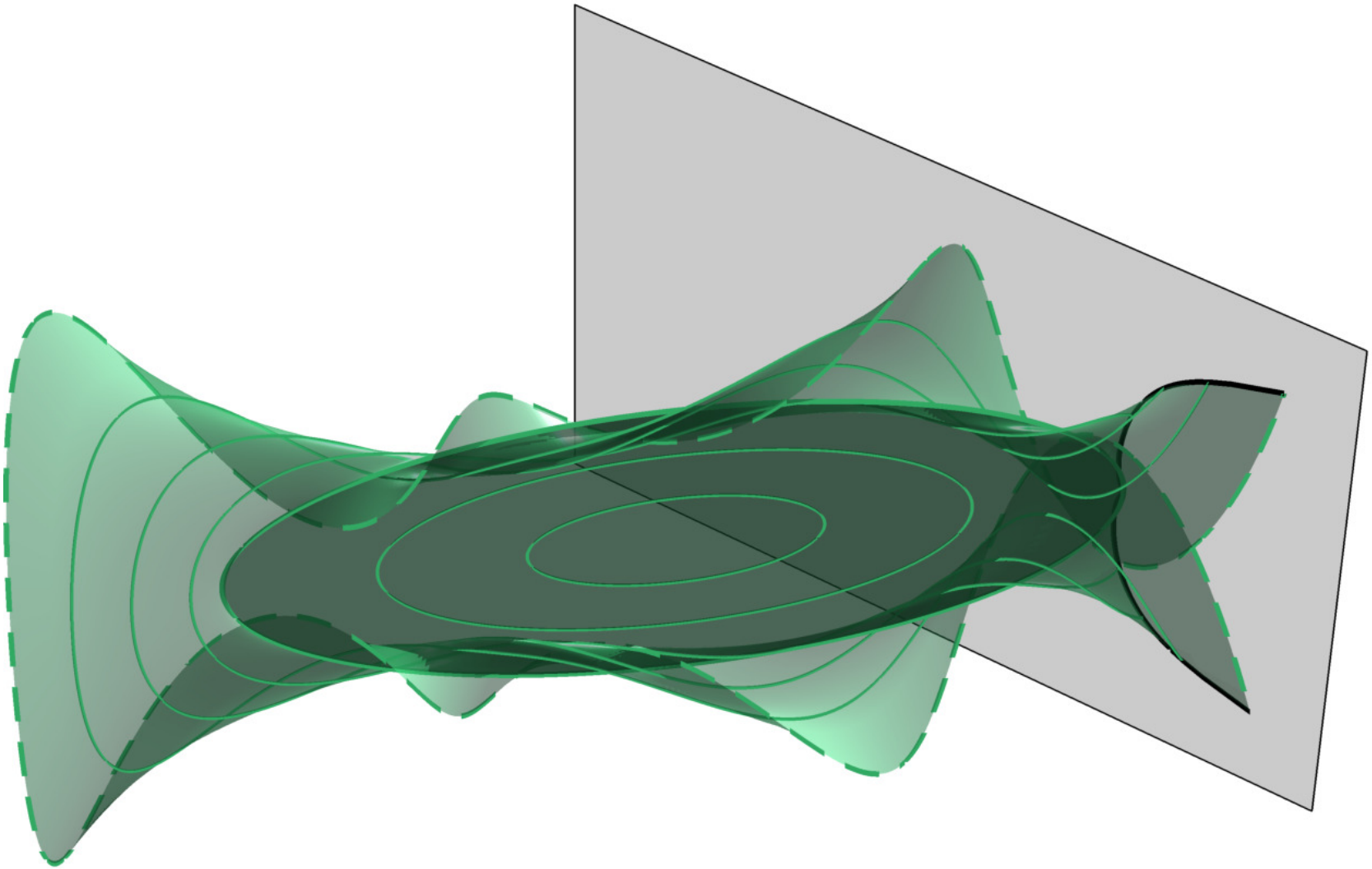


Figure 27: Nonsmooth mode 1

Invariant manifolds for $g_0 > 0$

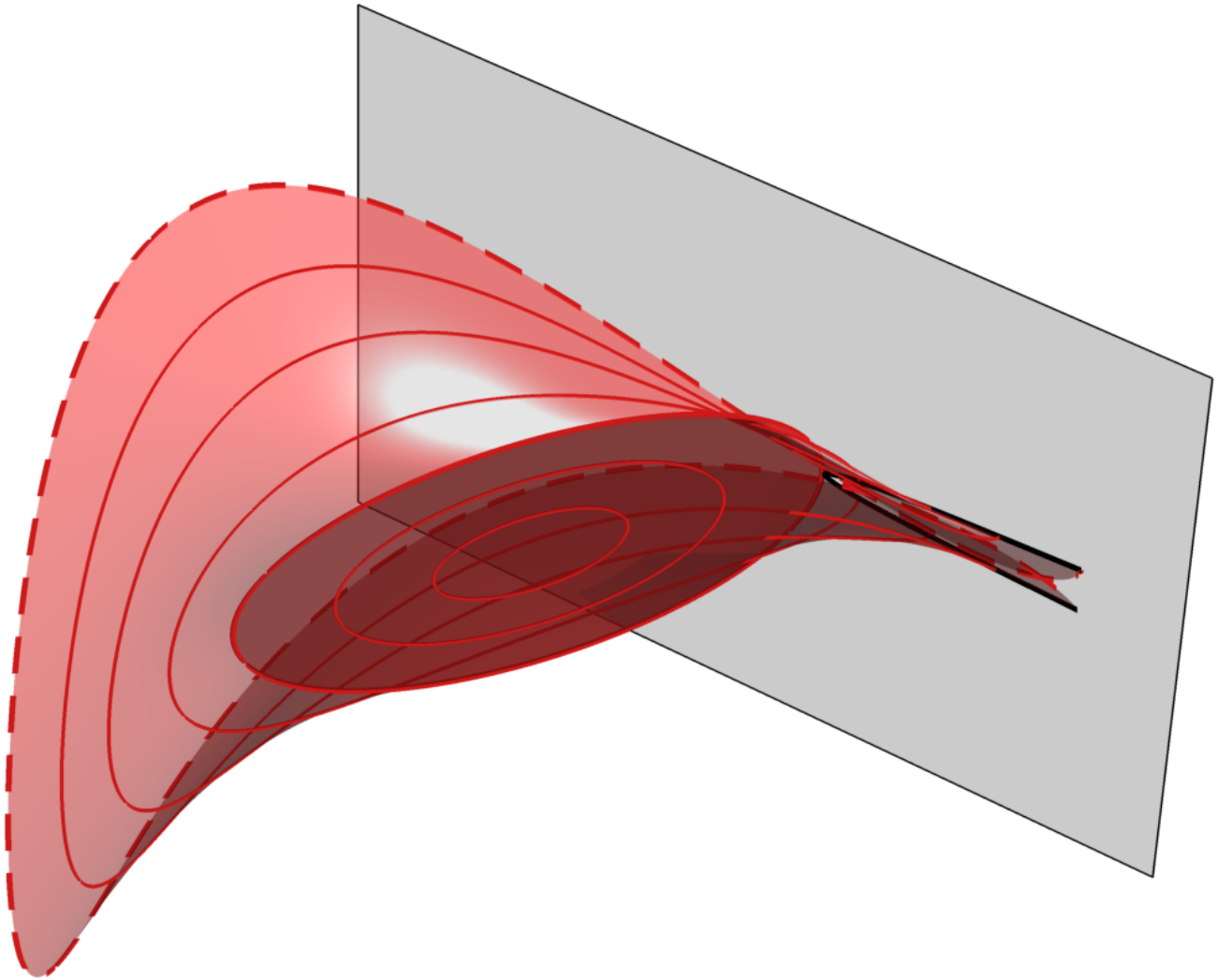


Figure 27: Nonsmooth mode 2

Invariant manifolds for $g_0 > 0$

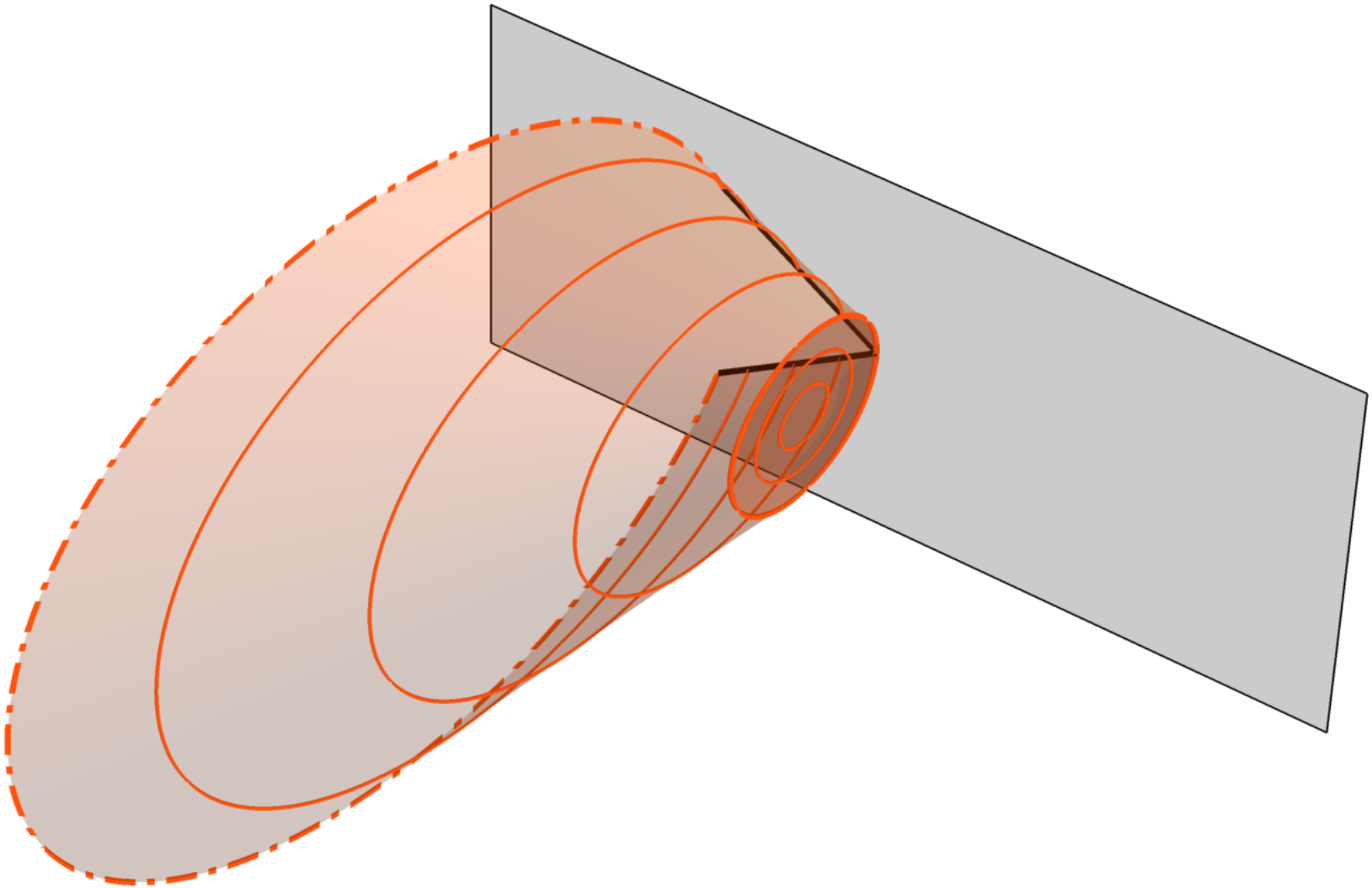


Figure 27: Nonsmooth mode 3

Invariant manifolds for $g_0 > 0$

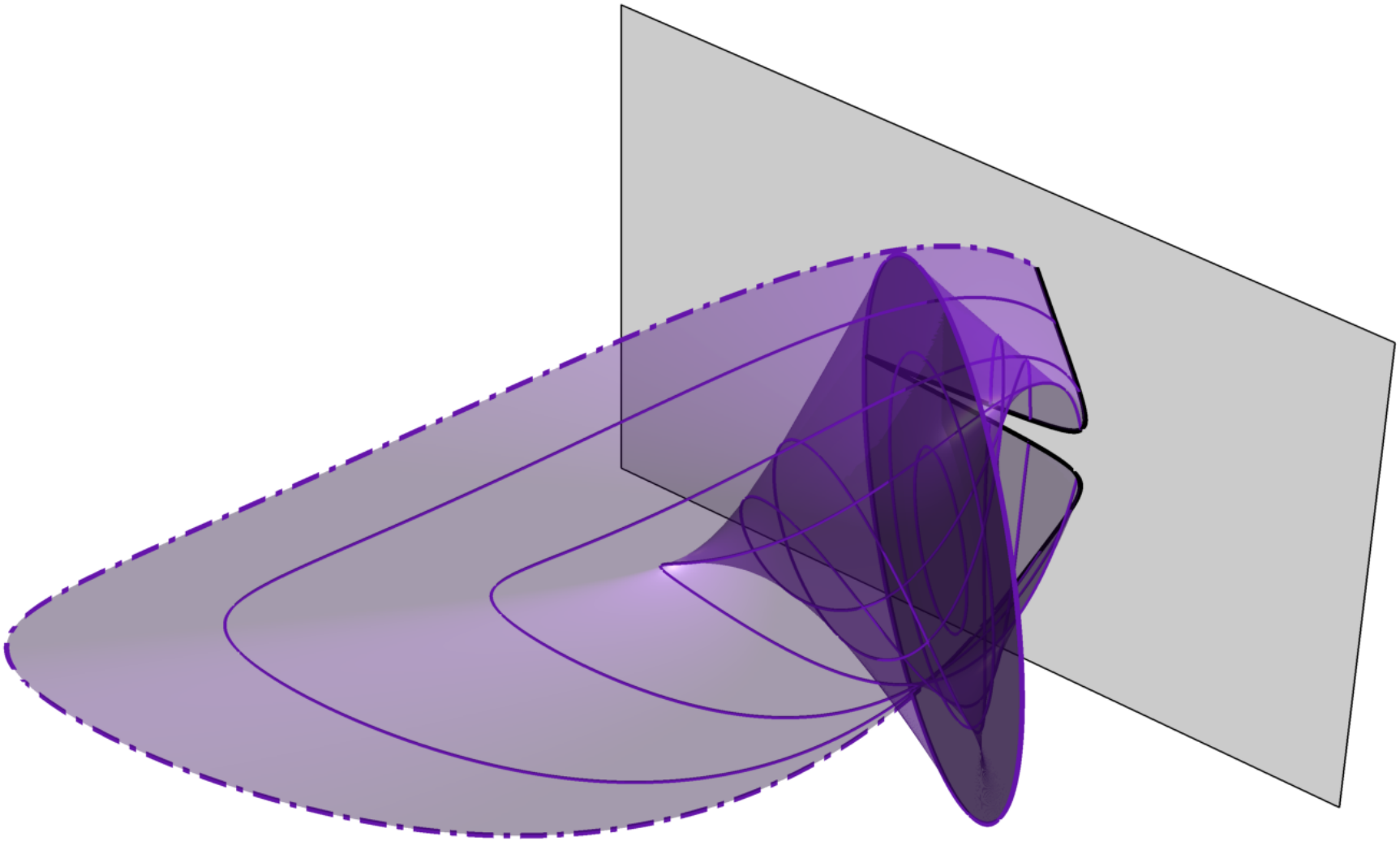


Figure 27: Nonsmooth mode 4 (H2)

Conical manifolds: $g_0 = 0$

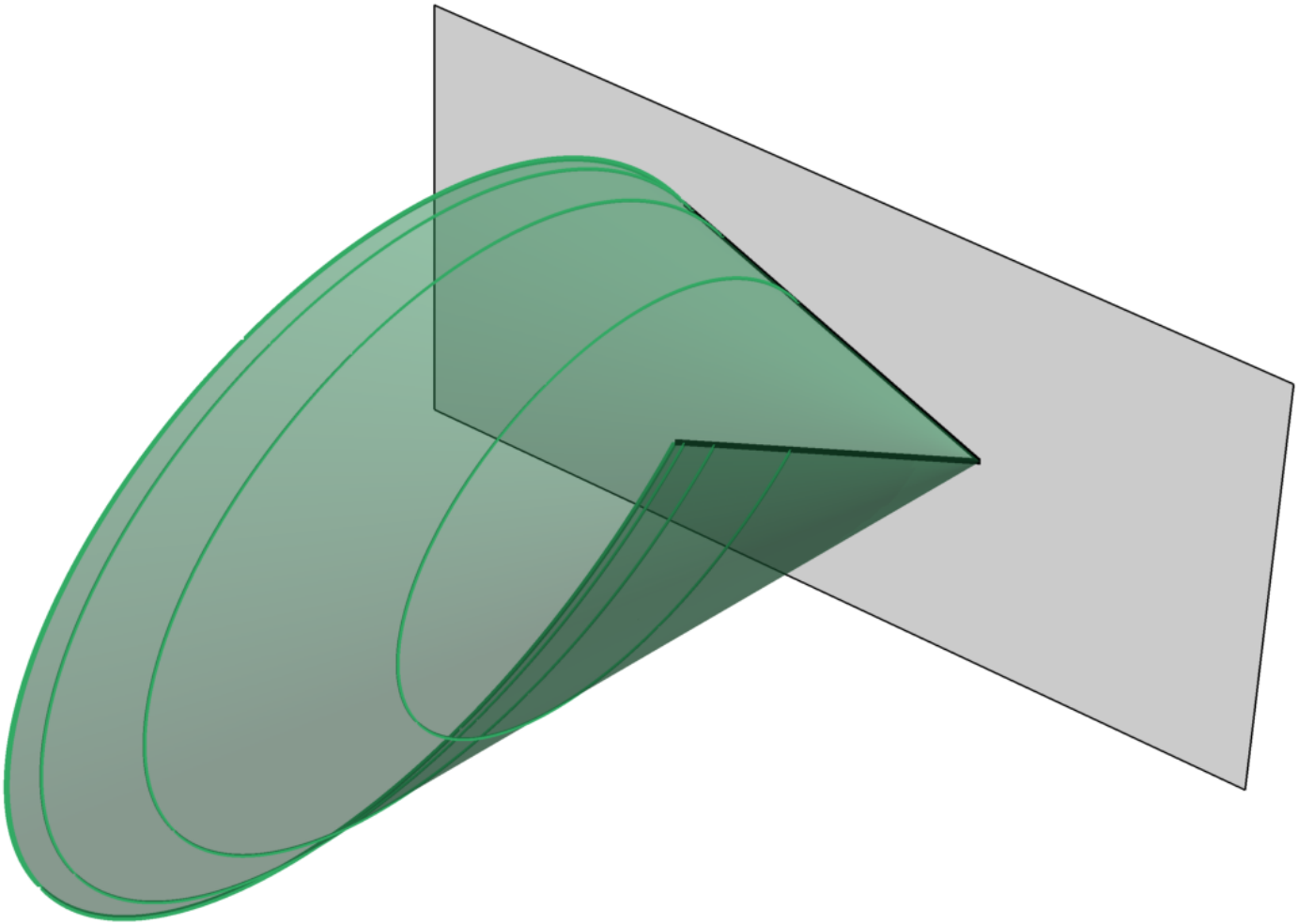


Figure 28: Nonsmooth mode for $T = 0.92$ s

Conical manifolds: $g_0 = 0$

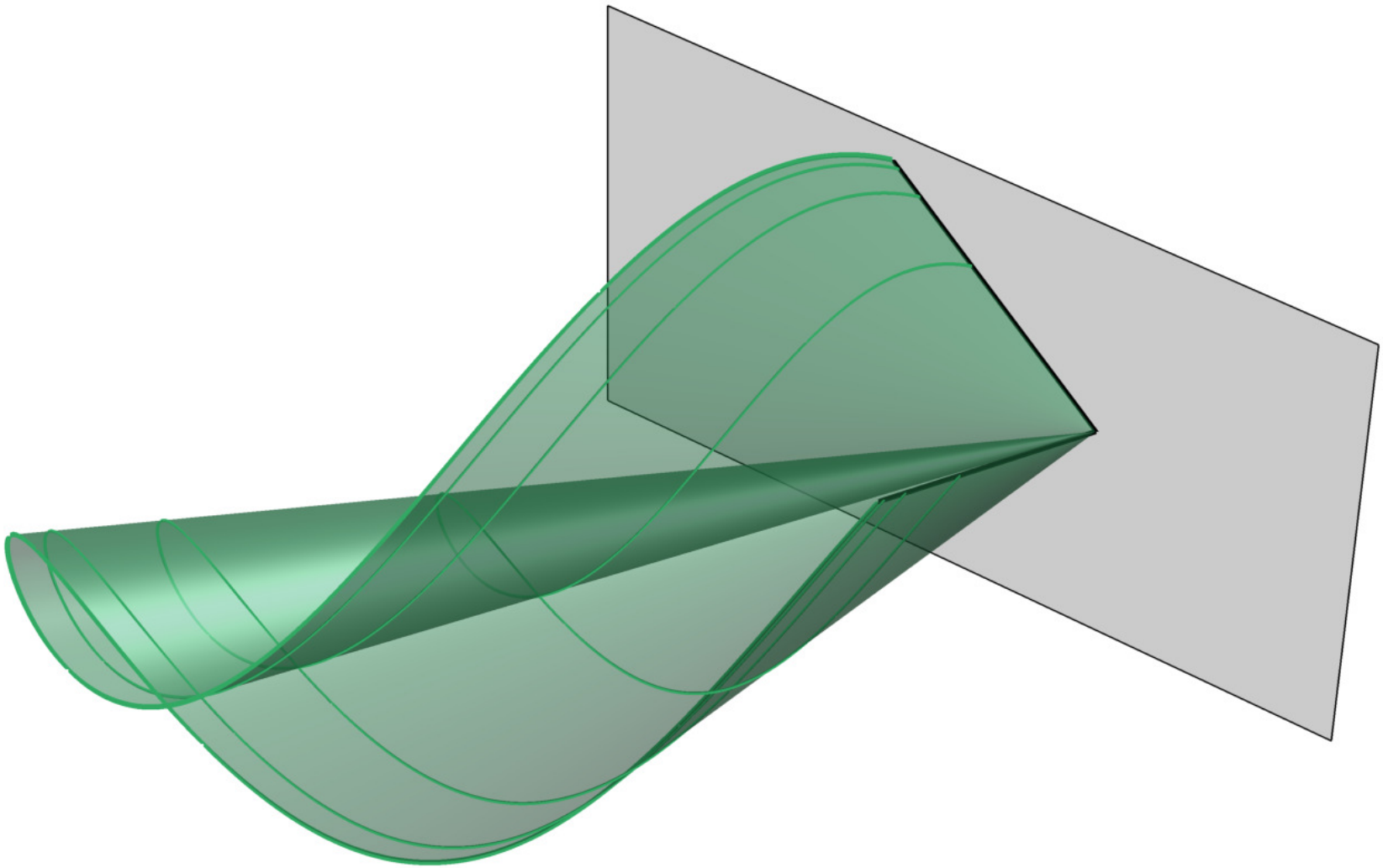


Figure 28: Nonsmooth mode for $T = 1.29$ s

“Disconnected” manifolds: $g_0 < 0$

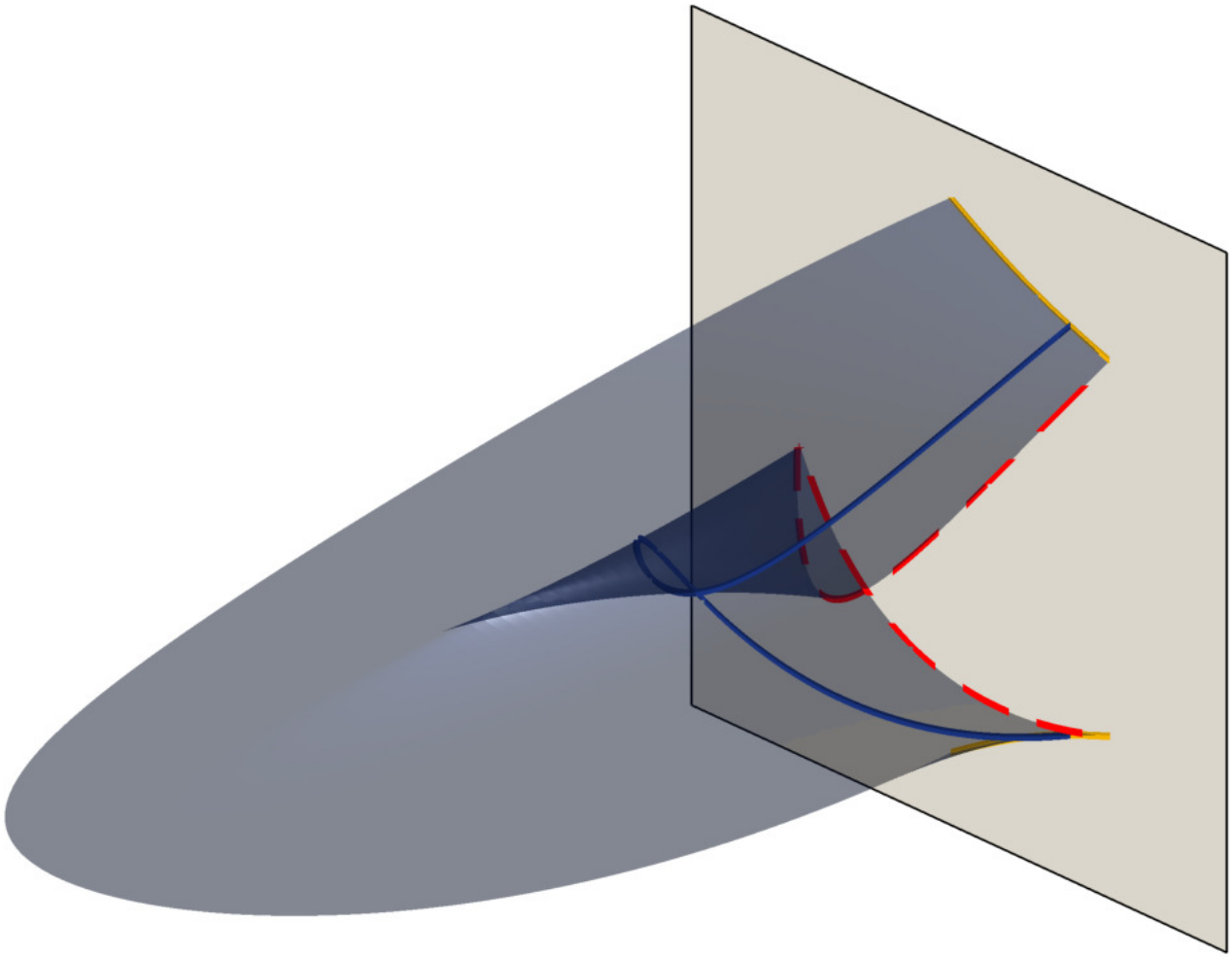


Figure 29: Nonsmooth mode for $T \approx 1.1$ s

“Disconnected” manifolds: $g_0 < 0$

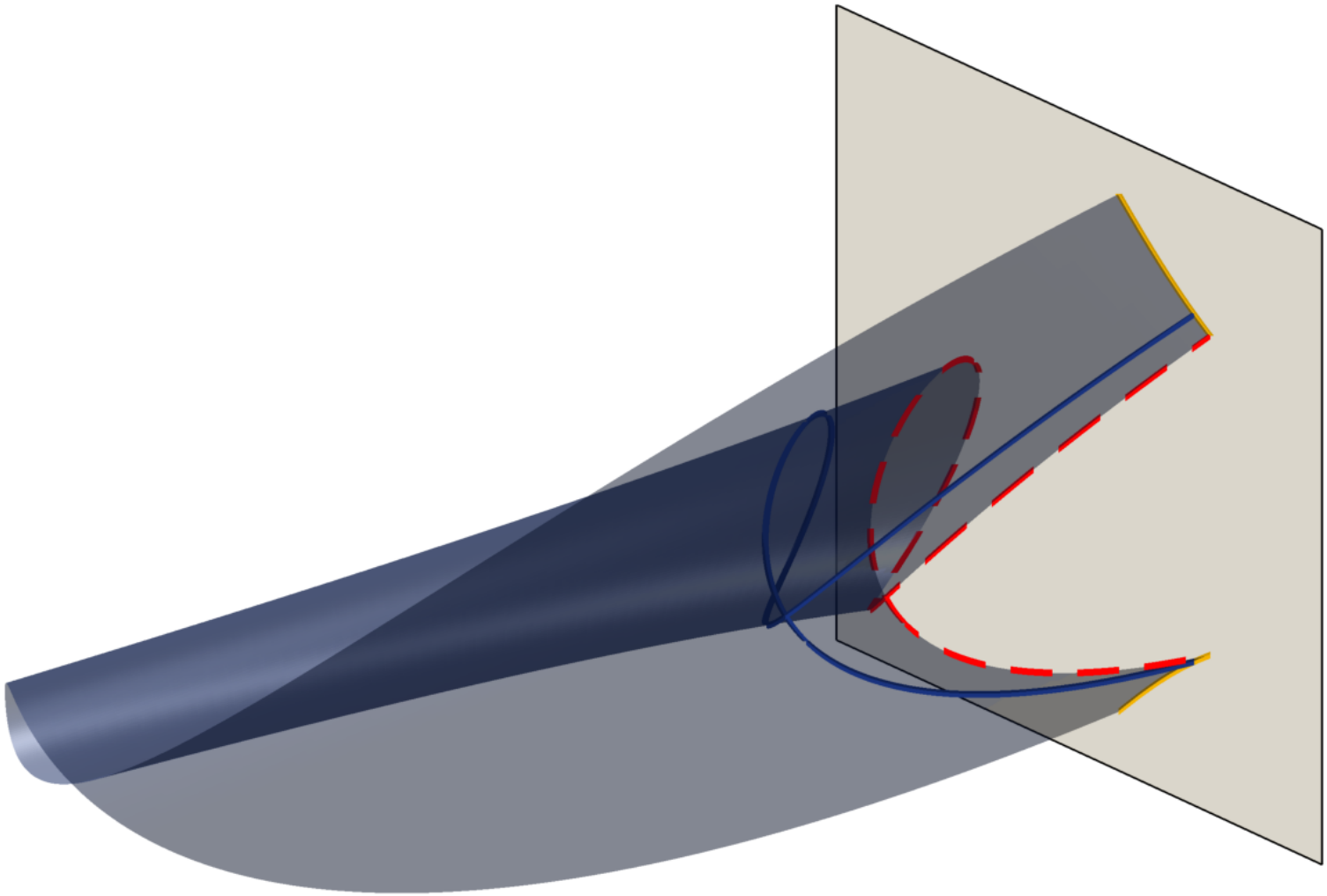


Figure 29: Nonsmooth mode for $T \approx 0.9$ s

k impacts per period: direct approach

Notations for two impacts at t_1 and T

$$\mathbf{NF}(T - t_1)\mathbf{NF}(t_1) - \mathbf{I} := \mathbf{A}(t_1, T) \tag{25}$$

Mathematical problem

Find $\mathbf{q}(0)$ and (t_1, T) such that $\mathbf{q}(0) \in \ker\left(\mathbf{NF}(T - t_1)\mathbf{NF}(t_1) - \mathbf{I}\right)$

- Idea
  1. Solve $\det\left(\mathbf{A}(t_1, T)\right) = 0$ for unknowns (t_1, T)
  2. calculate $\ker(\mathbf{A}(t_1, T))$
- Issues
  1. $\det(\mathbf{A}(t_1, T)) = 0$ is a cumbersome condition
  2. dimension of ker has to be greater than rank of the system of conditions $u_n(t_1) = u_n(T) = g$

Not possible to solve in practice for $n \geqslant 3$

k impacts per period

Main mathematical results[9,10,11]

- **Existence of** $t_1, \dots, t_k$
  There exist a known isomorphism $\boldsymbol{\varphi}$ and a known matrix $\boldsymbol{\Pi}$ such that

  $$\mathbf{q}(0) \in \ker(\mathbf{A} - \mathbf{I}) \xrightarrow{\sim} \ker(\boldsymbol{\Pi}), \quad \boldsymbol{\Pi} \in \mathrm{Skew}_k$$

  that is $\mathbf{A}(t_1, \dots, t_k)\mathbf{q}(0) = \mathbf{q}(0) \iff \boldsymbol{\Pi}(t_1, \dots, t_k)\boldsymbol{\varphi}(\mathbf{q}(0)) = \mathbf{0}$
- **Conditions** $u_n(t_1) = g, \dots, u_n(t_k) = g$
  There exists a known matrix $\boldsymbol{\Sigma}$ such that closing contact conditions read

  $$\boldsymbol{\Sigma}(t_1, \dots, t_k)\boldsymbol{\varphi}(\mathbf{q}(0)) = \mathbf{g}^\top$$

Procedure (with $\boldsymbol{\lambda} = \boldsymbol{\varphi}(\mathbf{q}(0))$)

Given k, find $\boldsymbol{\lambda}$ and $t_1, \dots, t_k$ such that

- $\mathbf{M}(t_1, \dots, t_k)\boldsymbol{\lambda} = 0 \longrightarrow k(k-1)/2$ equations
- $\boldsymbol{\Sigma}(t_1, \dots, t_k)\boldsymbol{\lambda} = \mathbf{g}^\top \longrightarrow k$ equations

for $k + k$ unknowns ($t_1, \dots, t_k$ and $\boldsymbol{\lambda}$). Formulation does **not depend on** n

---

[9] Thorin, Delezoide & Legrand. Non-smooth modal analysis of piecewise-linear impact oscillators. *SIAM Journal on Applied Dynamical Systems* (2017).
[10] Thorin & Legrand. *Advanced Topics in Nonsmooth Dynamics* (2018).
[11] Thorin, Delezoide & Legrand. Periodic solutions of n-dof autonomous vibro-impact oscillators with one lasting contact phase. *Nonlinear Dynamics* (2017).

k impacts per period

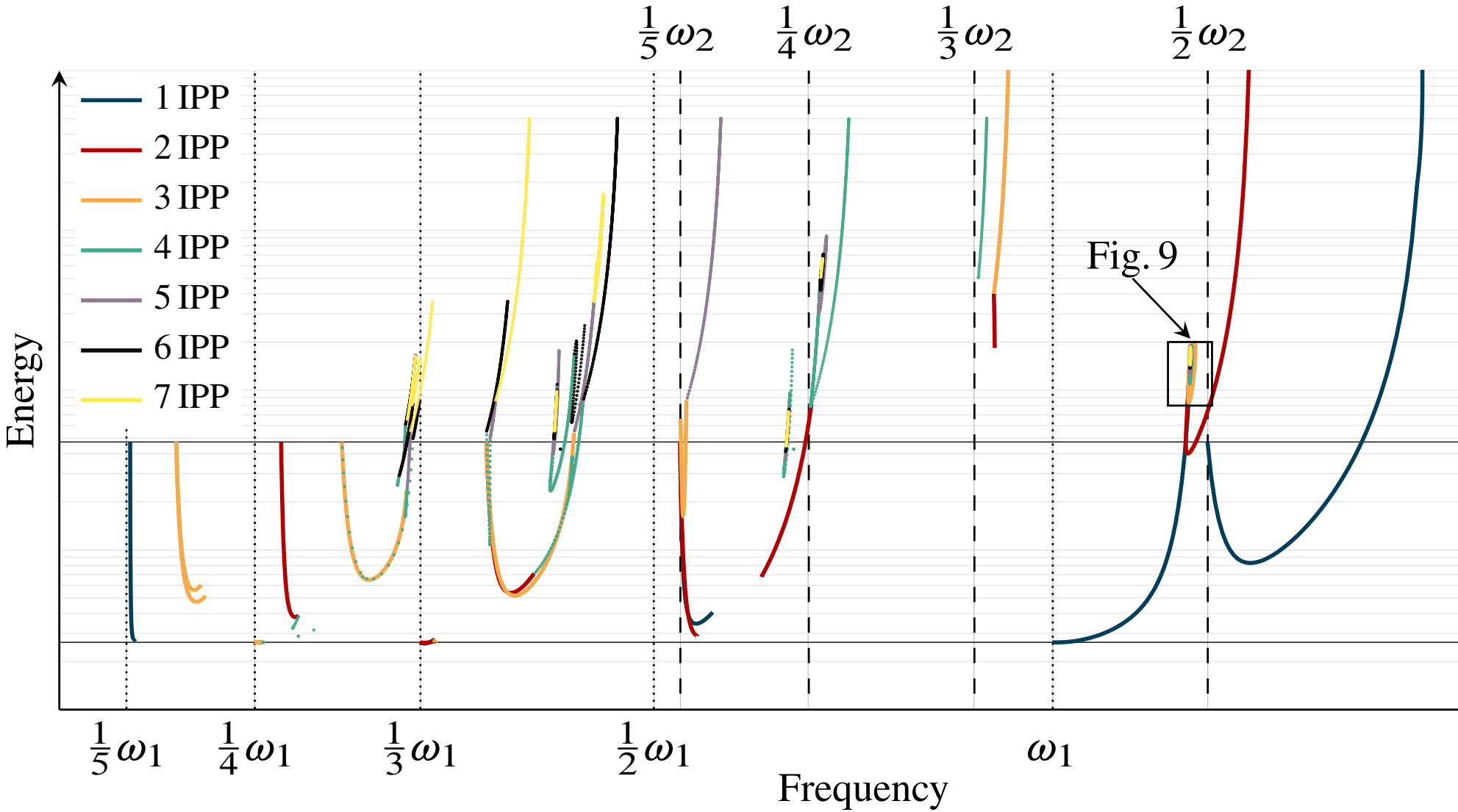


Figure 30: Spectrum of vibration

Impacts, really?

Impact law $e = 1$

- Unrealistic impact law: impulses in the contact force questionable in the context of continuum mechanics: chattering effect?
- Convergence in space [FEA]: what if dof $\to \infty$?

Difficulties

- non-periodic damped solutions
  - Impact law with $e \neq 1$: unsuccessful so far
  - Structural damping: unsuccessful so far
- Multiple contact points
- Comparison with forced-response

Frequency Domain

Unilateral contact in equality form

- System: bar + Signorini BC
- Fourier Transform[9,10]

$$\mathcal{F}\{u(x,t)\} = \hat{u}(x,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\exp(-i\omega t)\mathrm{d}t \tag{26}$$

- Local PDE: rod equation [in frequency domain]

$$\hat{u}_{xx}(x,\omega)) + \omega^2\hat{u}(x,\omega) = 0 \tag{27}$$

- Dirichlet $\hat{u}(0,\omega) = 0$ [in frequency domain]
- Signorini at $x = L$ [in time domain, with $p(t) \equiv u_x(L,t)$]

$$g_0 - u(L,t) \geq 0, \quad p(L,t) \leq 0, \quad (g_0 - u(L,t))p(L,t) = 0 \tag{28}$$

$$\Leftrightarrow \quad p(t) + \max[u(L,t) - g_0 - p(t), 0] = 0 \tag{29}$$

$$\Leftrightarrow \quad \min[u(L,t) - g_0, p(t)] = 0 \tag{30}$$

  - ▷ Signorini complementarity conditions expressed in *equality* form!

[9] Lu & Legrand. Nonsmooth modal analysis via the boundary element method for one-dimensional bar systems. *Nonlinear Dynamics* (2022).
[10] Lu & Legrand. Harmonic Balance based Nonsmooth Modal Analysis of unilaterally constrained discrete systems. *Nonlinear Dynamics* (2023).

Unilateral contact in equality form

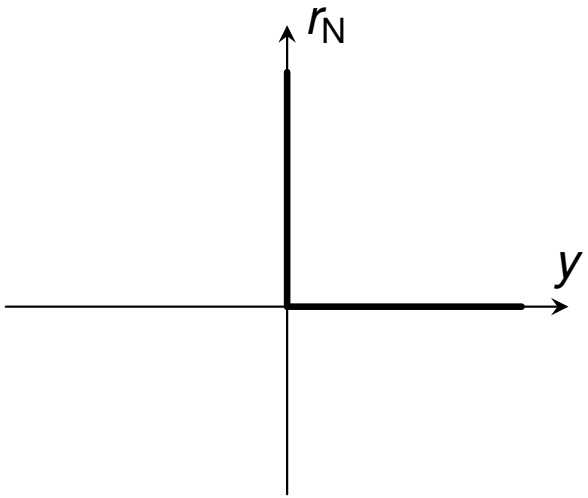
- System: bar + Signorini BC
- Fourier Transform[9,10]

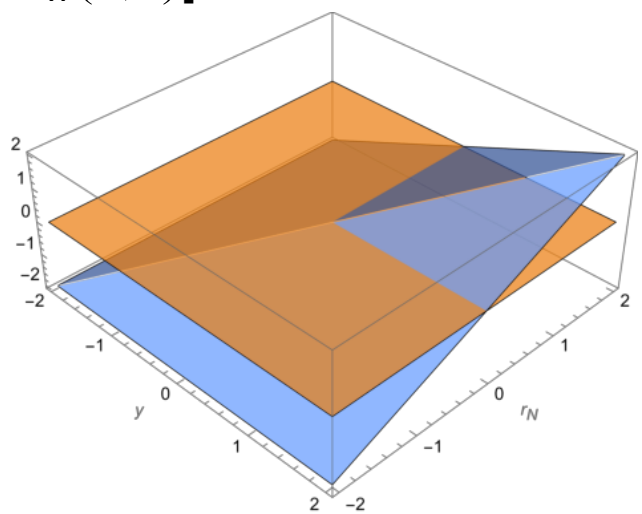
$$\mathcal{F}\{u(x,t)\} = \hat{u}(x,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\exp(-i\omega t)\mathrm{d}t \tag{26}$$

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[10] Lu & Legrand. Harmonic Balance based Nonsmooth Modal Analysis of unilaterally constrained discrete systems. *Nonlinear Dynamics* (2023).

Semi-discretization in space

- Trial and test functions

$$\hat{u}(x,\omega) \approx \sum_{i=1}^{N} \phi_i(x)\hat{u}_i(\omega), \quad \hat{u}_x(x,\omega) \approx \sum_{i=1}^{N} \phi_{i\,x}(x)\hat{u}_i(\omega) \tag{28}$$

$$v(x) \approx \sum_{i=1}^{N} \phi_i(x)v_i, \quad v_x(x) \approx \sum_{i=1}^{N} \phi_{i\,x}(x)v_i \tag{29}$$

- Shape functions $\{\phi_i(x)\}$: FE hat functions, linear modes, or Craig-Bampton modes...
- Semi-discretized weak form of governing (Helmholtz) equation

$$(\mathbf{K} - \omega^2\mathbf{M})\hat{\mathbf{u}}(\omega) = \boldsymbol{\phi}(L)\hat{p}(\omega) \tag{30}$$

where $\boldsymbol{\phi}(x) \equiv [\phi_1(x), \phi_2(x), \ldots, \phi_N(x)]$

- Condensation of the linear dynamics to the nonlinear interface

$$\hat{u}(L,\omega) \approx \hat{u}_N(\omega) = G_h(\omega)\hat{p}(\omega), \quad G_h(\omega) = \boldsymbol{\phi}(L)^\top \mathbf{G}(\omega)^{-1}\boldsymbol{\phi}(L) \tag{31}$$

where $\mathbf{G}(\omega) \equiv \mathbf{K} - \omega^2\mathbf{M}$

Semi-discretization in time: Fourier series

- Periodic solutions

$$p(t) \approx p_h(t) = \sum_{k=-M}^{M} a_k \exp(jk\Omega t), \quad u(L,t) \approx u_{Nh}(t) = \sum_{k=-M}^{M} b_k \exp(jk\Omega t) \tag{32}$$

expressed in frequency domain

$$\hat{p}(\omega) \approx \hat{p}_h(\omega) = \sum_{k=-M}^{M} a_k \delta_{k\Omega} \quad \text{and} \quad \hat{u}(L,\omega) \approx \hat{u}_{Nh}(\omega) = \sum_{k=-M}^{M} b_k \delta_{k\Omega} \tag{33}$$

- Linear relationship at the contact interface: (33) into (31)

$$b_k = G_h(k\Omega) a_k, \quad k = -M, \ldots, M \tag{34}$$

- Signorini boundary condition enforced in a *weak sense* [HBM]

$$g_k(\mathbf{a}, \mathbf{b}) = \int_0^T \exp(jk\Omega t) \min[u_{Nh}(t) - g_0, p_h(t)]\big) \, \mathrm{d}t = 0, \quad \forall k \tag{35}$$

  - (34) and (35) form a system of $4M$ nonsmooth equations in the $4M$ unknowns $(\mathbf{a}, \mathbf{b})$
  - No need of an impact law
  - Classical nonlinear solvers and quadrature schemes

Spectrum of vibration and approximate solutions

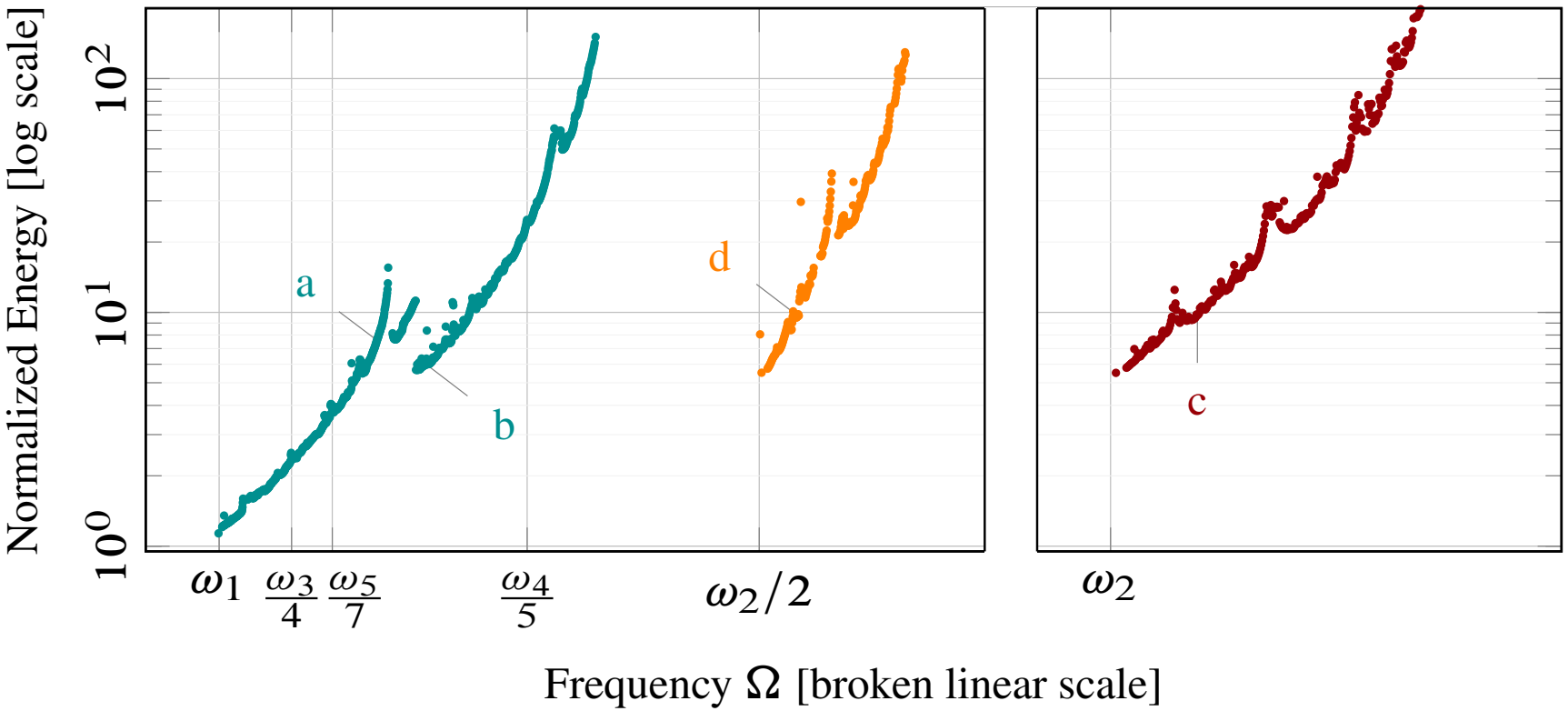
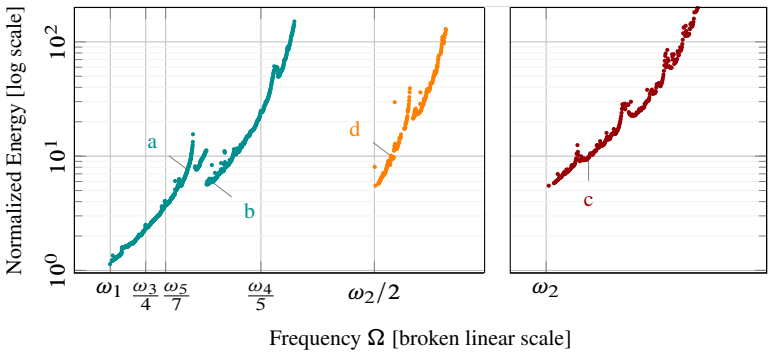
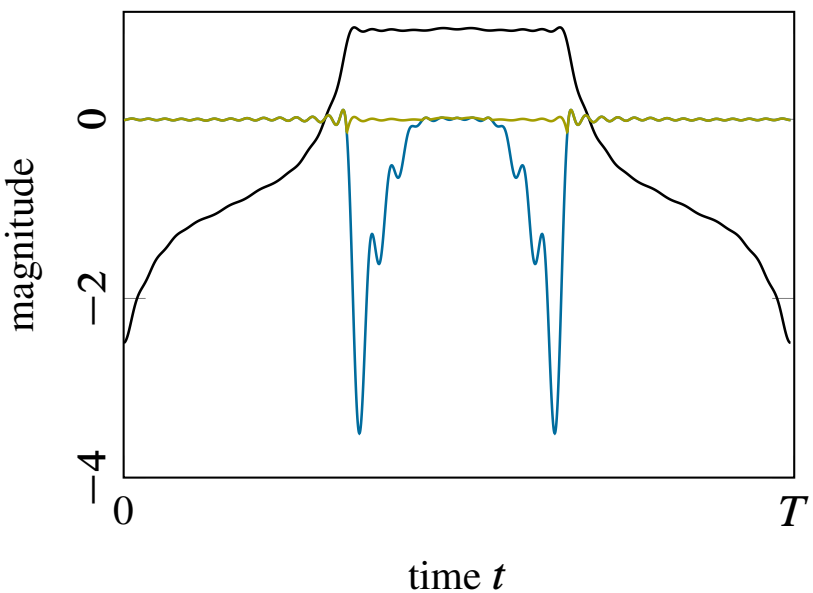


Figure 31: FEP: first mode [●], second mode [●], and subharmonic 2 of the second mode [●]

Spectrum of vibration and approximate solutions





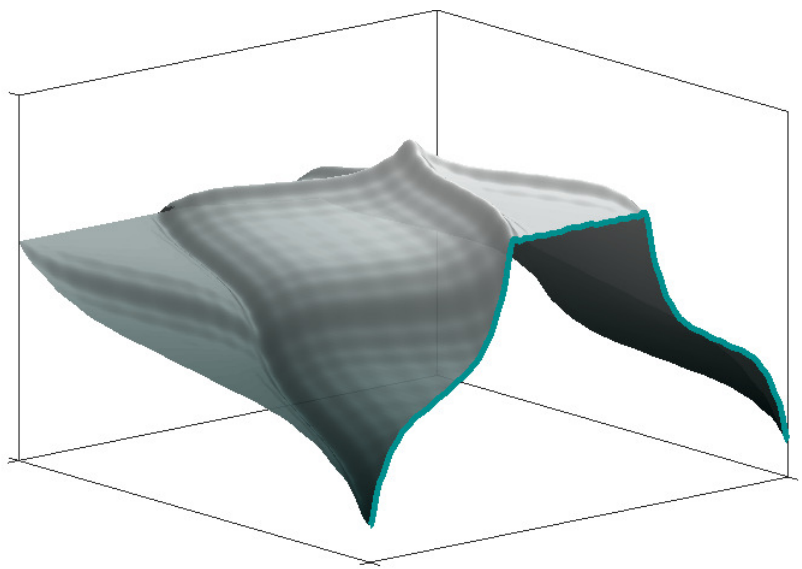
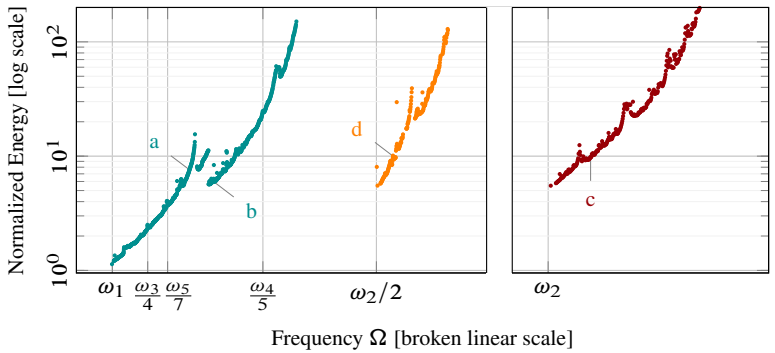
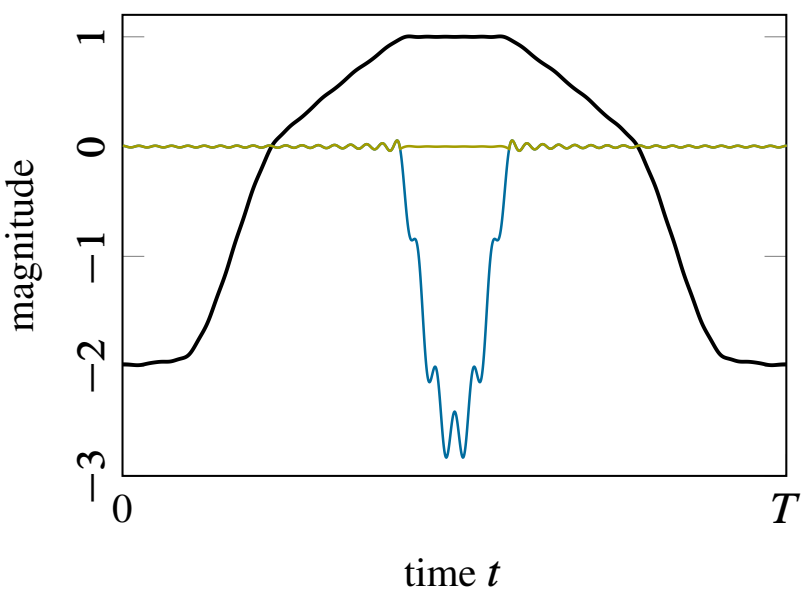


Figure 31: Displacement and contact force: point ‘a’ in backbone curve

Spectrum of vibration and approximate solutions





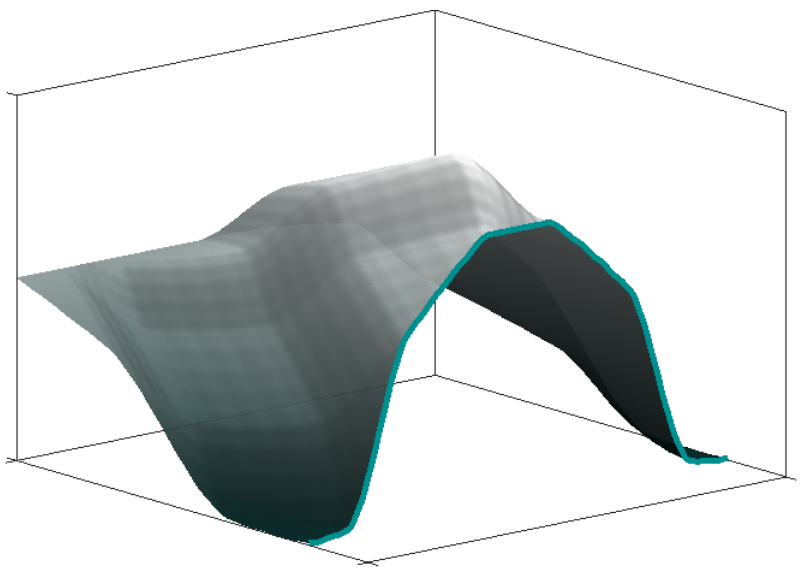
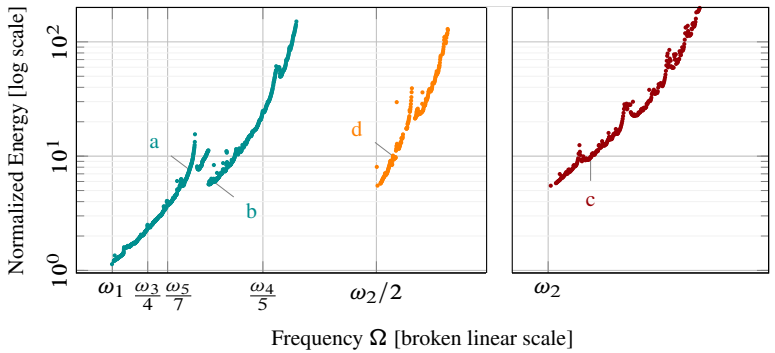
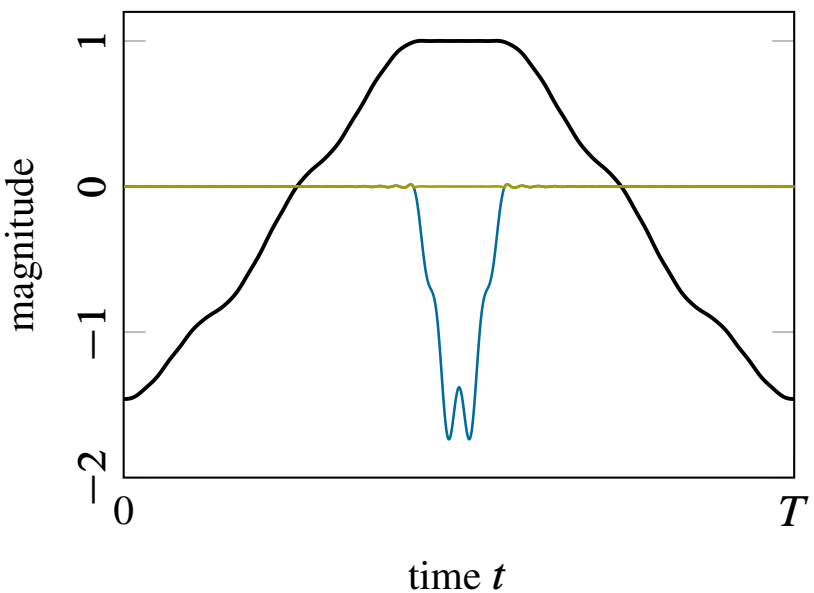


Figure 31: Displacement and contact force: point ‘b’ in backbone curve

Spectrum of vibration and approximate solutions





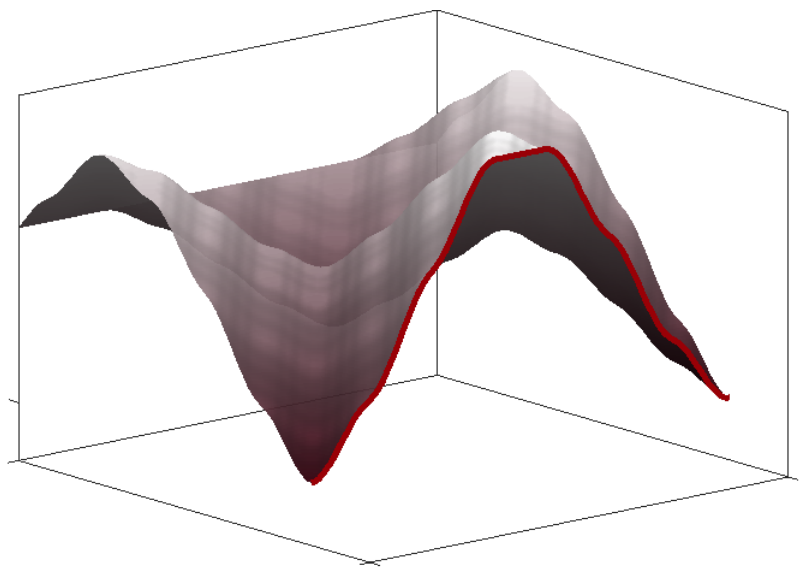
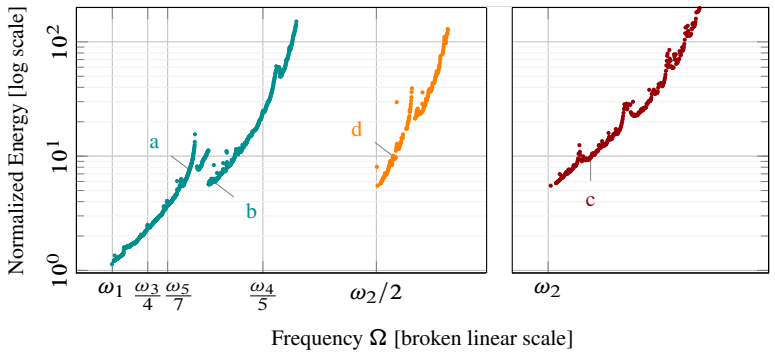
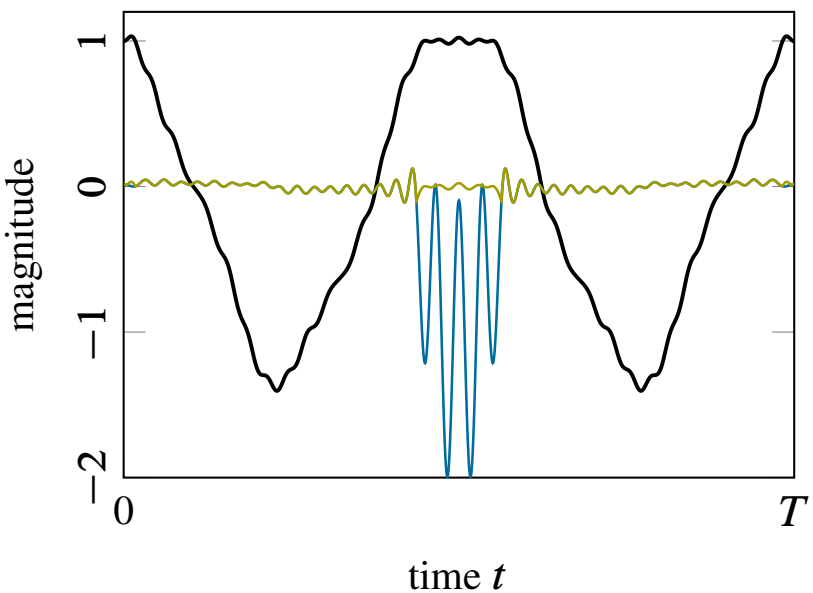


Figure 31: Displacement and contact force: point ‘c’ in backbone curve

Spectrum of vibration and approximate solutions





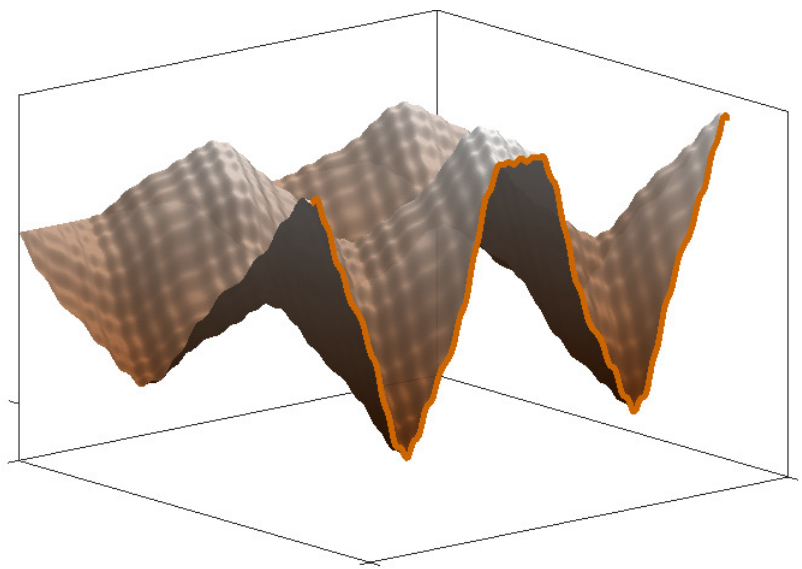


Figure 31: Displacement and contact force: point ‘d’ in backbone curve

Application to a two-bar system

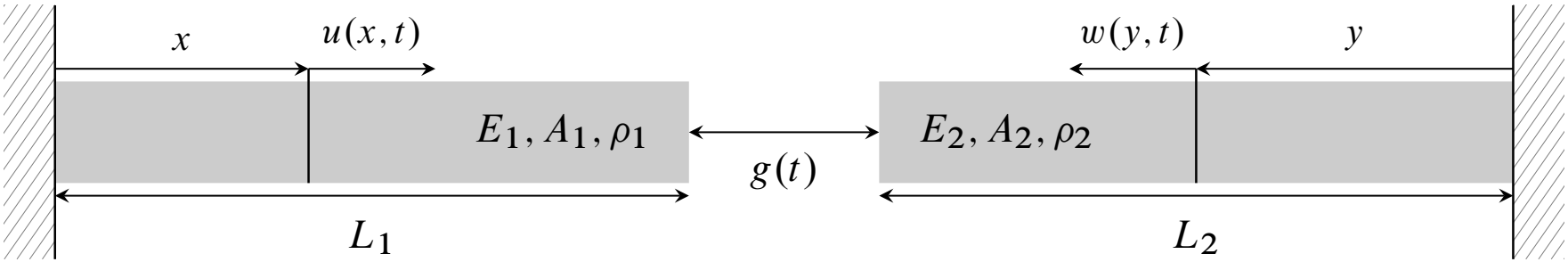
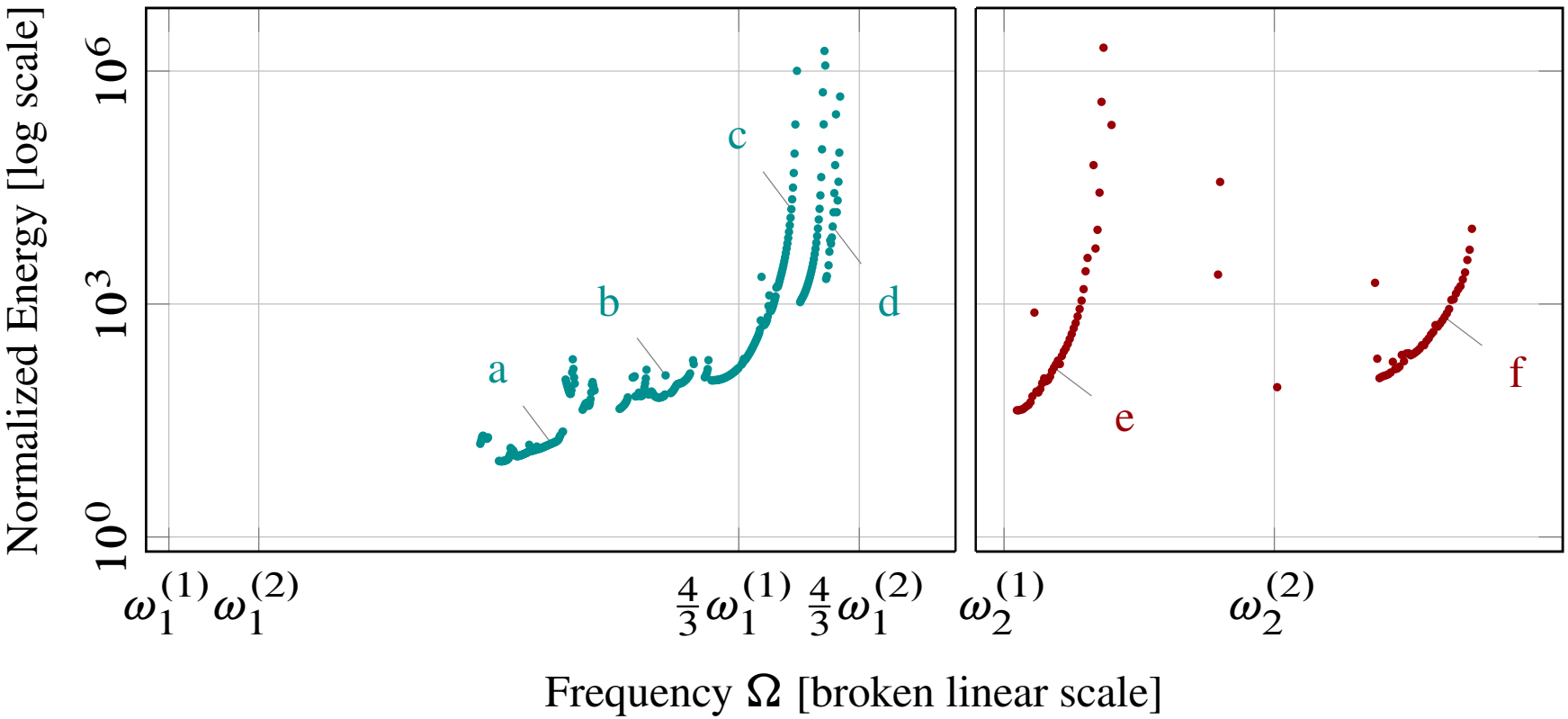


Figure 32: System of interest



Application to a two-bar system

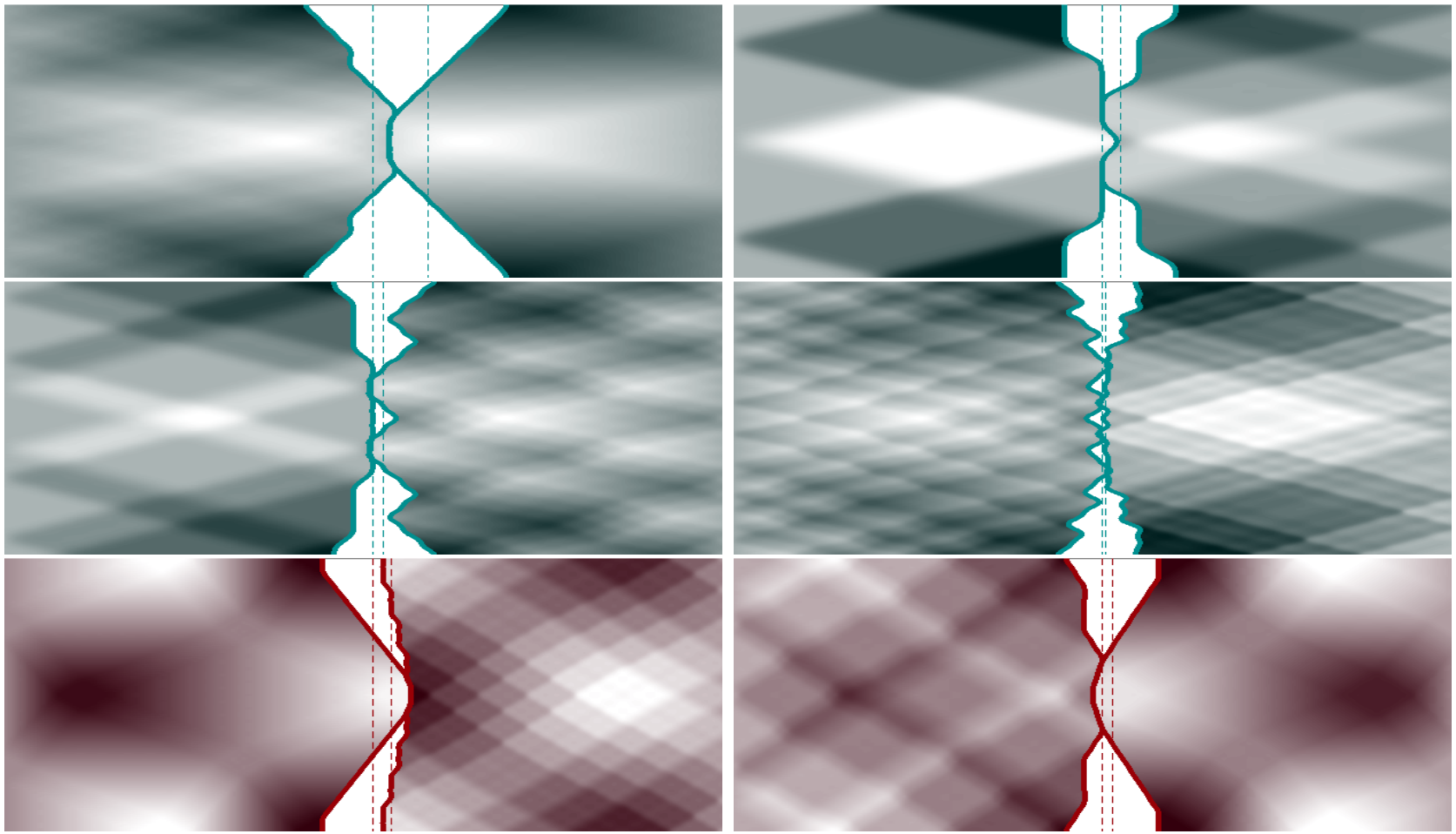
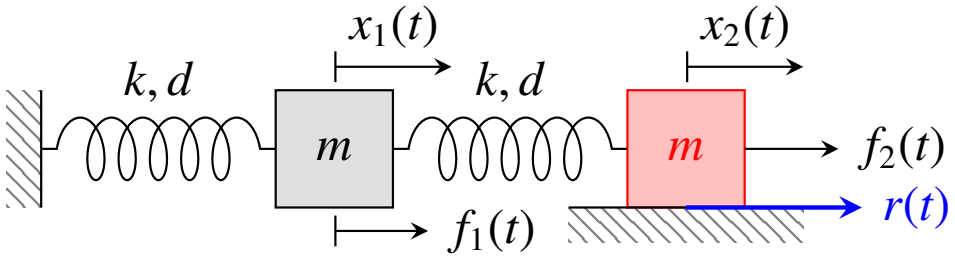


Figure 32: First and second NSMs, and internal resonances

Friction

Illustrative example: non-smooth dynamics

- System



- Dynamics: unknowns $(x_1(t), x_2(t), r(t))$

$$m\ddot{x}_1 + d(2\dot{x}_1 - \dot{x}_2) + k(2x_1 - x_2) - f_1 = 0 \qquad \text{[mass 1]} \qquad (36)$$

$$m\ddot{x}_2 + d(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) - f_2 - r = 0 \qquad \text{[mass 2]} \qquad (37)$$

$$\Psi_r(\dot{x}_2, r) = 0 \qquad \text{[Coulomb/Tresca law]} \qquad (38)$$

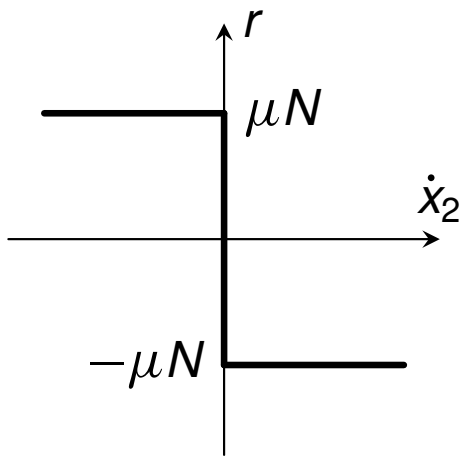
- Questions
  - expression of $\Psi_r(\dot{x}_2, r)$?
  - invertibility of $\Psi_r(\dot{x}_2, r)$?

Coulomb/Tresca Law

- Formulation[11]

$$\begin{cases} \dot{x}_2 = 0 & \Rightarrow \quad |r| \leq \mu N & \text{[sticking or sliding transition]} \\ \dot{x}_2 \neq 0 & \Rightarrow \quad r = -\mu N \dot{x}_2 / |\dot{x}_2| & \text{[sliding]} \end{cases} \tag{39}$$

- Admissible set



- Problem: multivalued mapping r as a function of $\dot{x}_2$ (or vice-versa)
  - ▷ no single-valued mapping $r(\dot{x}_2)$ (or $\dot{x}_2(r)$)
  - ▷ resolution: regularization of the friction law and/or removal of the sliding mass
  - ▷ zero-level set of the implicit function $\Psi_r(\dot{x}_2, r)$[12][13][14] $\rightarrow$ set $(\dot{x}_2, r)$ such that $\Psi_r(\dot{x}_2, r) = 0$

[11] Acary, Brémond & Huber. *Transactions of the European Network for Nonsmooth Dynamics* (Springer, 2018).
[12] Alart & Curnier. A mixed formulation for frictional contact problems prone to Newton like solution methods. *CMAME* (1991).
[13] Le van & Nguyen. A weighted residual relationship for the contact problem with Coulomb friction. *Computers & Structures* (2009).
[14] Ben Dhia & Zammali. Level-Sets and Arlequin framework for dynamic contact problems. *European Journal of Computational Mechanics* (2004).

Equality-based formalism: $\Psi_r(\dot{x}_2, r) = 0$ (1/2)

- Possible expression[15]

$$\Psi_r(\dot{x}_2, r) = \dot{x}_2 + \min(0, \rho(r + \mu N) - \dot{x}_2) + \max(0, \rho(r - \mu N) - \dot{x}_2); \quad \rho > 0 \quad (40)$$

- Zero-level set function = Coulomb set

$$\Psi_r(\dot{x}_2, r) = 0 \Leftrightarrow \left\{ \begin{array}{lll} r = \mu N & \text{if} & \dot{x}_2 < 0 \\ \dot{x}_2 = 0 & \text{if} & -\mu N \leq r \leq \mu N \\ r = -\mu N & \text{if} & \dot{x}_2 > 0 \end{array} \right\} \Leftrightarrow (39) \quad (41)$$

- Projector[16][17][18]

$$\text{proj}_{D(\mu N)}(r - \rho\dot{x}_2) = \left\{ \begin{array}{lll} +\mu N & \text{if} & r - \rho\dot{x}_2 > \mu N \\ r - \rho\dot{x}_2 & \text{if} & -\mu N \leq r - \rho\dot{x}_2 \leq \mu N \\ -\mu N & \text{if} & r - \rho\dot{x}_2 < -\mu N \end{array} \right. \quad (42)$$

and

$$\Psi_r(\dot{x}_2, r) = r - \text{proj}_{D(\mu N)}(r - \rho\dot{x}_2) = 0, \quad \rho > 0 \quad \Leftrightarrow \quad (41) \quad (43)$$

[15] Leung, Guoqing & Wanji. Smoothing Newton method for solving two- and three-dimensional frictional contact problems. *IJNME* (1998).

[16] Moreau. *Nonsmooth Mechanics and Applications* (Springer, 1988).

[17] Leine & Nijmeijer. *Dynamics and Bifurcations of Non-Smooth Mechanical Systems.* (Springer, 2006).

[18] Charroyer, Chiello & Sinou. Self-excited vibrations of a non-smooth contact dynamical system with planar friction based on the shooting method. *International Journal of Mechanical Sciences* (2018).

Equality-based formalism: $\Psi_r(\dot{x}_2, r) = 0$ (2/2)

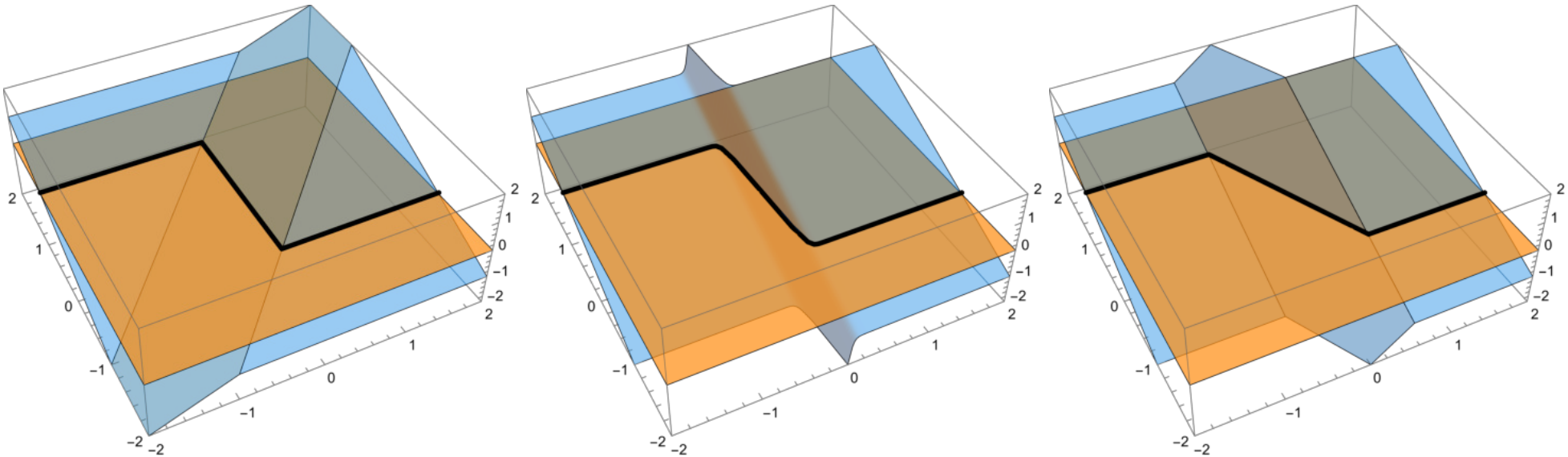


Figure 33: Function $\Psi_r(\dot{x}_2, r)$ (blue) and zero-level set (orange). Coulomb set $\equiv$ intersection

From left to right:

- $\Psi_r(\dot{x}_2, r) = \dot{x}_2 + \min(0, r + \mu N - \dot{x}_2) + \max(0, r - \mu N - \dot{x}_2)$
- $\Psi_r(\dot{x}_2, r) = r + \mu N \tanh(\beta \dot{x}_2)$
- $\Psi_r(\dot{x}_2, r) = r + \mu N(\beta \dot{x}_2 + \min(1 - \beta \dot{x}_2, 0) + \max(-1 - \beta \dot{x}_2, 0))$

Galerkin Formulation (1/2)

- Dynamics

$$\Psi_1(x_1, x_2) = m\ddot{x}_1 + d(2\dot{x}_1 - \dot{x}_2) + k(2x_1 - x_2) - f_1 \cos\omega t = 0 \quad (44)$$

$$\Psi_2(x_1, x_2, r) = m\ddot{x}_2 + d(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) - r = 0 \quad (45)$$

$$\Psi_r(\dot{x}_2, r) = \dot{x}_2 + \min(0, r + \mu N - \dot{x}_2) + \max(0, r - \mu N - \dot{x}_2) = 0 \quad (46)$$

$\rightarrow$ Non-smooth differential algebraic equation

- Approximate periodic solution

$$x_i(t) \approx x_{1h}(t) = \sum_{k=1}^{N_h} x_{ik}\phi_k(t), \quad i = 1, 2 \qquad r(t) \approx r_h(t) = \sum_{k=1}^{N_h} r_k\phi_k(t) \quad (47)$$

  - $\phi_k(t)$: family of shape functions
  - Fourier functions $\rightarrow$ HBM: $\phi_k(t) = (\cos k\omega t, \sin k\omega t)$ in the following
  - notation: $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{r}) \equiv (x_{1k}, x_{2k}, r_k)_{k=1,\ldots,N_h}$
  - $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{r})$: contributions of the chosen shape functions $\rightarrow$ to be calculated!
- Term-by-term time derivatives (for x_1 and x_2)

$$\dot{x}_i(t) \approx \dot{x}_{1h}(t) = \sum_k x_{ik}\dot{\phi}_k(t), \qquad \ddot{x}_i(t) \approx \ddot{x}_{ih}(t) = \sum_k x_{ik}\ddot{\phi}_k(t), \quad i = 1, 2 \quad (48)$$

Galerkin Formulation (2/2)

- Weighted residuals → numerical quadrature if necessary (FFT, trapezoidal rule)

$$\forall k,\ \int_0^T \phi_k(t)\ \Psi_1(x_{1h}(t), x_{2h}(t))\,\mathrm{d}t = 0 \qquad \text{[mass 1 dyn.: linear in } (\mathbf{x}_1, \mathbf{x}_2)\text{]} \tag{49}$$

$$\forall k,\ \int_0^T \phi_k(t)\ \Psi_2(x_{1h}(t), x_{2h}(t), r_h(t))\,\mathrm{d}t = 0 \qquad \text{[mass 2 dyn.: linear in } (\mathbf{x}_1, \mathbf{x}_2, \mathbf{r})\text{]} \tag{50}$$

$$\forall k,\ \int_0^T \phi_k(t)\ \Psi_\mathrm{r}(\dot{x}_{2h}(t), r_h(t))\,\mathrm{d}t = 0 \qquad \text{[friction: non-smooth in } (\mathbf{x}_2, \mathbf{r})\text{]} \tag{51}$$

- Systems of smooth and non-smooth equations: $\mathbf{g}(\mathbf{x}) = \mathbf{0}$

$$(49) \rightarrow \mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{A}_{12}\mathbf{x}_2 - \mathbf{f}_1 = \mathbf{0} \qquad \text{[smooth, exact]} \tag{52}$$

$$(50) \rightarrow \mathbf{g}_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{r}) = \mathbf{A}_{21}\mathbf{x}_1 + \mathbf{A}_{22}\mathbf{x}_2 - \mathbf{A}_{2r}\mathbf{r} - \mathbf{f}_2 = \mathbf{0} \qquad \text{[smooth, exact]} \tag{53}$$

$$(51) \rightarrow \mathbf{g}_\mathrm{r}(\mathbf{x}_2, \mathbf{r}) = \mathbf{0} \qquad \text{[non-smooth, num.]} \tag{54}$$

with

$$g_{rk}(\mathbf{x}_2, \mathbf{r}) = \int_0^T \phi_k(t)\ \Psi_\mathrm{r}(\dot{x}_{2h}(t), r_h(t))\,\mathrm{d}t \tag{55}$$

- Solvers
  - Non-smooth Newton or similar
  - Matlab®/Python® `fsolve/root` → Hybrid Powell: dogleg trust-region procedure

Illustration of $\mathbf{g}_{rk}(\mathbf{x}_2, \mathbf{r}) = \int_0^T \phi_k(t)\, \Psi_r(\dot{x}_{2h}(t), r_h(t))\, \mathrm{d}t$

- One-term approximation ($\phi = \cos t$) (for simplicity)

$$x_{2h}(t) = a\cos t, \quad \dot{x}_{2h}(t) = -a\sin t, \quad r_h(t) = b\cos t \tag{56}$$

- Non-smooth Coulomb/Tresca equation

$$g_r(a,b) = \int_0^T \phi(t)\, \Psi_r(\dot{x}_{2h}(t), r_h(t))\, \mathrm{d}t = \int_0^T \cos t\, \Psi_r(-a\sin t, b\cos t)\, \mathrm{d}t \tag{57}$$

$$= \int_0^T \cos t \Big[-a\sin t + \min[0, b\cos t + \mu N - a\sin t] \tag{58}$$

$$+ \max[0, b\cos t - \mu N - a\sin t]\Big]\, \mathrm{d}t = 0$$

- Riemann sum approximation (for simplicity)
  - discretization with N time steps over $[0, T]$ of length $h = T/N$
  - $t_i = (i-1)h$, $i = 0, \ldots, N-1$
  - approximate integral for a given (a, b)

$$g_r(a,b) \approx \frac{T}{N} \sum_{i=0}^{N-1} \cos t_i \Big[-a\sin t_i + \min[0, b\cos t_i + \mu N - a\sin t_i] \tag{59}$$

$$+ \max[0, b\cos t_i - \mu N - a\sin t_i]\Big]$$

Illustration of $\mathbf{g}_{rk}(\mathbf{x}_2, \mathbf{r}) = \int_0^T \phi_k(t)\, \Psi_r(\dot{x}_{2h}(t), r_h(t))\, \mathrm{d}t$

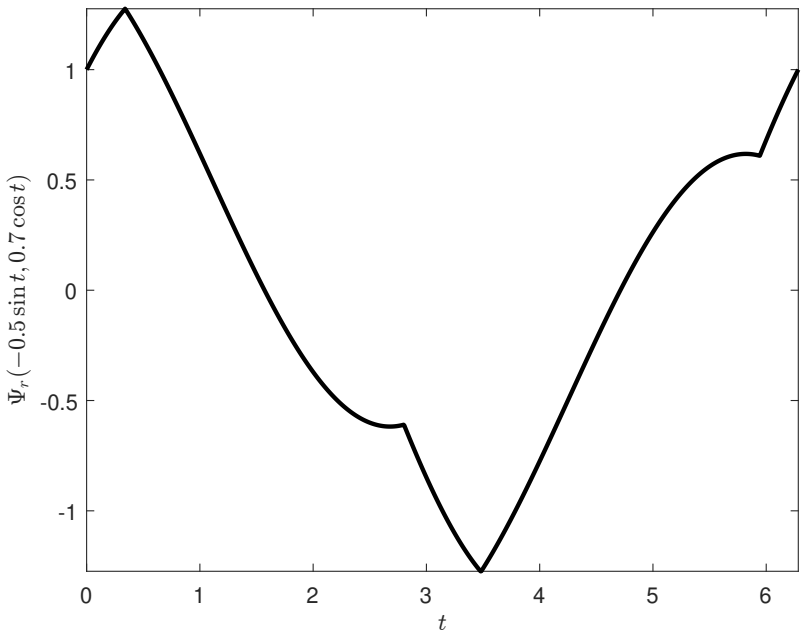


Figure 34: $\Psi_r(\dot{x}_{2h}(t), r_h(t))$, $t \in [0, 2\pi]$ with $a = 0.5$, $b = 0.7$ and $\mu N = 1$

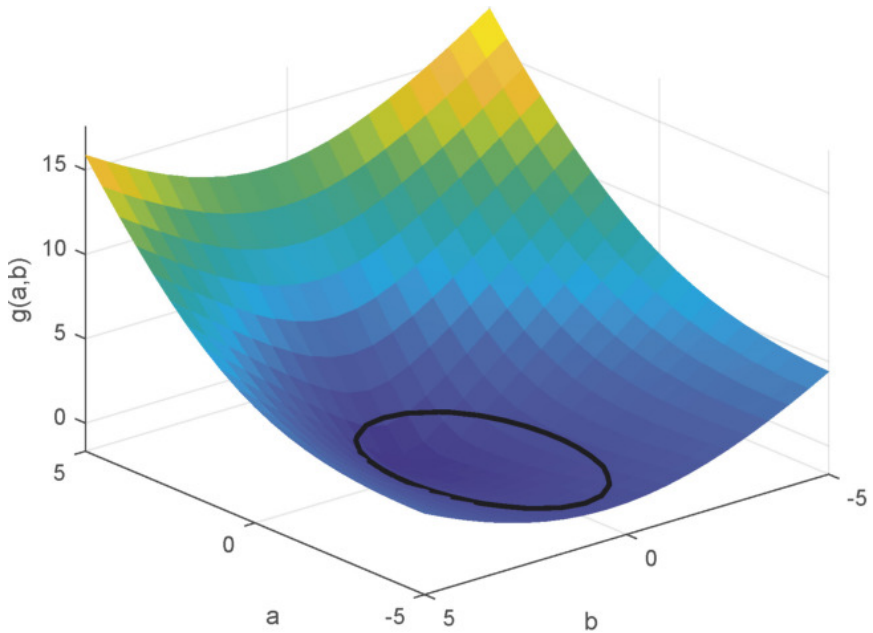
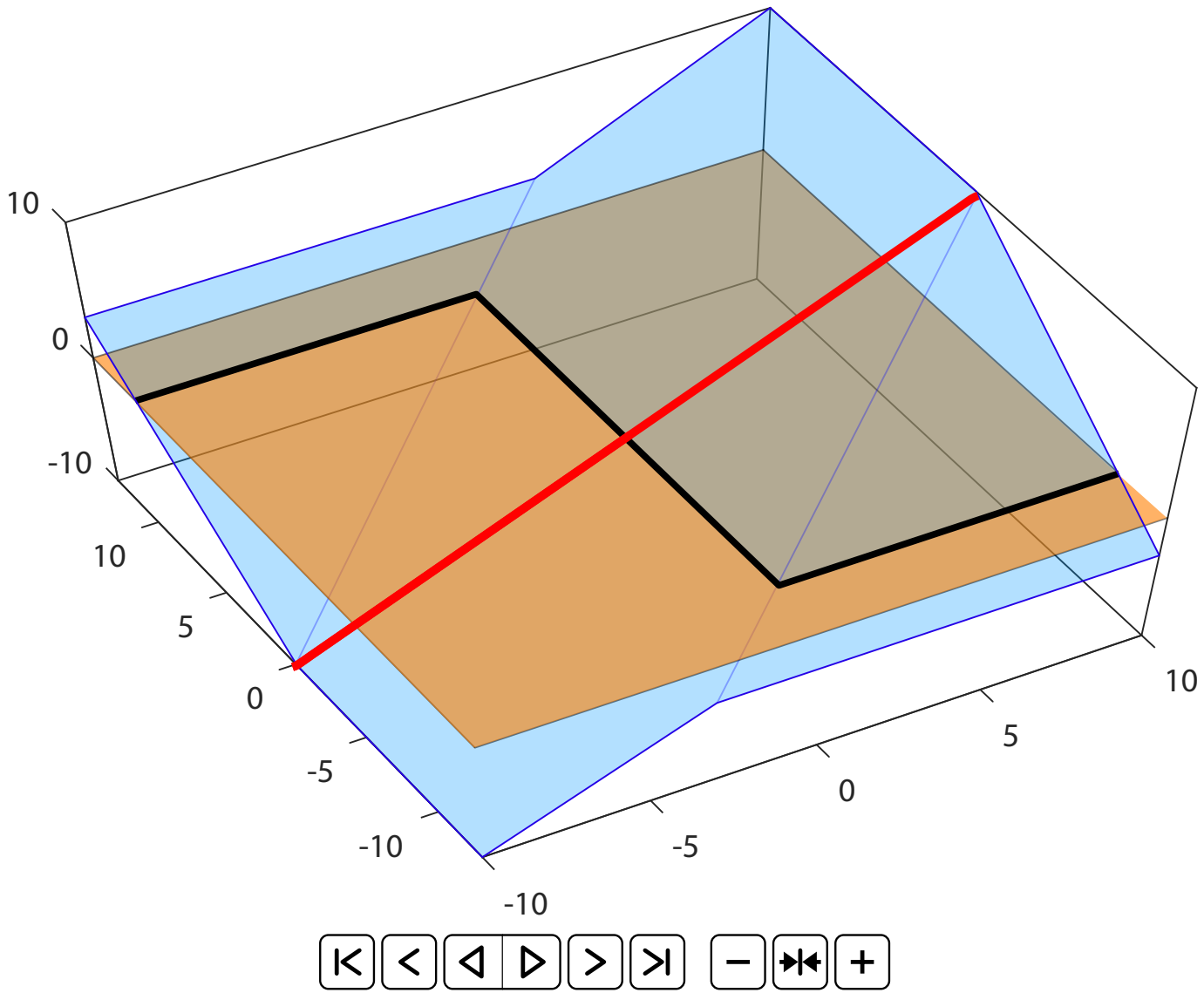
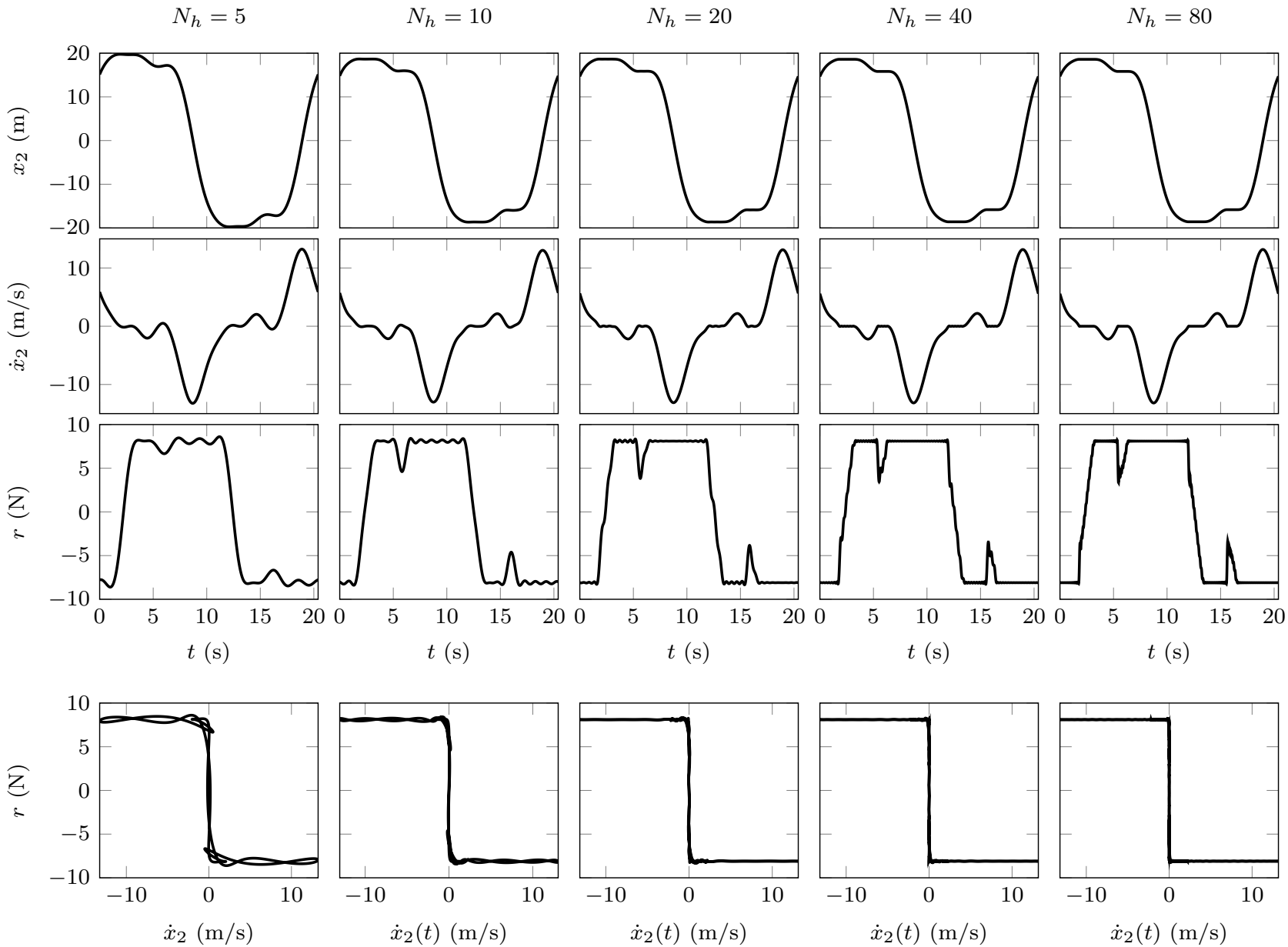


Figure 35: $g_r(a, b)$ over $(a, b) \in [-5, 5] \times [-5, 5]$ and $\mu N = 1$

Hybrid Powell with 10 odd harmonics



Convergence Analysis vs N_h (ϕ_k = Fourier)



Challiculties!

Nonlinear mode with unilateral contact

- Definition?
  - With $e = 1$: conservative but non-smooth. With linear viscous damping?
  - For $e < 1$: dissipation localized at impacts. What a damped nonlinear mode should be? Energy does not decay smoothly and the motion may accumulate impacts and come to rest in finite time.
  - General theory covering several contact points and non-simultaneous impacts still missing.
- Mass at the contact interface in FE framework
  - Impulses in the contact force
  - BEM? Nitsche? Modified-mass method?
- Computation
  - Time-stepping can dissipate or distort energy $\rightarrow$ effect on backbone computation not fully quantified
  - Harmonic balance suffers from Gibbs-type behavior at velocity jumps, and truncation error control is weak
  - Shooting and continuation need event handling hard to make robust
  - Validated numerics with error bounds on non-smooth modes largely absent

Square root singularity I

Context

- Oscillator vibrating near a rigid stop at position g
- At contact, velocity is reversed with a restitution coefficient e: $v^+ = -e\,v^-$
- Grazing impact: trajectory touches the stop with zero velocity $\rightarrow$ square root singularity[19][20]

Perturbed dynamics

- "Free" trajectory overshoots the stop by a small amount ε
- Virtual penetration does not occur: the impact happens first
- Near grazing: zero velocity, nonzero acceleration a
- The trajectory is locally parabolic:

$$x(t) \approx g + \tfrac{1}{2}at^2 - \varepsilon$$

- First return time (to obstacle):

$$\mathrm{FRT}(\varepsilon) \approx T_{\mathrm{lin}} - \sqrt{2\varepsilon}$$

- Dynamics affected by this term $\sqrt{2\varepsilon}$. Instability?

Square root singularity II

Why it matters

- Atypical bifurcations
  - Effective Floquet multipliers tend to infinity near grazing
  - Extreme sensitivity: a stable periodic orbit can abruptly turn into a chaotic attractor
- Abrupt transitions
  - Period-adding cascades and sudden jumps in amplitude
  - Not predicted by classical bifurcations (saddle-node, period-doubling)
- Numerical difficulties
  - Integration schemes and continuation methods lose accuracy near grazing
  - Map of the dynamics is not differentiable
- Stability of grazing orbits?
- No known derivations in the continuous case (1D bar)

[19] Nordmark. Existence of periodic orbits in grazing bifurcations of impacting mechanical oscillators. *Nonlinearity* (2001).

[20] Nordmark. Non-periodic motion caused by grazing incidence in an impact oscillator. *Journal of Sound and Vibration* (1991).

Nonlinear mode with friction

- Definition?
  - Coulomb friction → discontinuous vector field (Filippov type)
  - Smooth theory behind Shaw-Pierre manifolds and SSMs does not apply
- Origin: not an isolated equilibrium but an interval
  - Unclear what the right analogue of an invariant manifold is across a sliding surface.
  - Whole sets of sticking equilibria exist → mode "emanating from a fixed point" has no obvious meaning
- Computation
  - Harmonic balance: Truncation of harmonics has a particularly strong effect on stick phases
  - Rigorous bounds on how far a computed backbone can be trusted?
  - Continuation through sliding bifurcations and detection of pattern changes
- Stability analysis[21]
- Extension to continuous systems → Partial Differential Algebraic Equation[22]

$$\ddot{u}(x,t) - u_{xx}(x,t) = r(x,t), \quad \forall x \in]0,1[$$
$$\dot{u}(x,t) + \min(r(x,t) + \mu N - \dot{u}(x,t), 0) + \max(r(x,t) - \mu N - \dot{u}(x,t), 0) = 0$$
$$u(0,t) = 0, \quad u_x(1,t) = 0$$

[21] Schütz, Bayer & Leine. *Koopman-based stability analysis of differential-algebraic equations with applications to frictional multibody systems.* 2026. eprint: 2607.17339.

[22] Simeon & Arnold. Coupling DAEs and PDEs for Simulating the Interaction of Pantograph and Catenary. *Mathematical and Computer Modelling of Dynamical Systems* (2000).

NLM with unilateral and frictional contact coupling

Impacts, friction, clearances, and piecewise-linear stiffness break the smoothness assumptions behind most manifold theory:

- Mass at the interface?
- Restitution coefficient?
- Stochastic aspects

$\rightarrow$ Completely open!!!