CONTINUATION EXPÉRIMENTALE
de la théorie à la pratique



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1. MOTIVATIONS

2. TWO CONTROL STRATEGIES

2.1. PHASE LOCKED LOOP

2.2. CONTROL BASED CONTINUATION

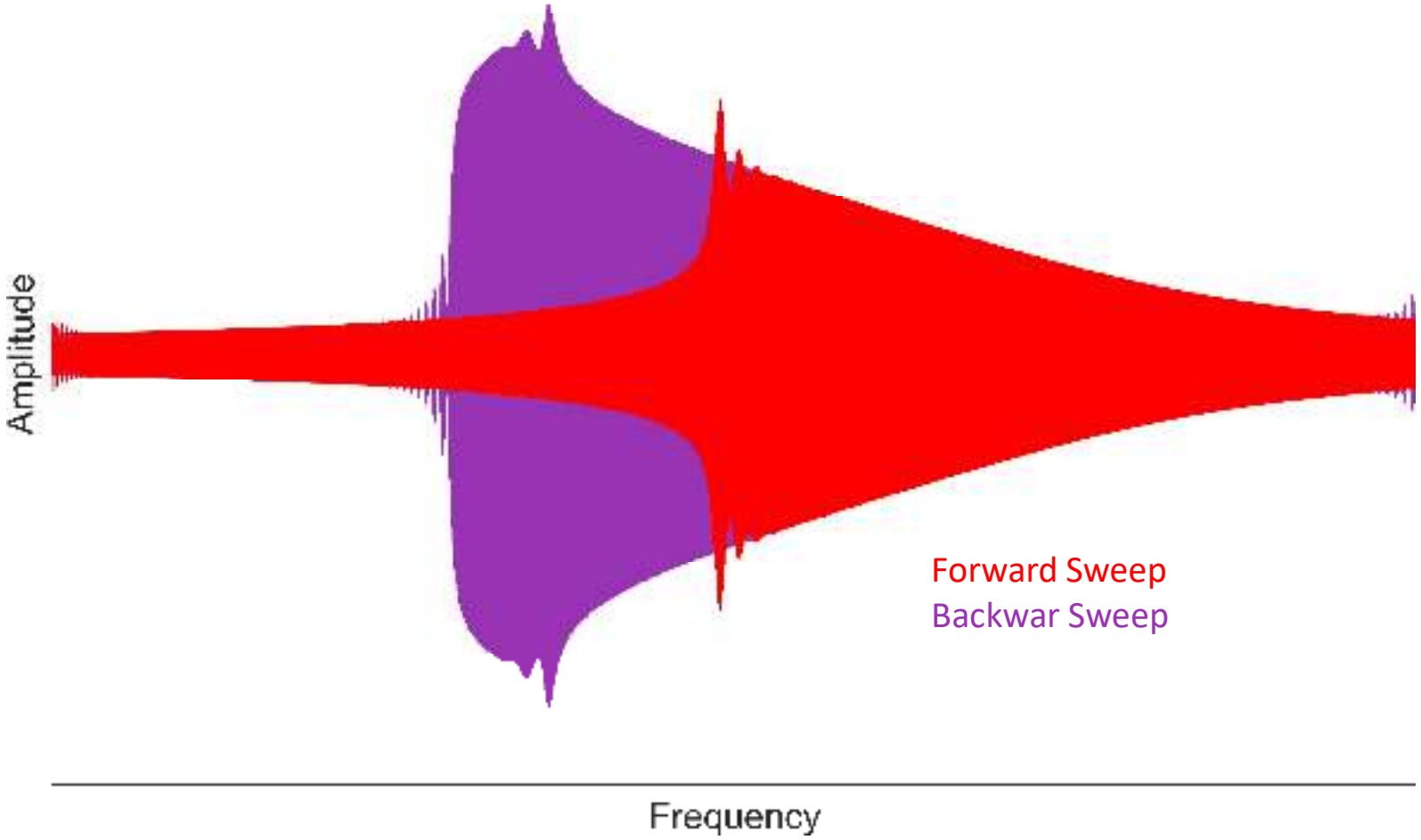
3. PRACTICAL CONSIDERATIONS

4. IMPLEMENTATION ON µC

5. CONCLUSION & PERSPECTIVES

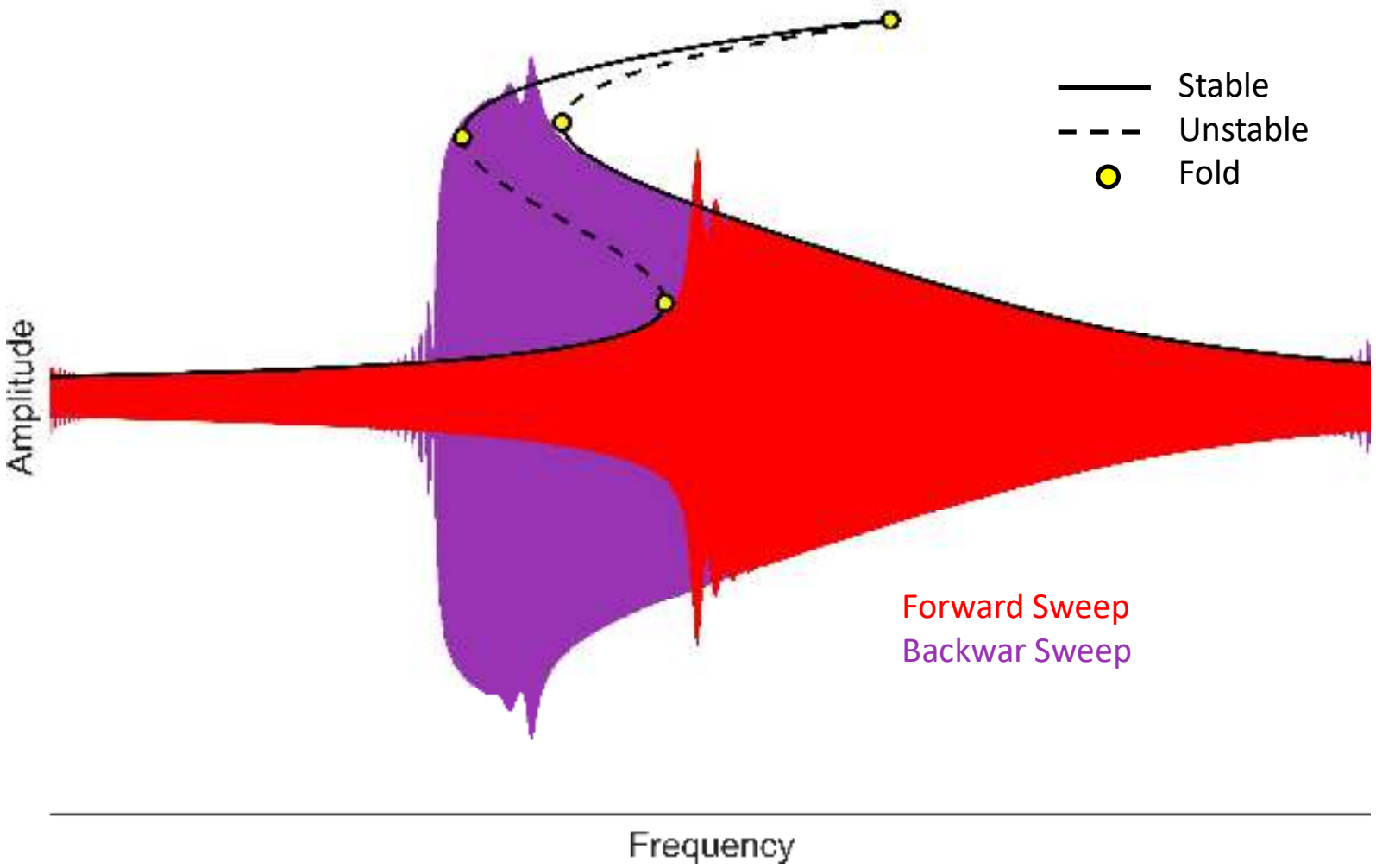
MOTIVATIONS

$$\ddot{x} + x + 0.02\dot{x} - 0.4x^3 + 0.2x^5 = 0.03\cos(\theta)$$



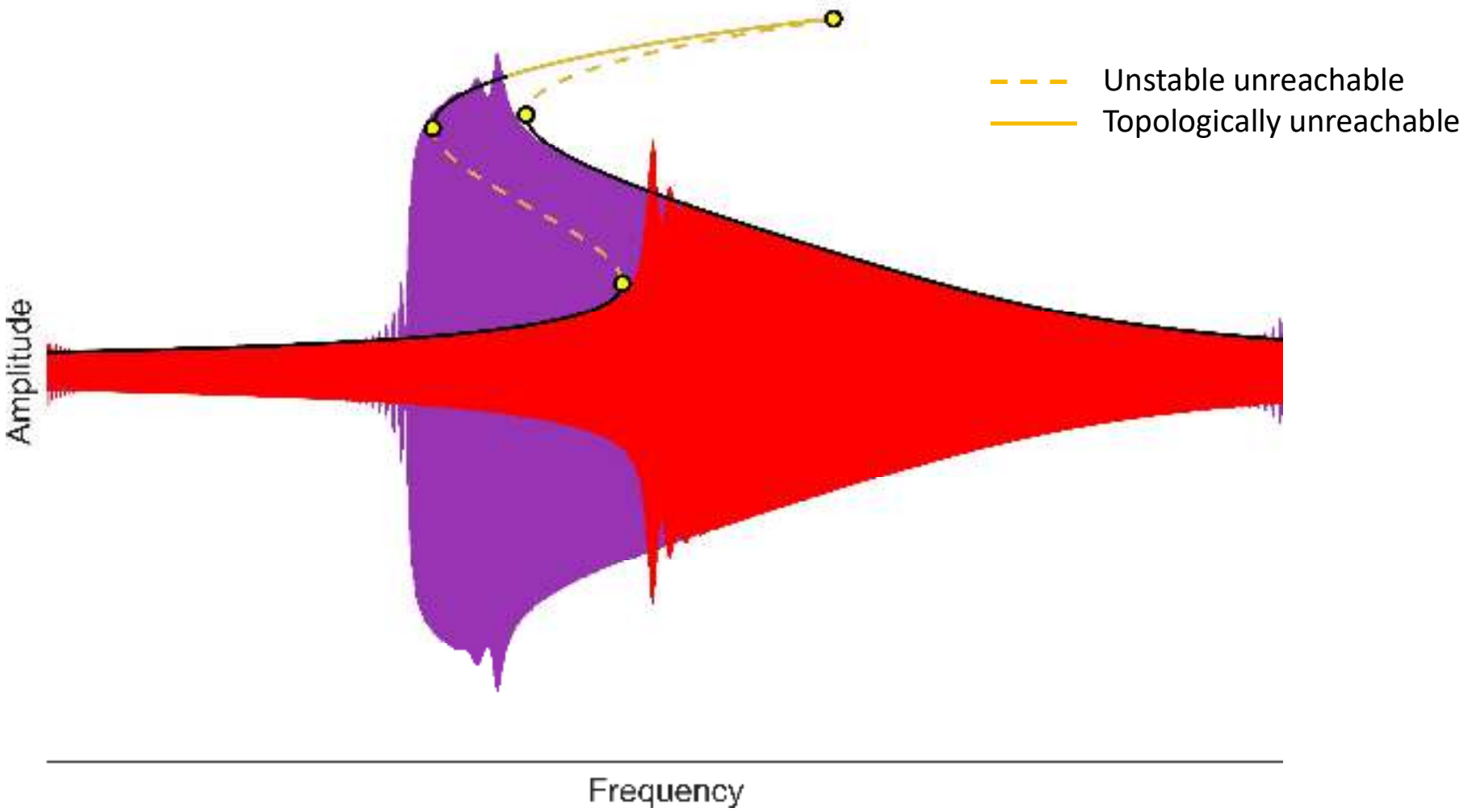
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NUMERICAL CONTINUATION METHODS

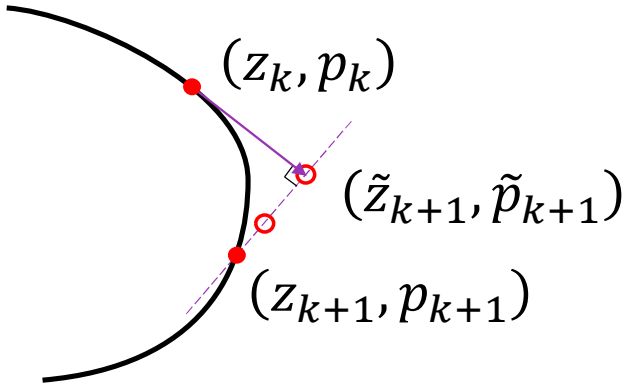
Dynamical system

$$\dot{x} = f(x, t, p)$$

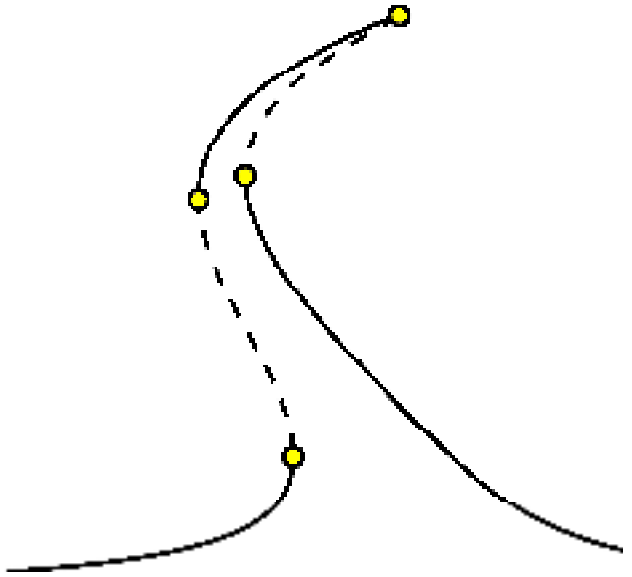
Discretization

- BVP (shooting)
- HBM
- …

Path following

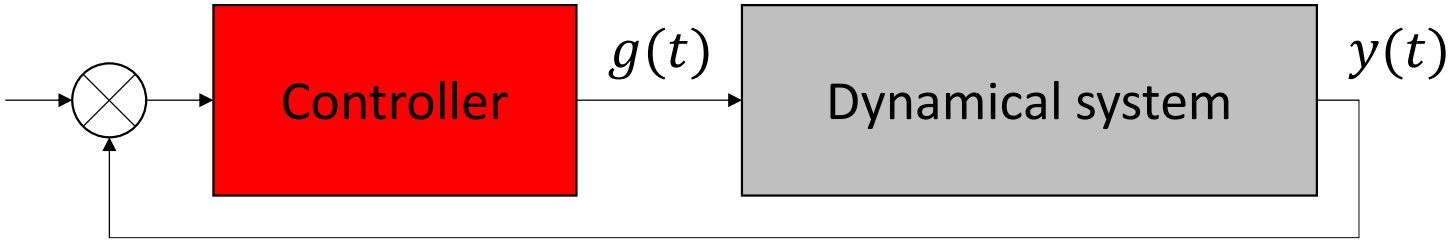


Bifurcation diagram



EXPERIMENTAL CONTINUATION METHODS

- **Feedback controlled** experiment [1]



- A key point: **non-invasivity**

A solution of the feedback controlled system IS ALSO a solution of the open loop system

[1] Sieber, J., & Krauskopf, B. (2008). Control based bifurcation analysis for experiments. *Nonlinear Dynamics*

CONTROL PROBLEM

- Consider a dynamical system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t)\big) + \boldsymbol{b}g(t)$$

$$y(t) = \boldsymbol{c}\boldsymbol{x}(t)$$

$\boldsymbol{x}(t) \in \mathbb{R}^n$: state vector

$y(t) \in \mathbb{R}$: measured output

$g(t) = g(t+T) \in \mathbb{R}$: periodic forcing

- The system admits a periodic solution

$$\tilde{\boldsymbol{x}}(t) = \tilde{\boldsymbol{x}}(t+T), \qquad \tilde{y}(t) = \tilde{y}(t+T)$$

If the periodic solution is UNSTABLE how to make it STABLE?

STABILITÉ DES SOLUTIONS PÉRIODIQUES

- Perturbation of the periodic orbit

$$\boldsymbol{x}(t) = \tilde{\boldsymbol{x}}(t) + \boldsymbol{q}(t)$$

- Linearized system

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}(t)\boldsymbol{q}(t)$$

with $\boldsymbol{A}(t) = \boldsymbol{A}(t+T) \equiv \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\tilde{\boldsymbol{x}}}$

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- Following **Floquet theory**, the system admits a fundamental solution matrix

$$\dot{\boldsymbol{\Phi}}(t) = \boldsymbol{A}(t)\boldsymbol{\Phi}(\mathrm{t}), \qquad \boldsymbol{\Phi}(0) = \mathbb{I}^n$$

- … extracting the **monodromy matrix**

$$\boldsymbol{\mathcal{M}} \equiv \boldsymbol{\Phi}(T)$$

Eigenvalues of the monodromy matrix = **Floquet multipliers** μ_j

- A periodic solution is **unstable** if : $\exists \left|\mu_j\right| > 1$

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A PLL, WHAT'S THAT ?

- Consider a forced dynamical system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t)\big) + \boldsymbol{b}G\cos(\omega t)$$

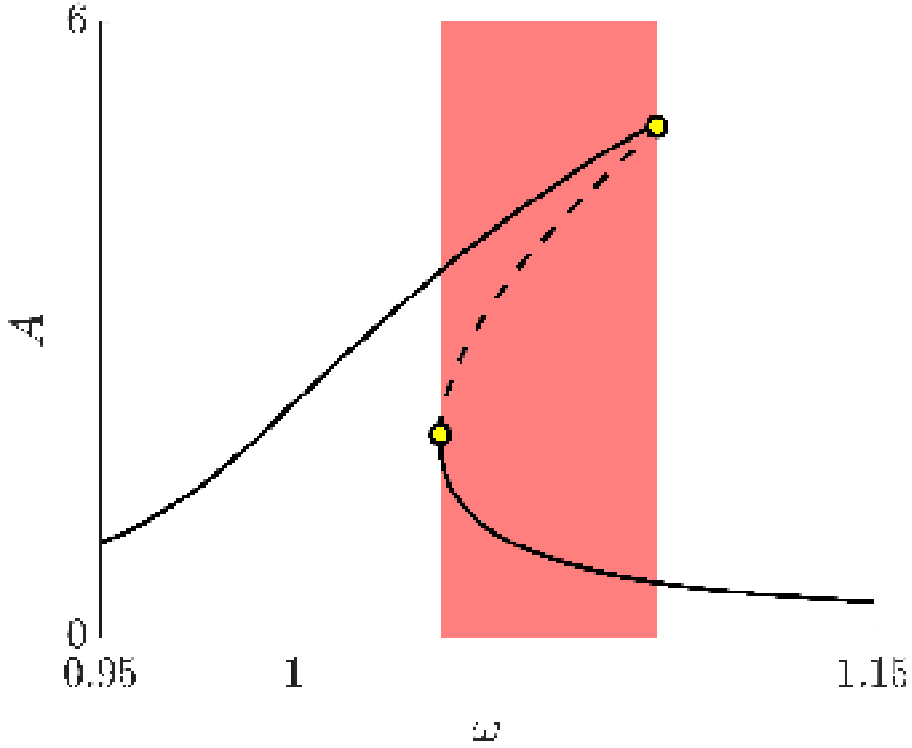
$$y(t) = \boldsymbol{c}\boldsymbol{x}(t)$$

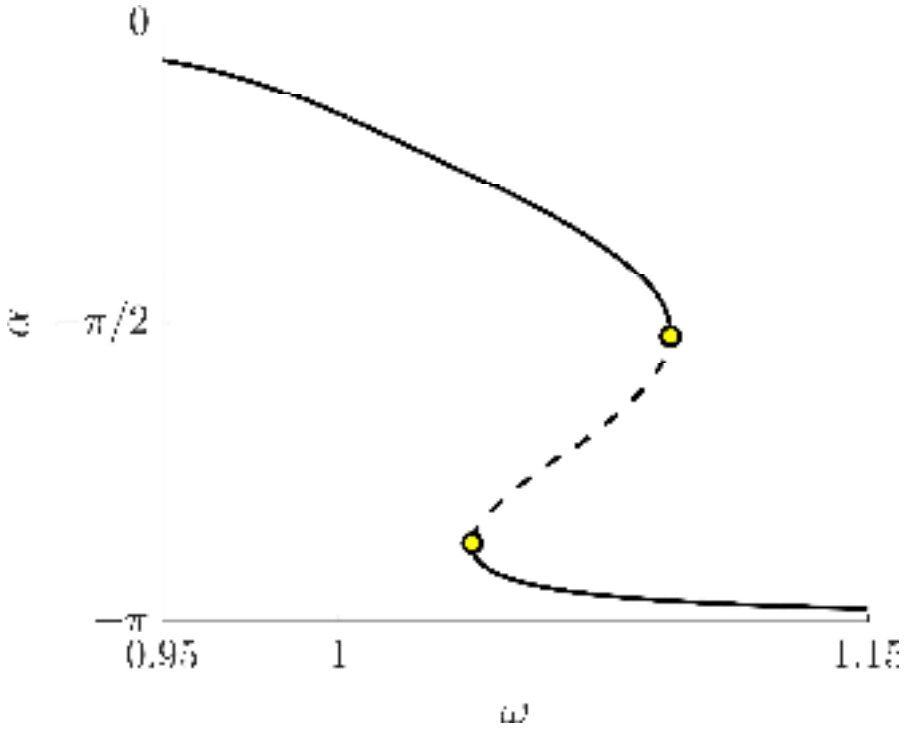
- This system admits a periodic solution $\tilde{y}(t)$ of the form

$$\tilde{y}(t) = A\cos(\omega t + \alpha) + H.O.H.$$

- The objective of a PLL is to **control the phase lag** α

WHY CONTROLLING THE PHASE LAG ?





$\Longrightarrow$ If we can impose the phase lag, we can *travel* allong the **whole branch**

TWO CONSTITUTIVE ELEMENTS

① A phase lag estimator

② A phase lag controller

SYNCHRONOUS DEMODULATION

- Consider that the output signal $y(t)$ can be expressed as

$$\mathrm{y(t)} = Y_0 + \sum_n Y_n \sin(n\omega t + \alpha_n)$$

- Multiplying by $\sin(\omega t)$, $\cos(\omega t)$

$$y\cos(\omega t) = \frac{Y_1}{2}\sin\alpha_1 + Y_0\cos(\omega t) + \cdots$$

$$y\sin(\omega t) = \frac{Y_1}{2}\cos\alpha_1 + Y_0\sin(\omega t) + \cdots$$

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- Low pass filtering

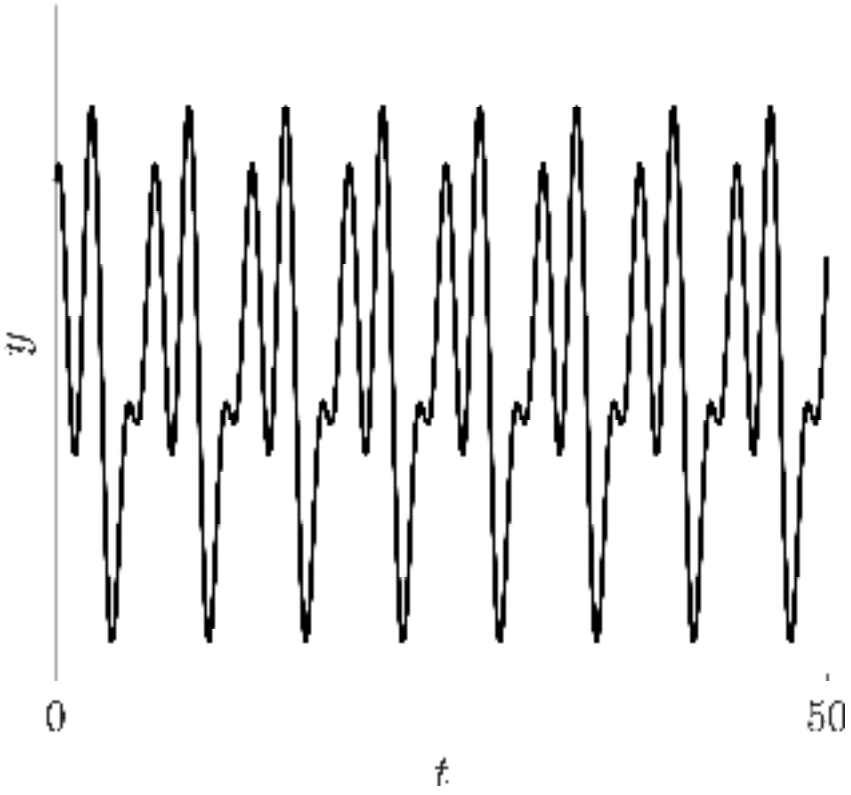
$$\dot{y}_1 = \omega_c(x\cos(\omega t) - y_1)$$

$$\dot{y}_2 = \omega_c(x\sin(\omega t) - y_1)$$

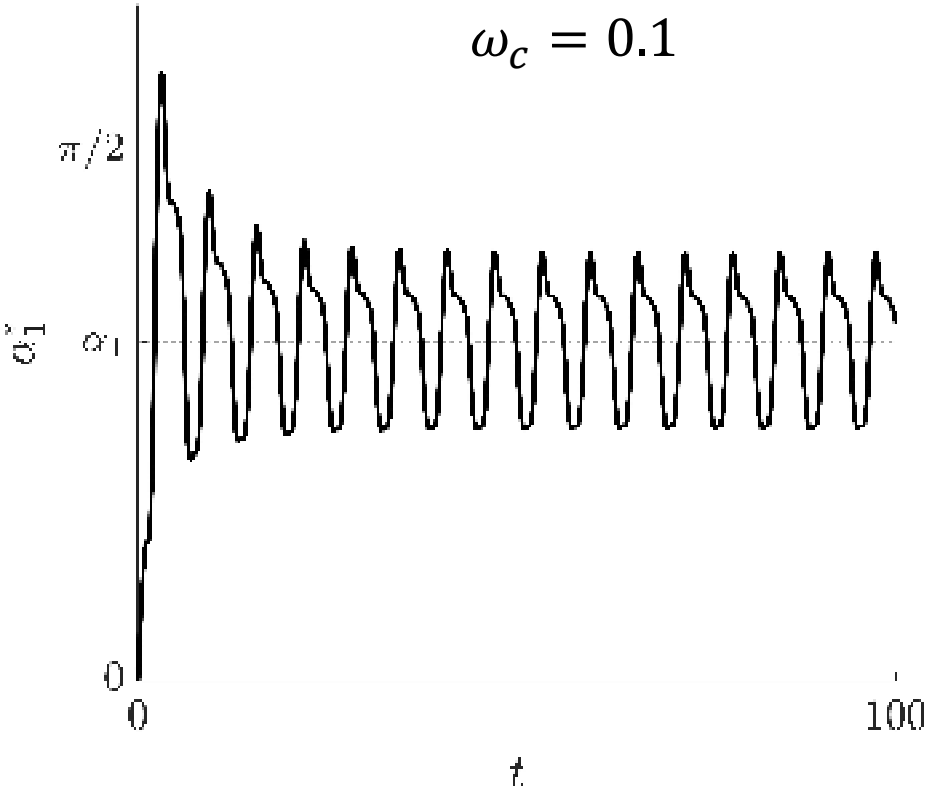
- The estimated phase lag α_1^* reads

$$\alpha_1^*\big(y_1(t), y_2(t)\big) = \tan^{-1}\left(\frac{y_1(t)}{y_2(t)}\right)$$

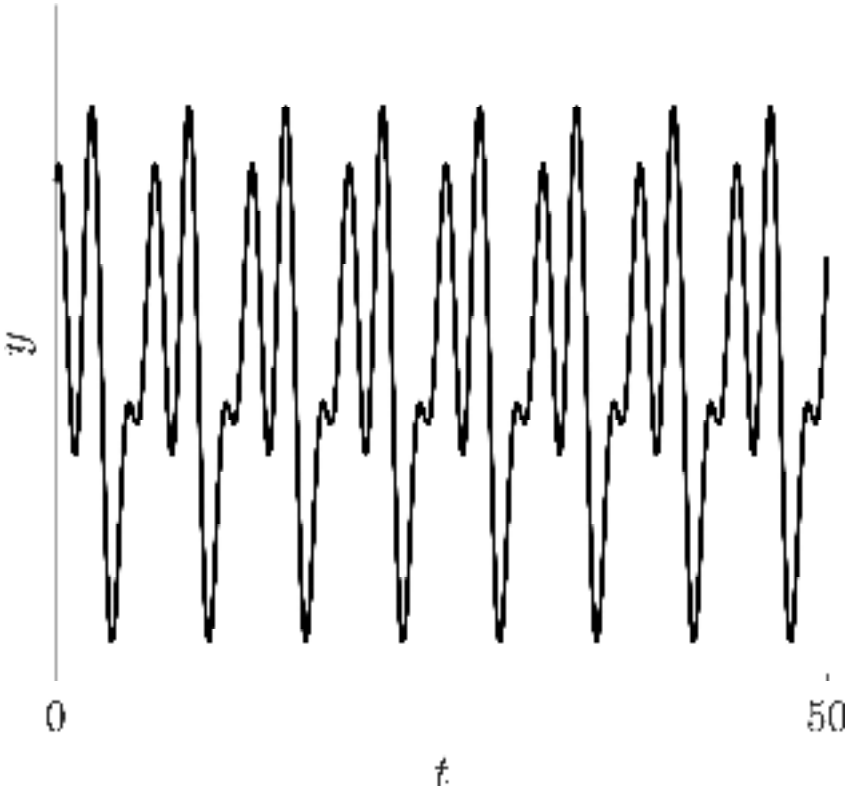
SYNCHRONOUS DEMODULATION



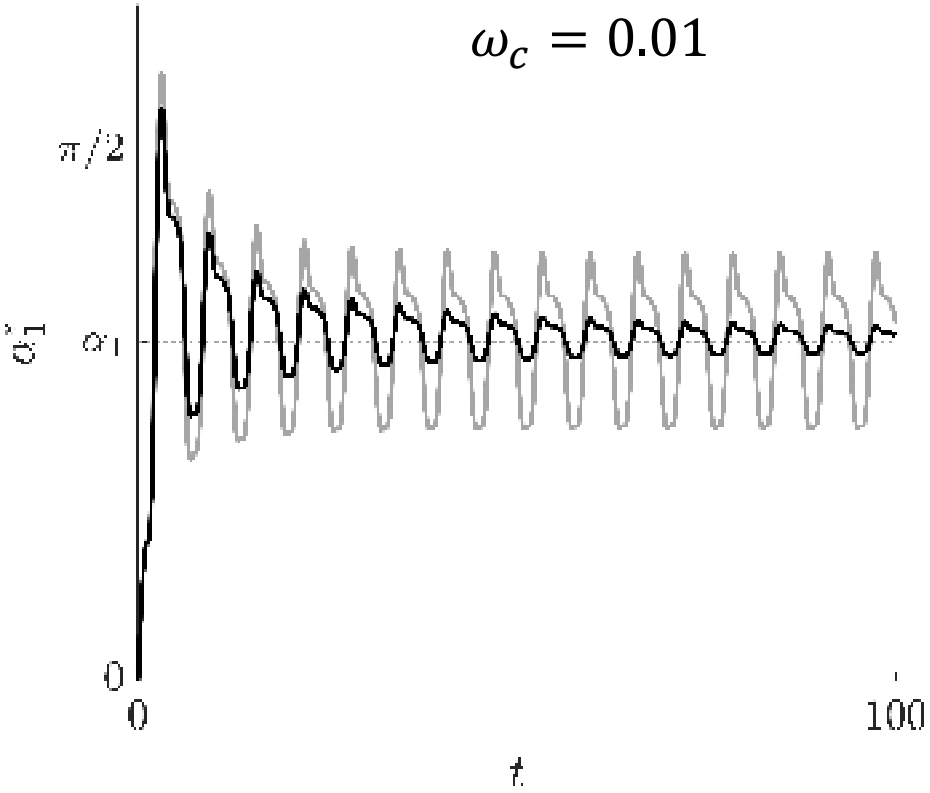
$\omega_c = 0.1$



SYNCHRONOUS DEMODULATION



$\omega_c = 0.01$



A NICE ALTERNATIVE : THE ADAPTIVE FILTER [1]

- Real time estimation of a Fourier representation of the output $y(t)$

$$y^* = Y_0 + \sum_{n=1}^{N_h} Y_{c,n}\cos(n\omega t) + Y_{s,n}\sin(n\omega t) = \boldsymbol{Q}^T\boldsymbol{Y}$$

- Governing equation

$$\dot{\boldsymbol{Y}} = \mu\boldsymbol{Q}(y - y^*)$$

- Estimated phase lag

$$\alpha_1^*\big(\boldsymbol{Y}(t)\big) = \tan^{-1}\left(\frac{Y_{s,1}(t)}{Y_{c,1}(t)}\right)$$

[1] Abeloos, G., Renson, L., Collette, C., & Kerschen, G. (2021). Stepped and swept control-based continuation using adaptive filtering. *Nonlinear Dynamics*

A NICE ALTERNATIVE : THE ADAPTIVE FILTER [2]

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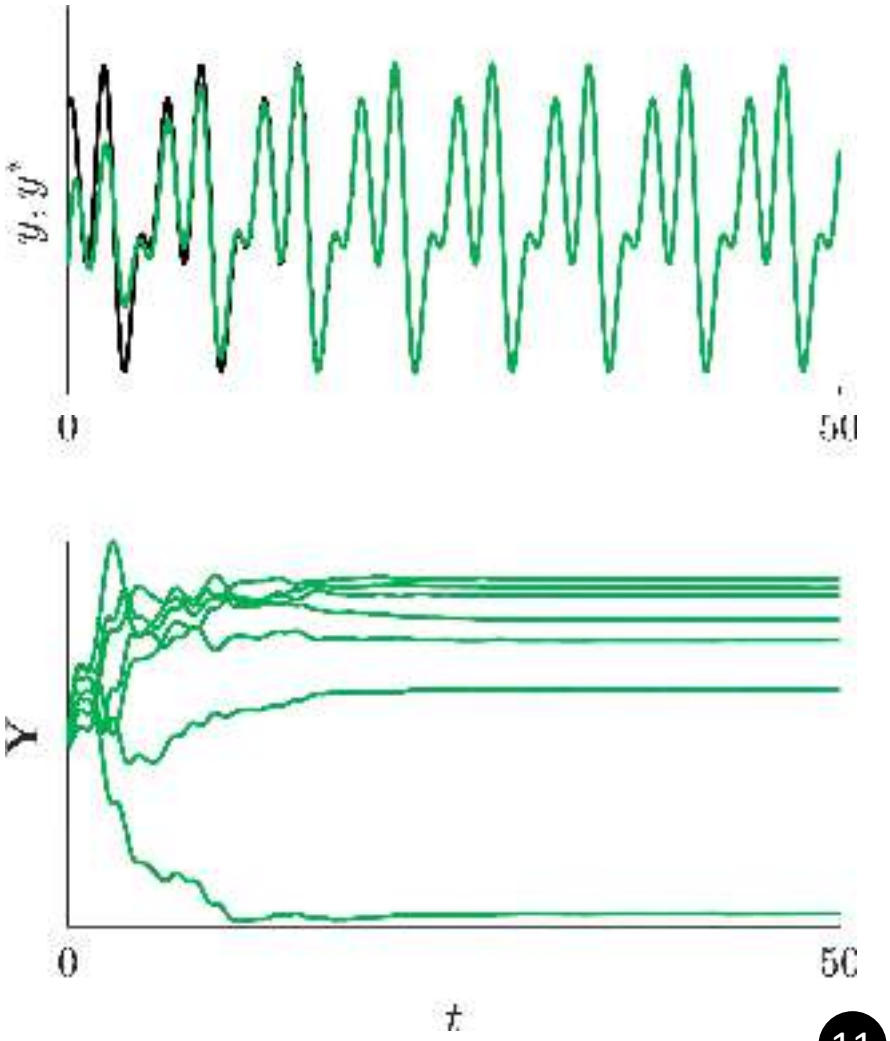
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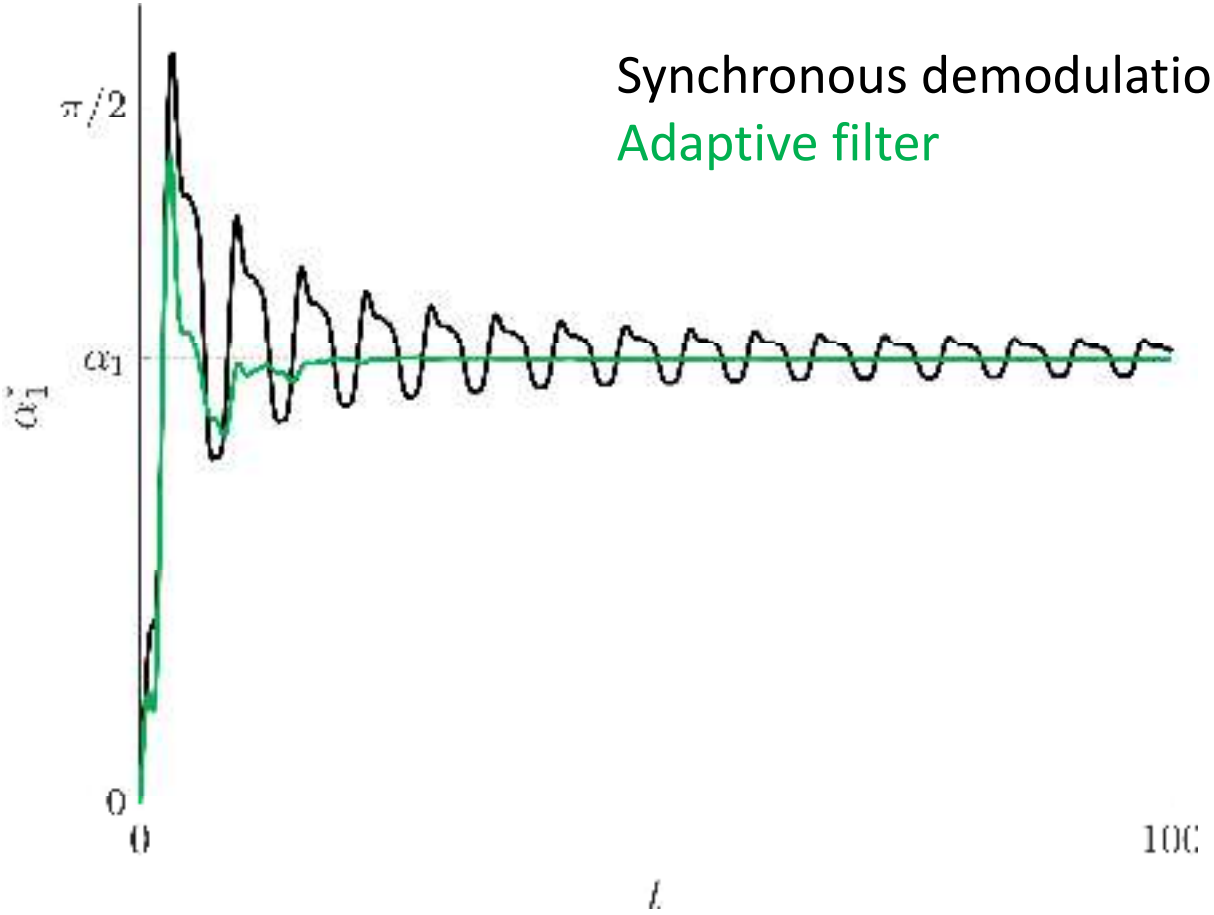
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FILTRE ADAPTATIF VS DÉMODULATION SYNCHRONE



THE PHASE CONTROLLER

- A PLL controls the instantaneous phase of the excitation $\theta(t)$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t)\big) + \boldsymbol{b}G\cos\big(\theta(t)\big)$$

$$\dot{\theta}(t) = \omega_0 + K\int\big(\alpha_1^*\big(\boldsymbol{Y}(t)\big) - \phi\big)dt$$

with

- ϕ: target phase lag,
- ω_0 : central frequency of the PLL,
- K : integral control gain

PLL NON-INVASIVITY

- Summary of the controlled system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t)\big) + \boldsymbol{b}G\cos\big(\theta(t)\big)$$

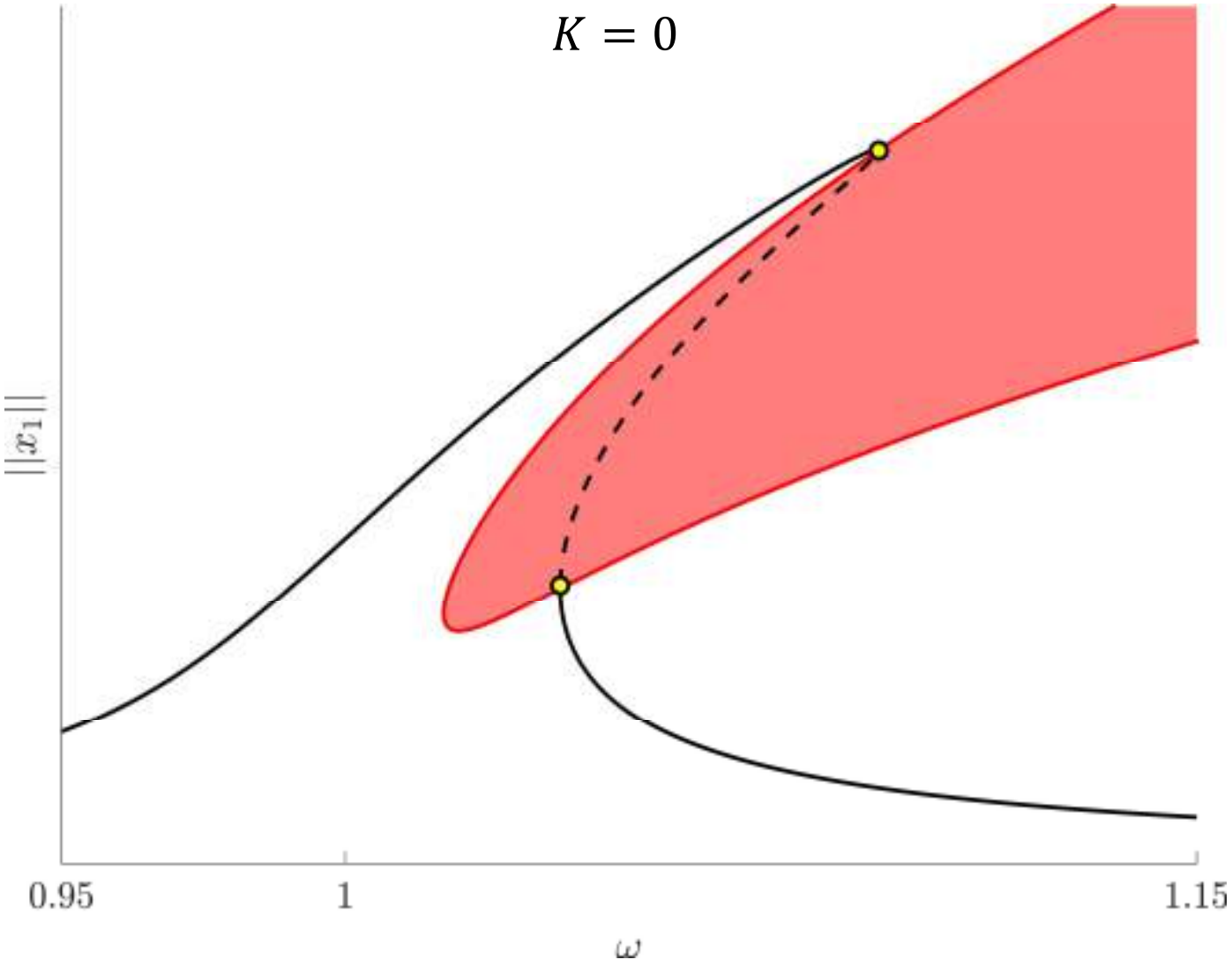
$$\dot{\boldsymbol{Y}}(t) = \mu\boldsymbol{Q}\big(\boldsymbol{c}\boldsymbol{x}(t) - \boldsymbol{Q}^T\boldsymbol{Y}(t)\big)$$

$$\dot{\theta}(t) = \omega_0 + K\int\big(\alpha_1^*\big(\boldsymbol{Y}(t)\big) - \phi\big)dt$$

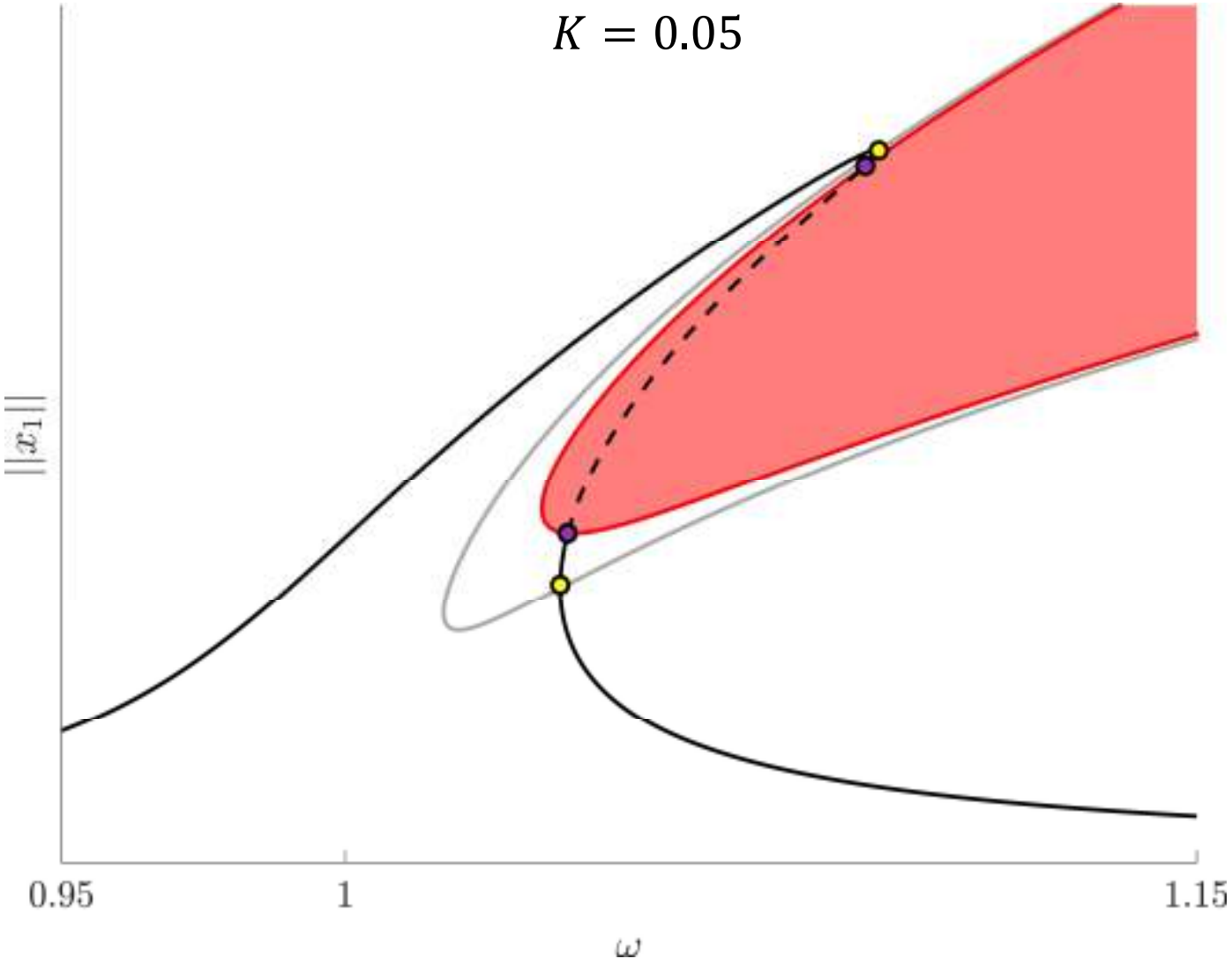
- The PLL is **non-invasive** iff $\dot{\theta}(t) = \omega$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t)\big) + \boldsymbol{b}G\cos(\omega t)$$

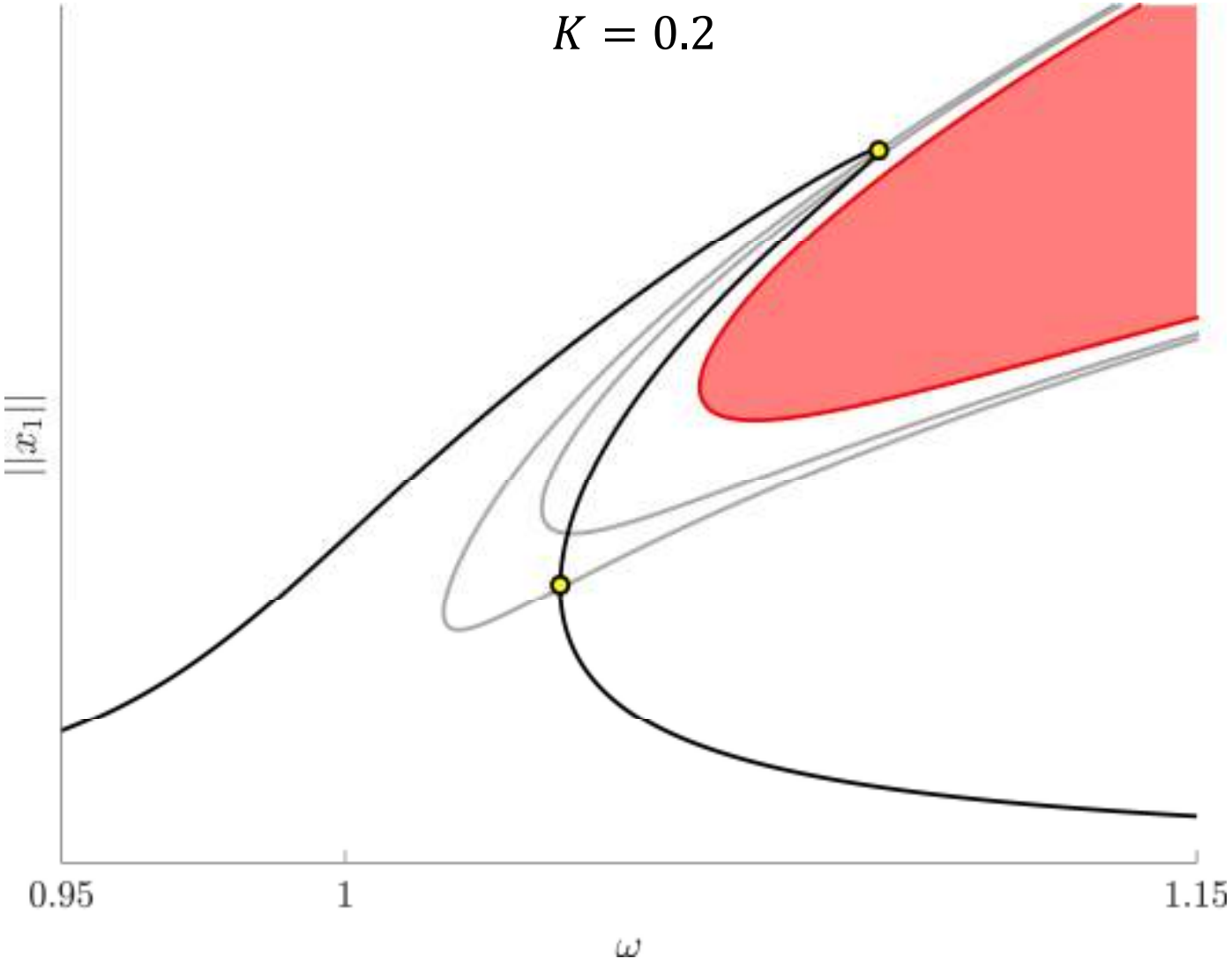
STABILIZATION OF PERIODIC SOLUTIONS



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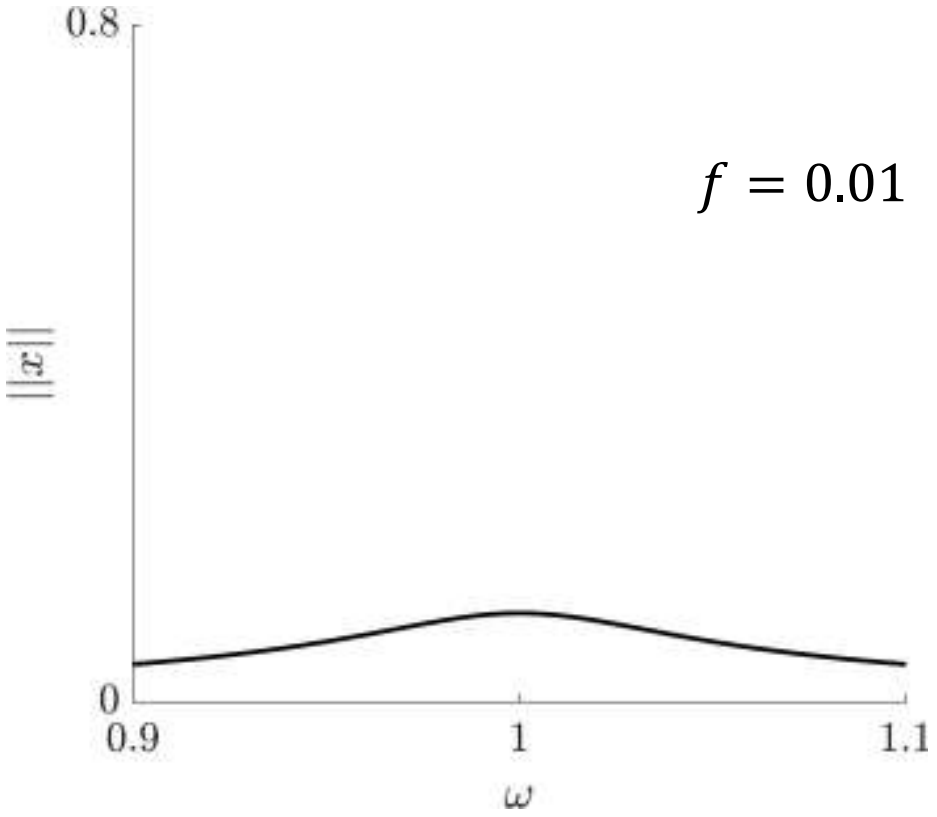


PLL MAY FAIL!

- Consider the system [3]

$$\ddot{x} + x + c_1\dot{x} + c_3\dot{x}^3 + \dot{x}^5 = G\cos(\omega t)$$

$$f = 0.01$$

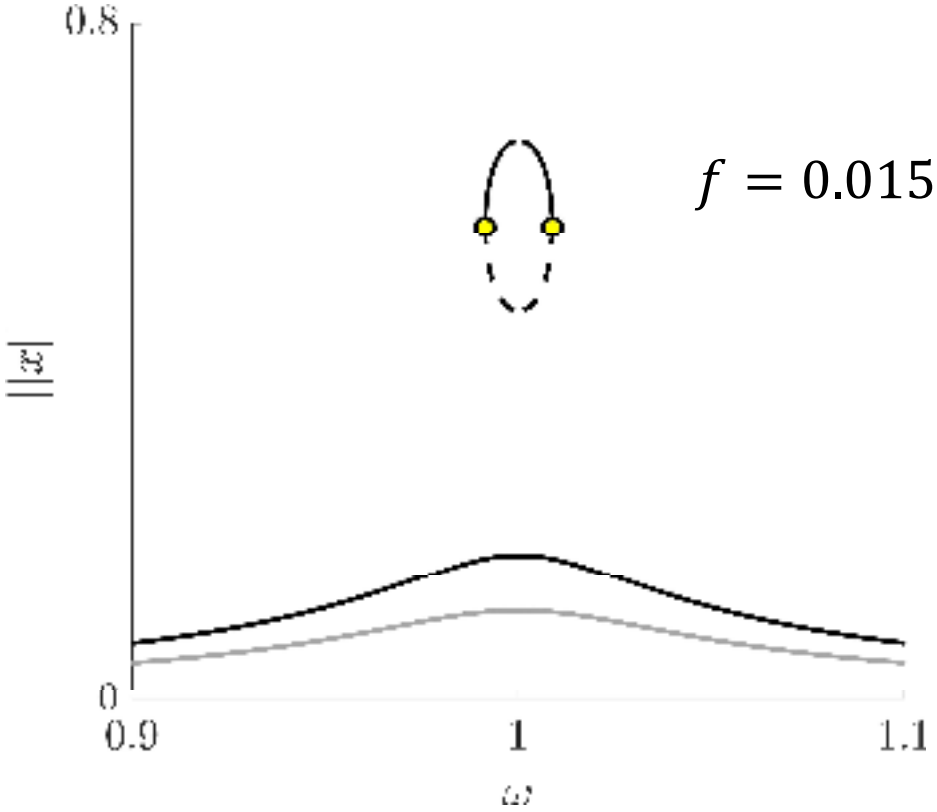


[3] Habib, G., Cirillo, G. I., & Kerschen, G. (2018). Isolated resonances and nonlinear damping. *Nonlinear Dynamics*

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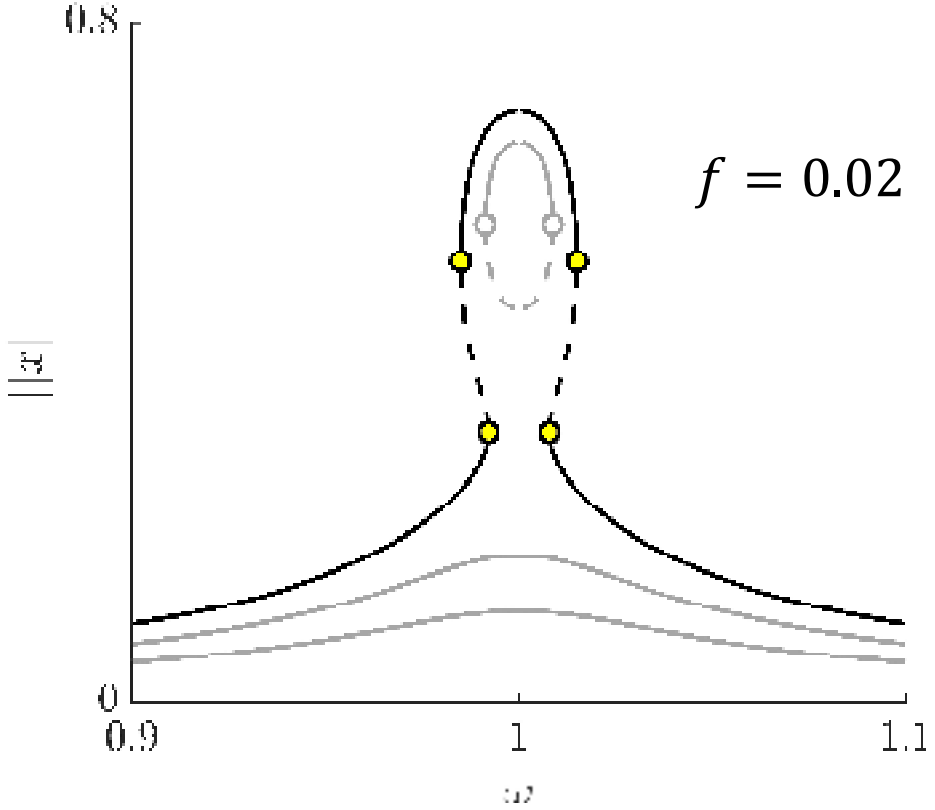
$$f = 0.015$$

[3] Habib, G., Cirillo, G. I., & Kerschen, G. (2018). Isolated resonances and nonlinear damping. *Nonlinear Dynamics*

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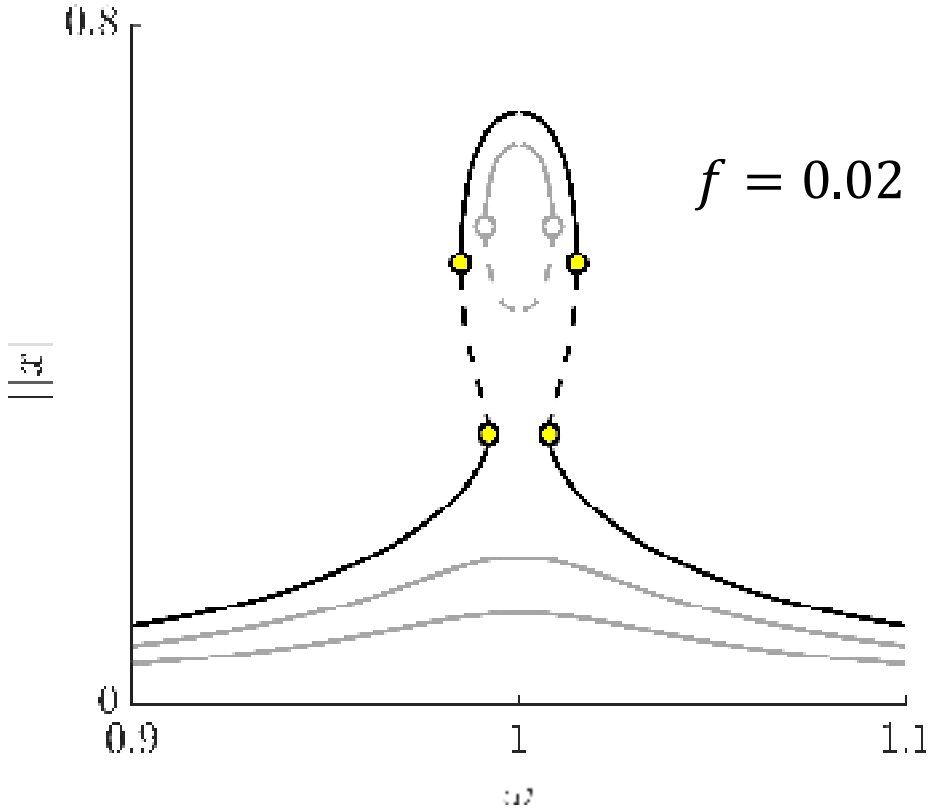
$f = 0.02$

[3] Habib, G., Cirillo, G. I., & Kerschen, G. (2018). Isolated resonances and nonlinear damping. *Nonlinear Dynamics*

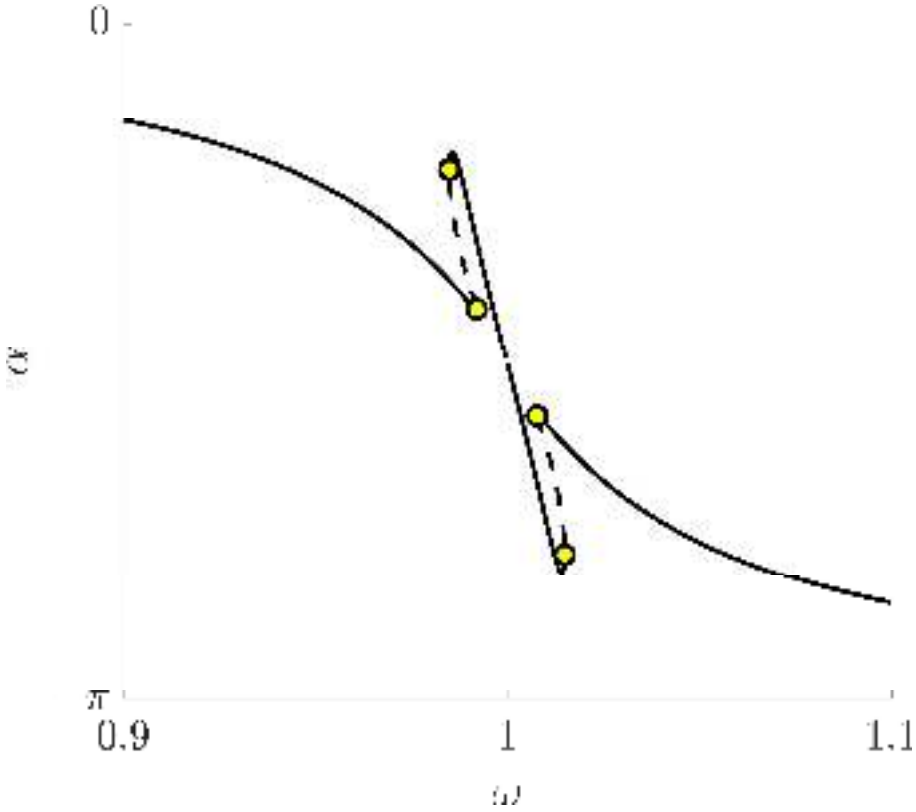
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$f = 0.02$



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CBC, WHAT'S THAT ?

- Substitute the external forcing by a **feedback control** signal

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t)\big) + \boldsymbol{b}g(t) \\ y(t) = \boldsymbol{c}\boldsymbol{x}(t) \end{cases} \longrightarrow \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t)\big) + \boldsymbol{b}K\big(r(t) - y(t)\big) \\ y(t) = \boldsymbol{c}\boldsymbol{x}(t) \end{cases}$$

K: control gain

$r(t)$: reference trajectory

- The control signal aims at constraining the ouput $y(t)$ to a presecribed **reference trajectory** $r(t)$ such that

NON-INVASIVITY AND STABILIZABILITY

- The control is **non-invasive** iff at a periodic solution $\tilde{y}(t)$

$$K\big(r(t) - \tilde{y}(t)\big) = g(t)$$

- In other words

$$r(t) = \frac{g(t)}{K} - \boldsymbol{c}\tilde{\boldsymbol{x}}(t)$$

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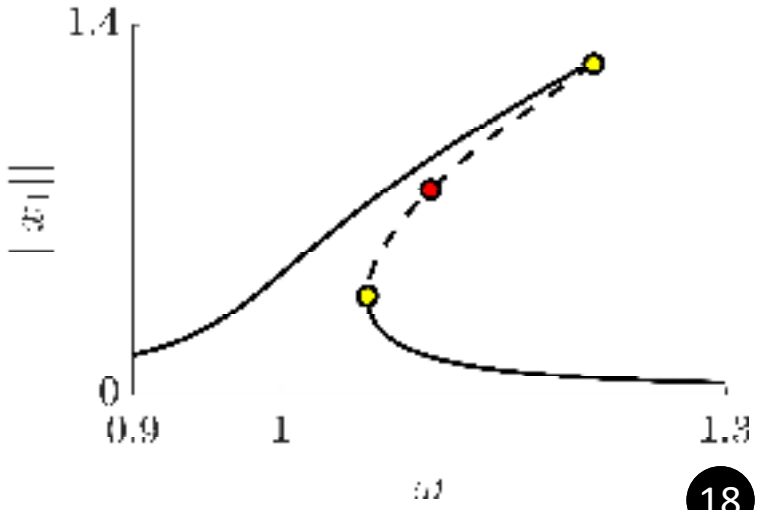
$$r(t) = \frac{g(t)}{K} - \boldsymbol{c}\tilde{\boldsymbol{x}}(t)$$

- The feedback controlled system reads

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t)\big) + \boldsymbol{b}g(t) + \boldsymbol{b}\boldsymbol{c}K\big(\tilde{\boldsymbol{x}}(t) - \boldsymbol{x}(t)\big)$$

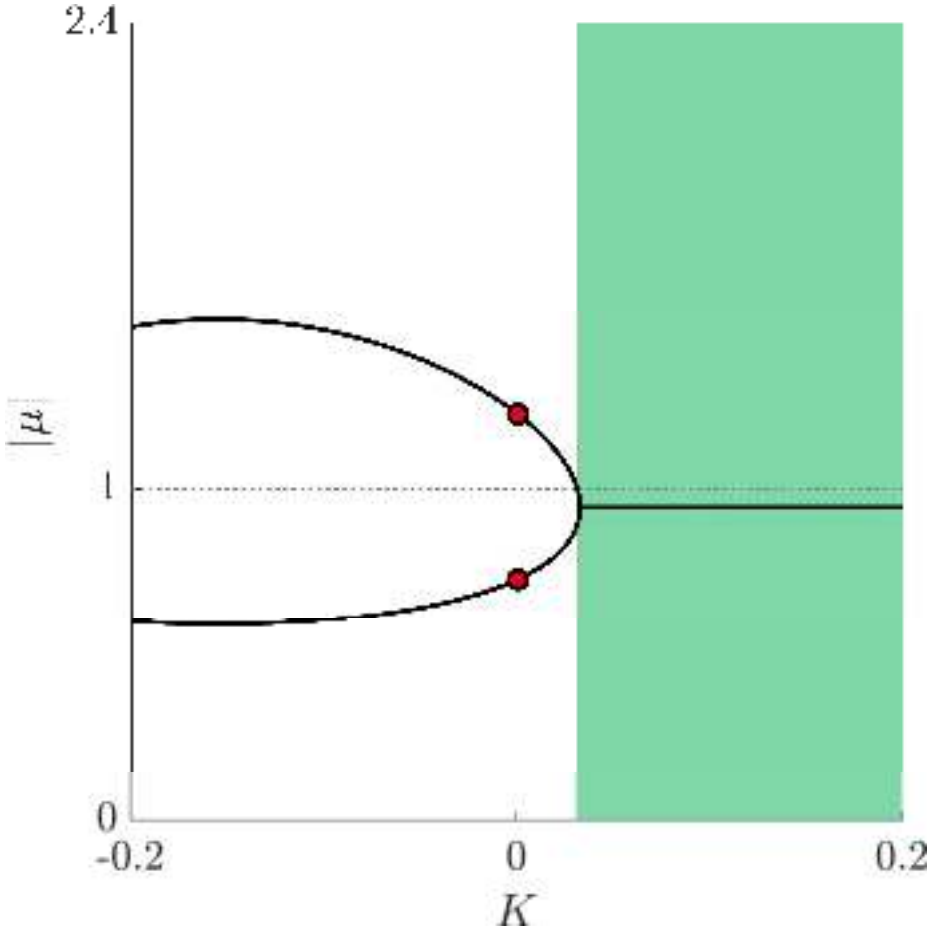
- Linearization in the vicinity of $\tilde{\boldsymbol{x}}(t)$, $\quad \boldsymbol{x}(t) = \tilde{\boldsymbol{x}}(t) + \boldsymbol{q}(t)$

$$\dot{\boldsymbol{q}}(t) = [\boldsymbol{A}(t) - \boldsymbol{b}\boldsymbol{c}K]\boldsymbol{q}(t)$$

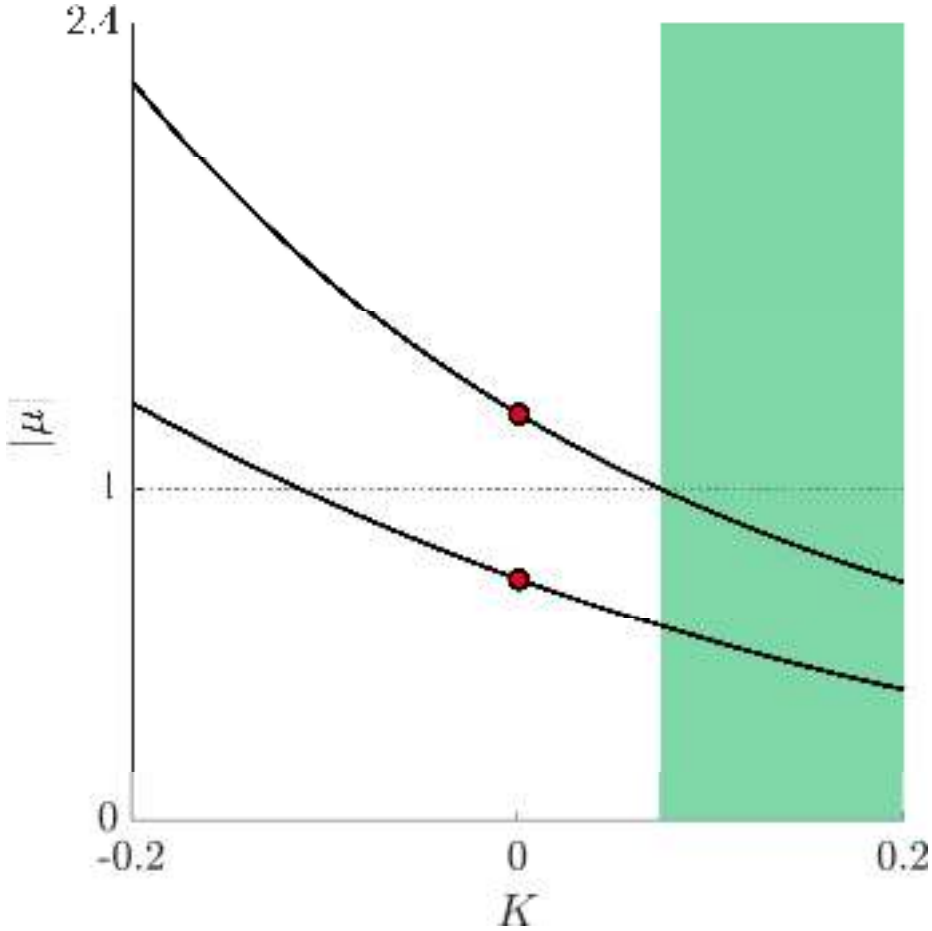


EFFECT OF THE CONTROL GAIN ON STABILITY

- Displacement feedback: $\boldsymbol{c} = (1 \quad 0)$



- Velocity feedback: $\boldsymbol{c} = (0 \quad 1)$



… YES, BUT!

- We **do no know** $\tilde{\boldsymbol{x}}(t)$ *à priori*

- … **how to construct the reference signal** $r(t)$?

FOURIER REPRESENTATION OF $r(t)$

- Fouier series expression of the reference signal $r(t)$

$$r(t) = R_0 + \sum_{n=1}^{N_h} R_{c,n}\cos(n\omega t) + R_{s,n}\sin(n\omega t) = \boldsymbol{Q}^T\boldsymbol{R}$$

- The controlled system now reads

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}K(\boldsymbol{Q}^T\boldsymbol{R} - y)$$

$$y = \boldsymbol{c}\boldsymbol{x}$$

NON-INVASIVITY CONDITION

- The control is **non-invasive** iff for $\boldsymbol{x}(t) = \widetilde{\boldsymbol{x}}(t), y(t) = \tilde{y}(t)$

$$K(\boldsymbol{Q}^T\boldsymbol{R} - \tilde{y}) = g(t)$$

- Also expressing the periodic solution $\tilde{y}(t)$ in Fourier series

$$\tilde{y} = \boldsymbol{Q}^T\widetilde{\boldsymbol{Y}}$$

- The non-invasivity condition now reads

$$K\boldsymbol{Q}^T\left(\boldsymbol{R} - \widetilde{\boldsymbol{Y}}\right) = \boldsymbol{Q}^T\boldsymbol{G}, \qquad g(t) = \boldsymbol{Q}^T\boldsymbol{G}$$

$$\Longrightarrow K\left(\boldsymbol{R} - \widetilde{\boldsymbol{Y}}\right) = \boldsymbol{G}$$

NON-INVASIVITY CONDITION

- *Deeper* investigation of the non-invasivity condition

$$K \begin{pmatrix} R_0 - \tilde{Y}_0 \\ R_{c,1} - \tilde{Y}_{c,1} \\ R_{s,1} - \tilde{Y}_{s,1} \\ R_{c,2} - \tilde{Y}_{c,2} \\ \vdots \\ R_{s,N_h} - \tilde{Y}_{s,N_h} \end{pmatrix} = \begin{pmatrix} 0 \\ G_{c,1} \\ G_{s,1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

- Thus we can express the equivalent forcing amplitude $\tilde{G}$

$$\tilde{G} = K\sqrt{\left(R_{c,1} - \tilde{Y}_{c,1}\right)^2 + \left(R_{s,1} - \tilde{Y}_{s,1}\right)^2}$$

CONSIDER (AGAIN) AN ADAPTIVE FILTER

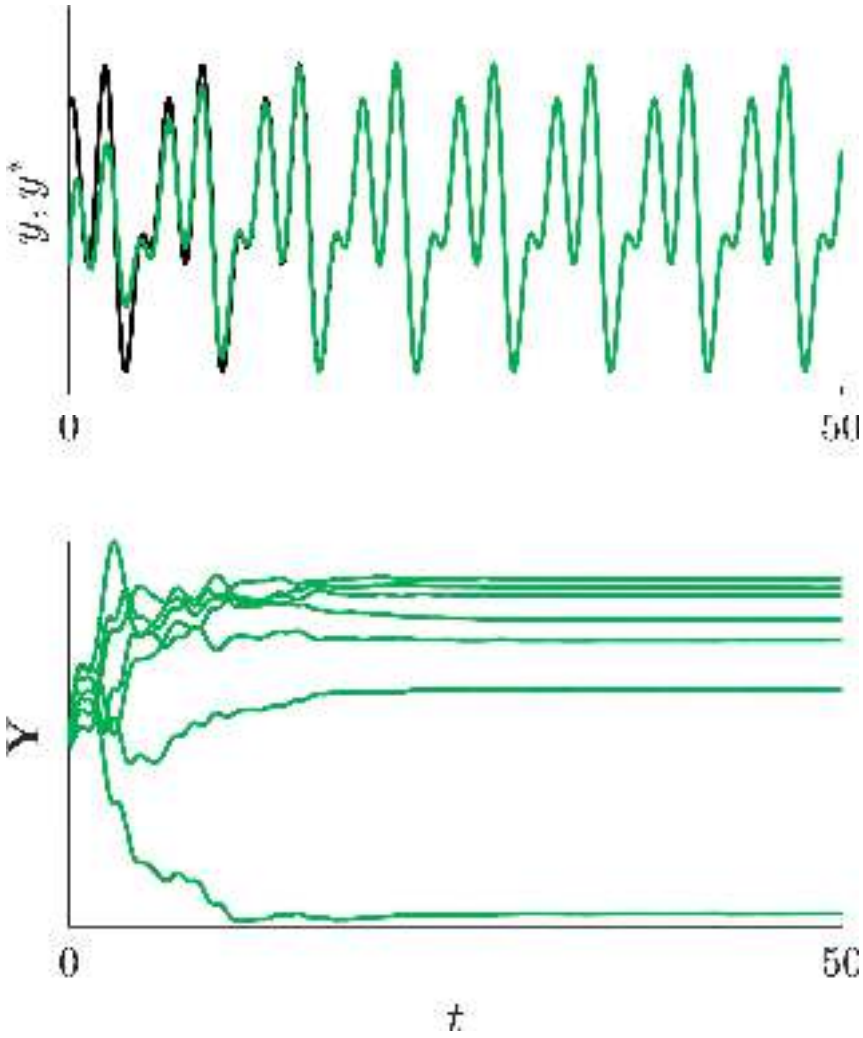
- Real time estimation of a Fourier representation of the output $y(t)$

$$y^* = Y_0 + \sum_{n=1}^{N_h} Y_{c,n}\cos(n\omega t) + Y_{s,n}\sin(n\omega t)$$

$$= \boldsymbol{Q}^T\boldsymbol{Y}$$

- Governing equation

$$\dot{\boldsymbol{Y}} = \mu\boldsymbol{Q}(y - y^*)$$



ELIMINATING NON-FUNDAMENTAL HARMONICS

- In order to cancel **non-fundamental harmonics** let us write $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{Y})$

$$\boldsymbol{R}(\boldsymbol{Y}) = \begin{pmatrix} Y_0 & R_{c,1} & R_{s,1} & Y_{c,2} & \cdots & Y_{s,N_h} \end{pmatrix}^T$$

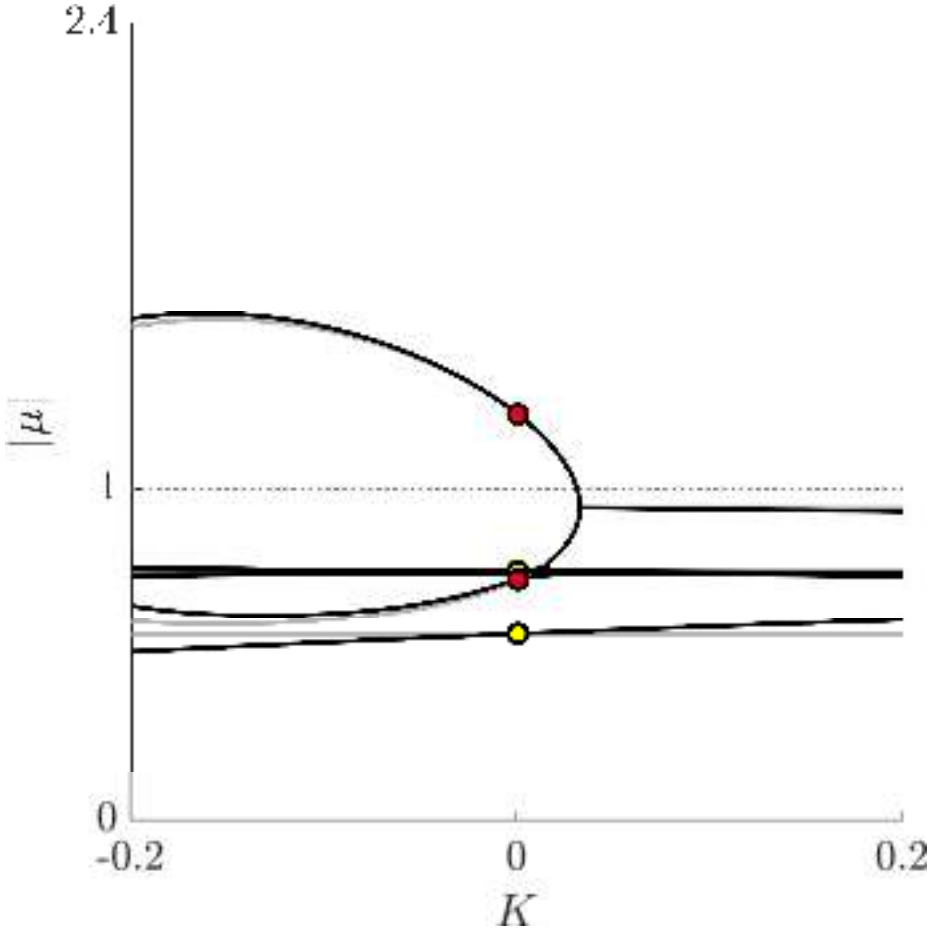
- Stability analysis

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}K(\boldsymbol{Q}^T\boldsymbol{R}(\boldsymbol{Y}) - \boldsymbol{c}\boldsymbol{x}) \\ \dot{\boldsymbol{Y}} = \mu\boldsymbol{Q}(\boldsymbol{c}\boldsymbol{x} - \boldsymbol{Q}^T\boldsymbol{Y}) \end{cases} \xrightarrow{linearizing} \begin{pmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{W}} \end{pmatrix} = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{b}\boldsymbol{c}K & \boldsymbol{b}K\boldsymbol{Q}^T \left.\dfrac{\partial \boldsymbol{R}}{\partial \boldsymbol{Y}}\right|_{\widetilde{Y}} \\ \mu\boldsymbol{Q}\boldsymbol{c} & -\mu\boldsymbol{Q}\boldsymbol{Q}^T \end{bmatrix} \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{W} \end{pmatrix}$$

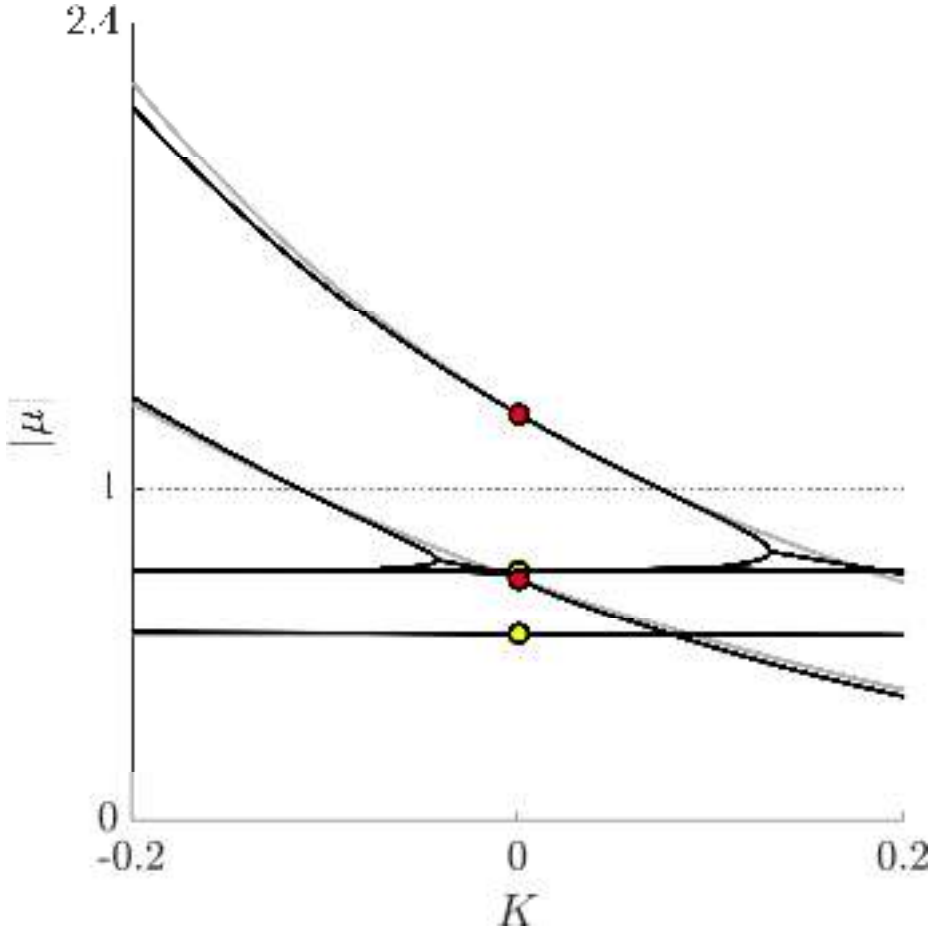
Avec $\left.\dfrac{\partial \boldsymbol{R}}{\partial \boldsymbol{Y}}\right|_{\widetilde{Y}} = \begin{bmatrix} 1 & & \\ & \mathbb{O}^2 & \\ & & \mathbb{I}^{2(N_h-1)} \end{bmatrix}$

INFLUENCE ON STABILITY

- Displacement feedback: $\boldsymbol{c} = (1 \quad 0)$

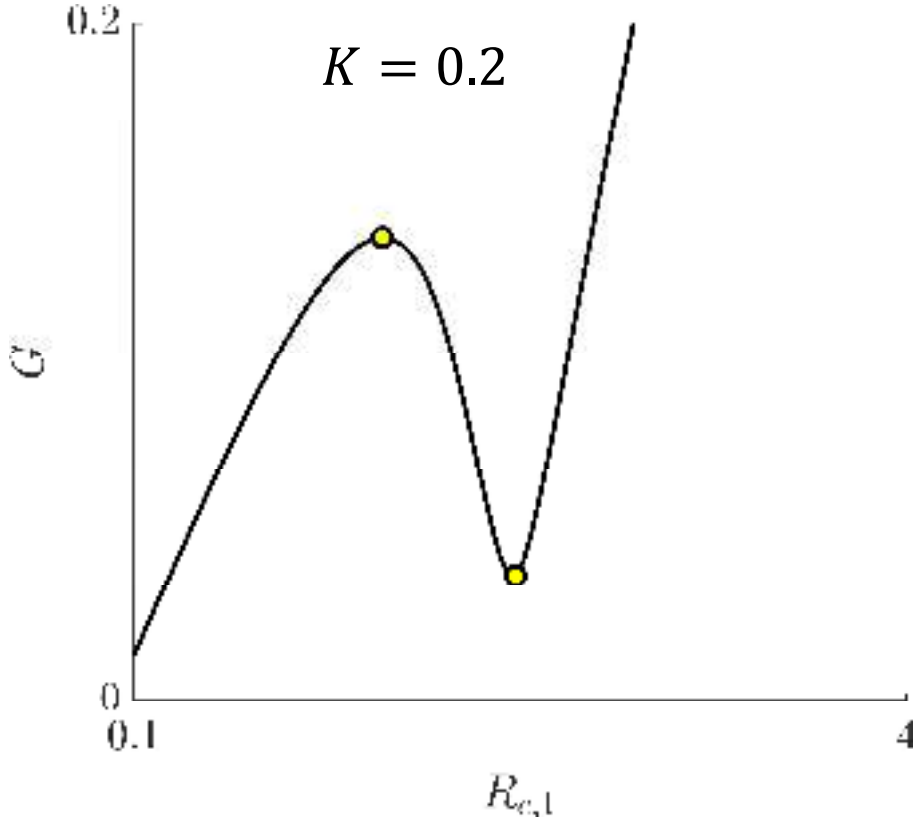


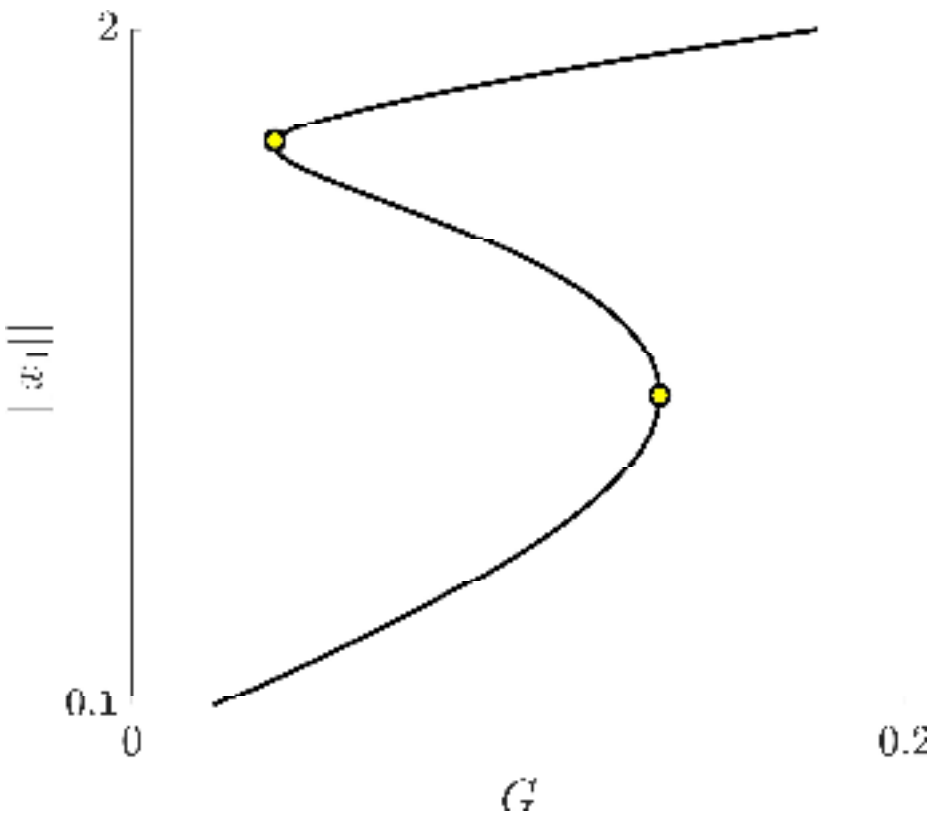
- Velocity feedback: $\boldsymbol{c} = (0 \quad 1)$



UNFOLDING THE BIFURCATION DIAGRAM

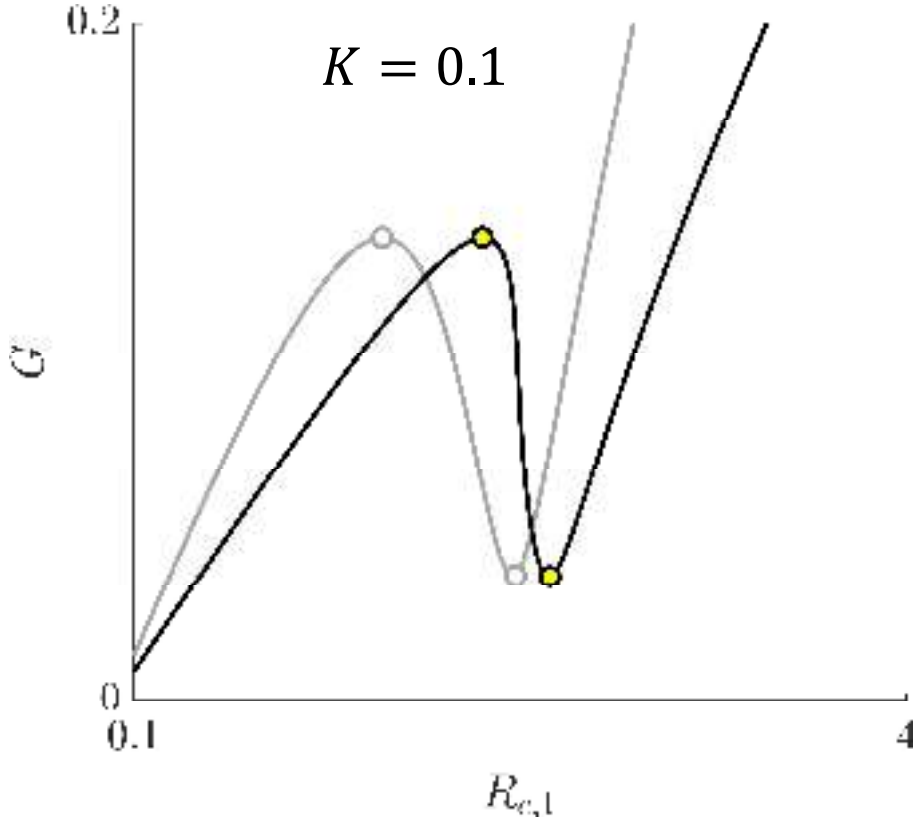
- We finaly have two remaining unknowns: $R_{c,1}$ et $R_{s,1}$
- The *easiest* solution consists in setting one of the to unknowns to 0

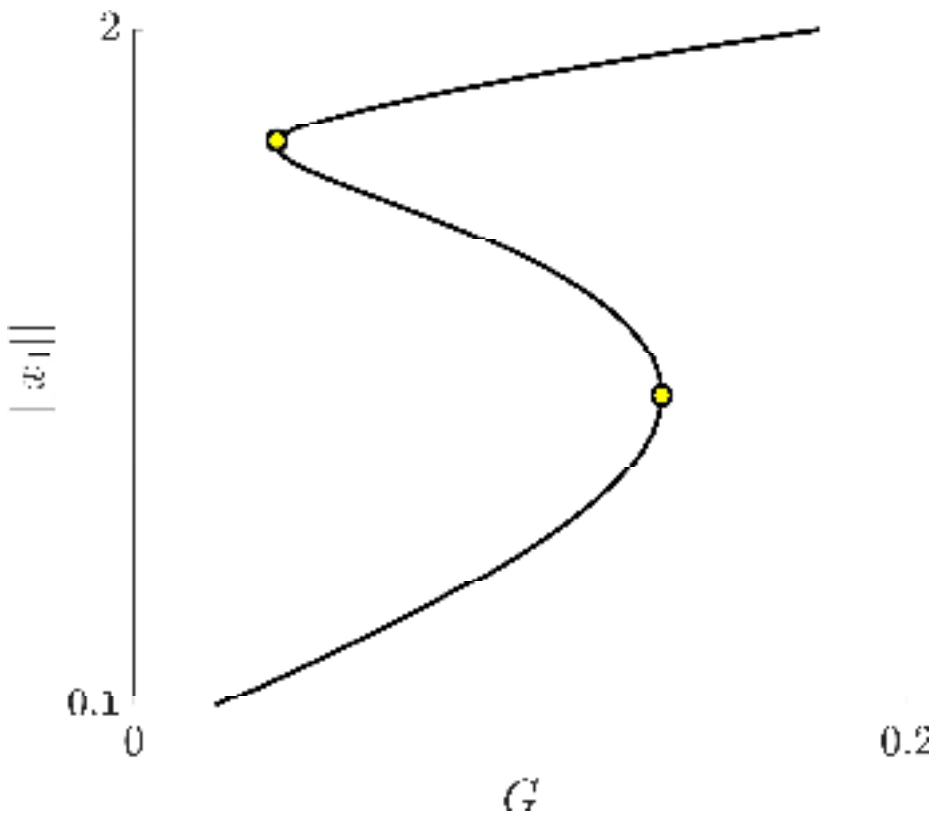




UNFOLDING THE BIFURCATION DIAGRAM

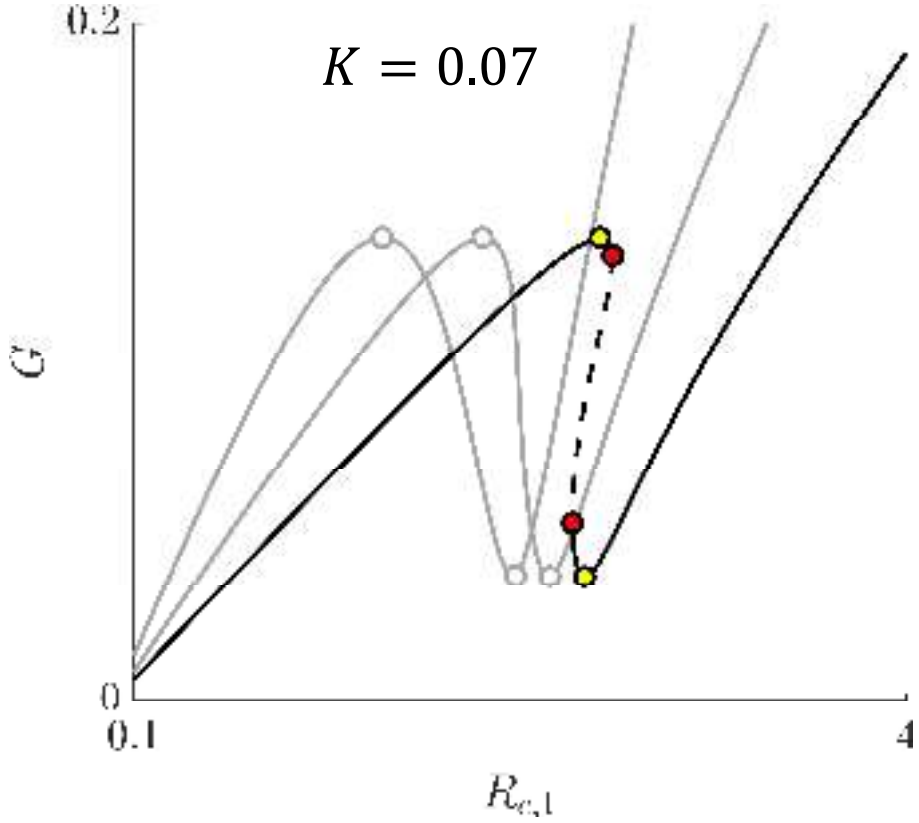
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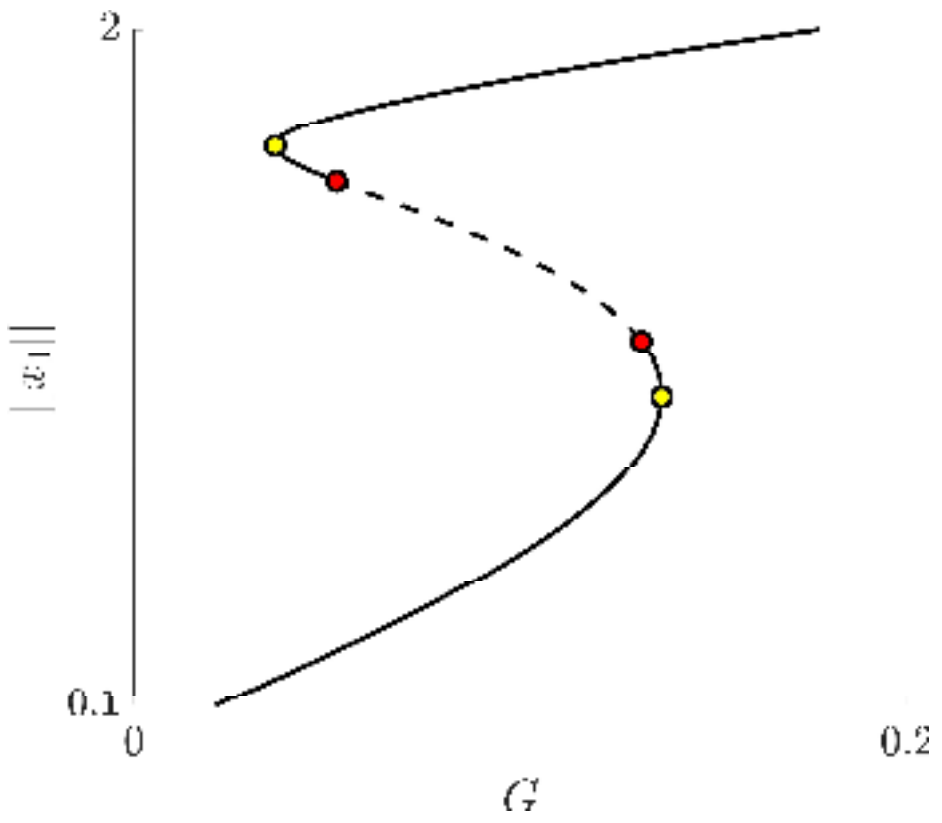




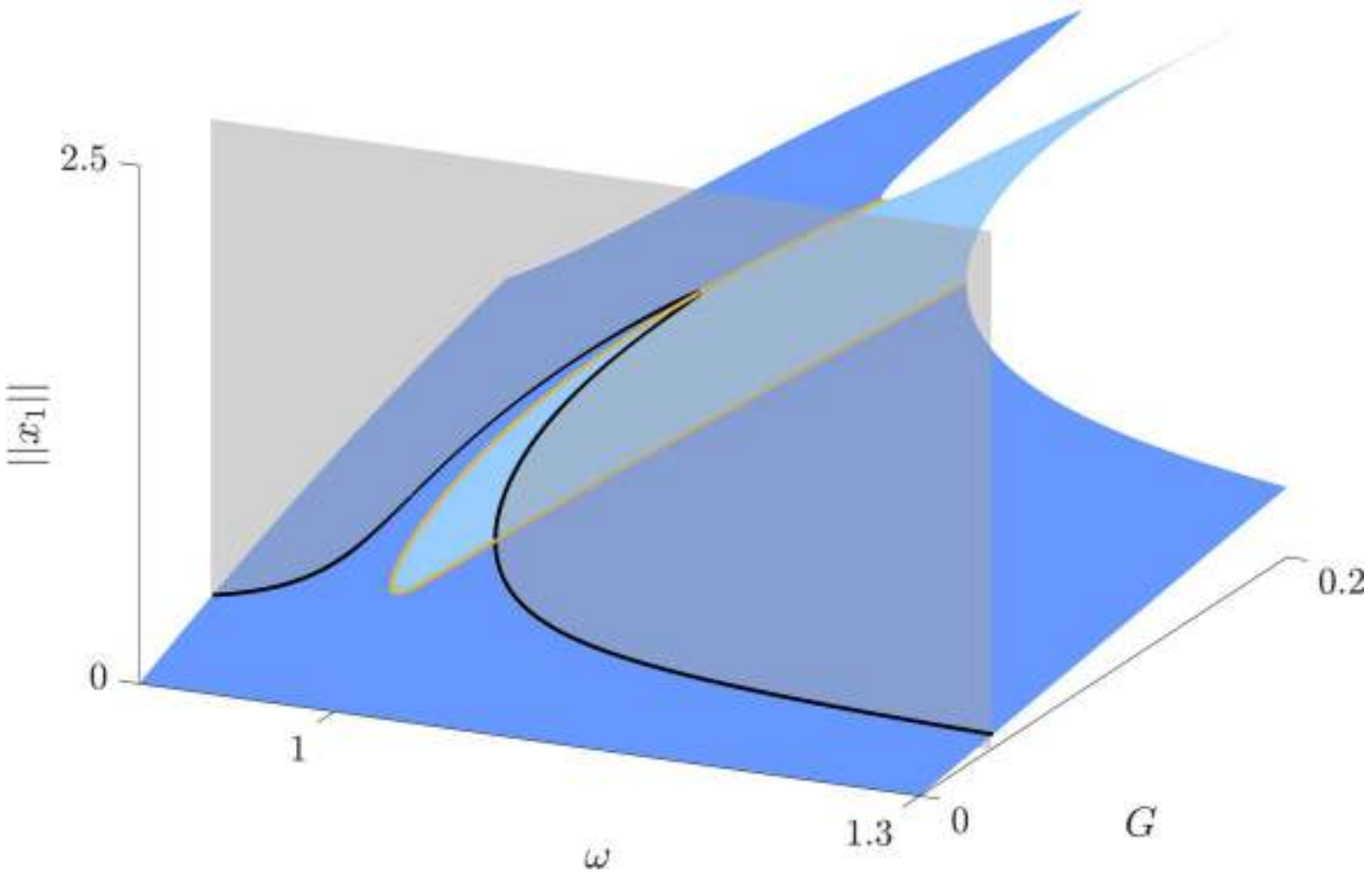
UNFOLDING THE BIFURCATION DIAGRAM

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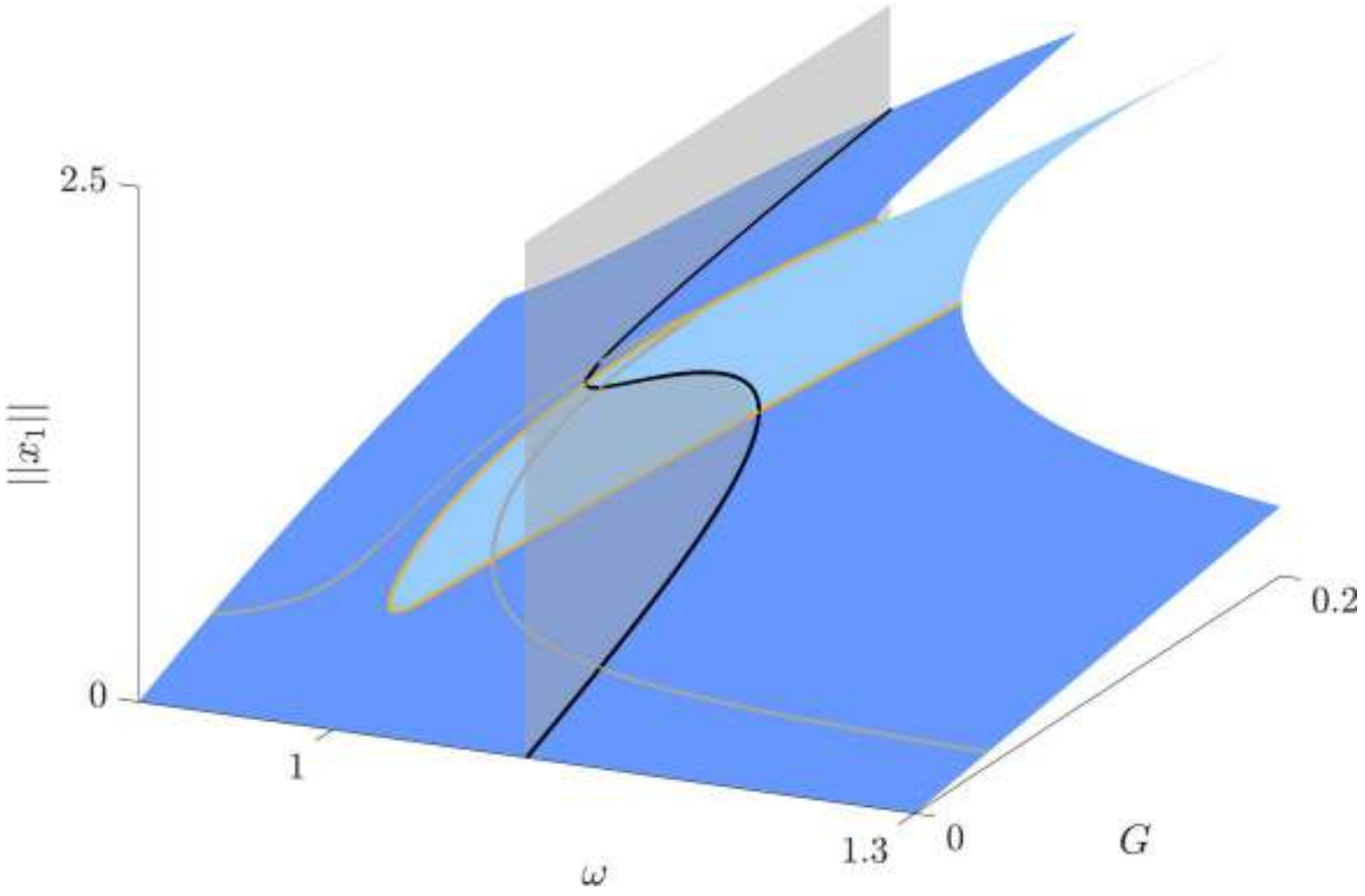




UNFOLDING THE BIFURCATION DIAGRAM



UNFOLDING THE BIFURCATION DIAGRAM



1. MOTIVATIONS

2. TWO CONTROL STRATEGIES

   2.1. PHASE LOCKED LOOP

   2.2. CONTROL BASED CONTINUATION

3. PRACTICAL CONSIDERATIONS

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5. CONCLUSION & PERSPECTIVES

SLOW FLOW AND AVERAGING

- Let us consider a *generic* 1 d.o.f system

$$\ddot{x} + x + h(x, \dot{x}) = G\cos(\omega t)$$

- Van der Pol transformation

$$x(t) = a(t)\sin\big(t + \alpha(t)\big), \qquad \dot{x}(t) = a(t)\cos\big(t + \alpha(t)\big)$$

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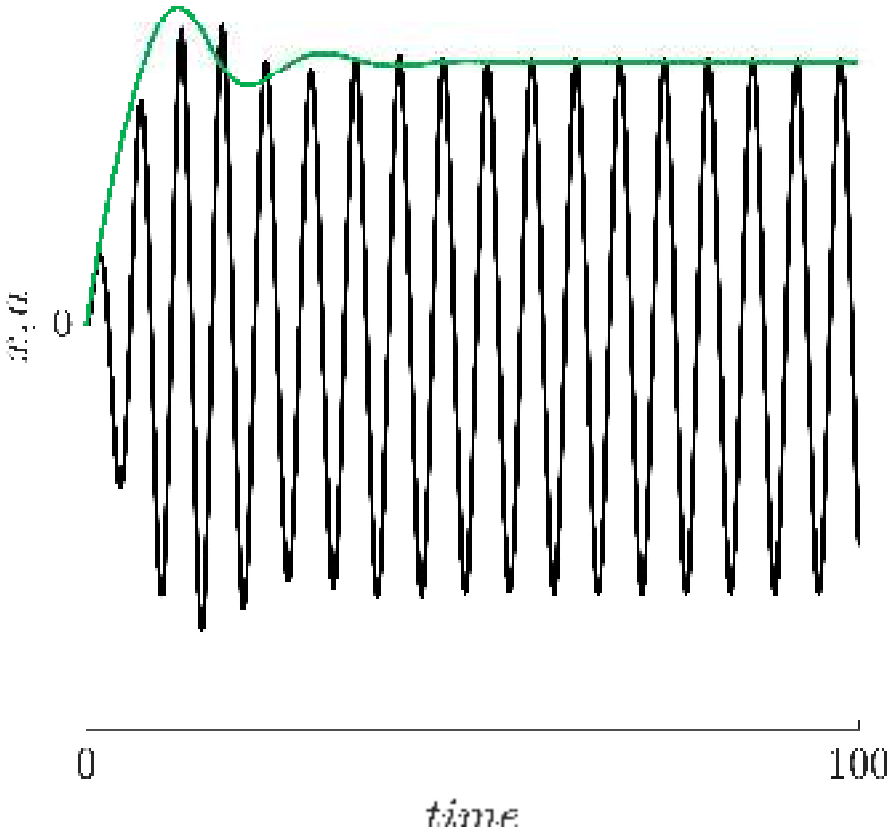
$$x(t) = a(t)\sin\big(t + \alpha(t)\big), \qquad \dot{x}(t) = a(t)\cos\big(t + \alpha(t)\big)$$

- Si $(h, G) \backsim \mathcal{O}(\epsilon)$, $\epsilon \ll 1$ Avec $\psi(t) = (\omega - 1)t - \alpha(t)$

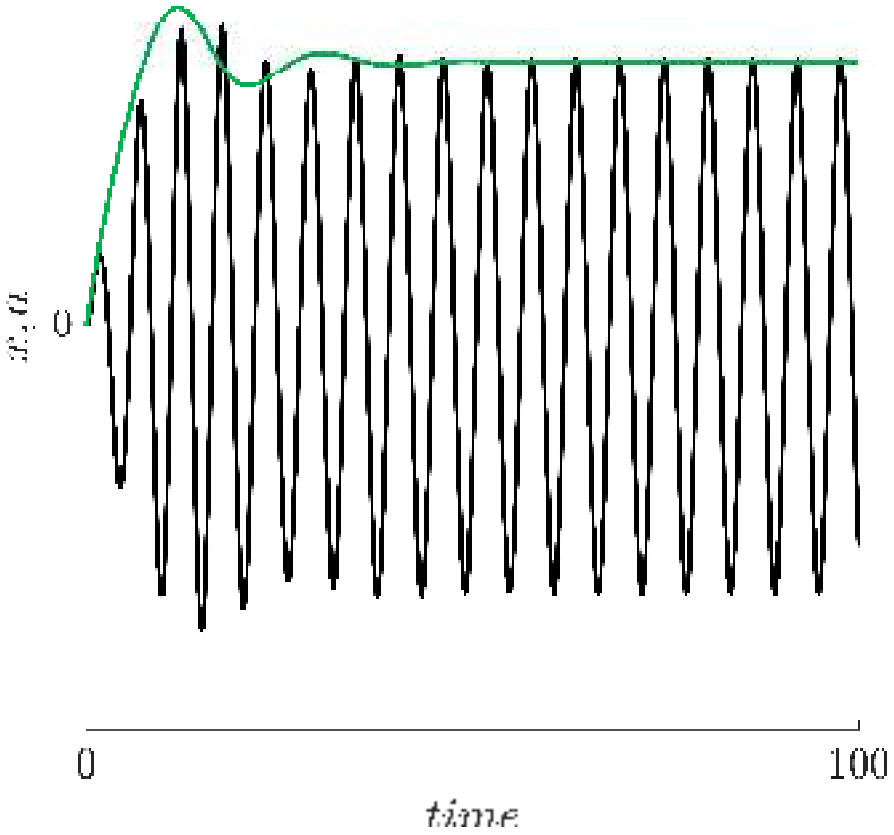
$$\dot{a}(t) \approx h_1(a) + \frac{1}{2}g\cos(\psi) \qquad h_1(a) = \frac{\omega}{2\pi}\int_0^T h(a, \alpha)\cos(t + \alpha)\,dt$$

$$\dot{\psi}(t) \approx \omega - 1 - \frac{1}{a}\left(h_2(a) + \frac{1}{2}g\sin(\psi)\right) \qquad h_2(a) = \frac{\omega}{2\pi}\int_0^T h(a, \alpha)\sin(t + \alpha)\,dt$$

ODE VS SLOW FLOW



ODE VS SLOW FLOW



- Caracteristic matrix

$$\boldsymbol{J}_u = \begin{pmatrix} -f_{1,a} & f_2 - a(\omega - 1) \\ \frac{1}{a}(\omega - 1 - f_{2,a}) & -\frac{f_{1,a}}{a} \end{pmatrix}$$

- Eigenvalues of $\boldsymbol{J}_u$: Floquet exponents λ_i

$$\mu_i = e^{\lambda_i T}$$

SLOW FLOW OF CBC SYSTEM

- Let us first consider **displacement** measurement and feedback

$$\ddot{x} + x + h(x, \dot{x}) = K(\boldsymbol{Q}^T \boldsymbol{R} - x)$$

$$\dot{\boldsymbol{Y}} = \mu \boldsymbol{Q}(x - \boldsymbol{Q}^T \boldsymbol{Y})$$

- … Averaging … Stability analysis

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_u - \frac{K}{2}\begin{pmatrix} 0 & a \\ 1/a & 0 \end{pmatrix} & \mathbb{O}^{2\times(2N_h+1)} \\ \boldsymbol{c} & -\frac{\mu}{2}\mathbb{I}^{2N_h+1} \end{pmatrix}$$

- **Strong coupling** of the eigenvalues with the controled system

SLOW FLOW OF CBC SYSTEM

- We now consider **velocity** measurement and feedback

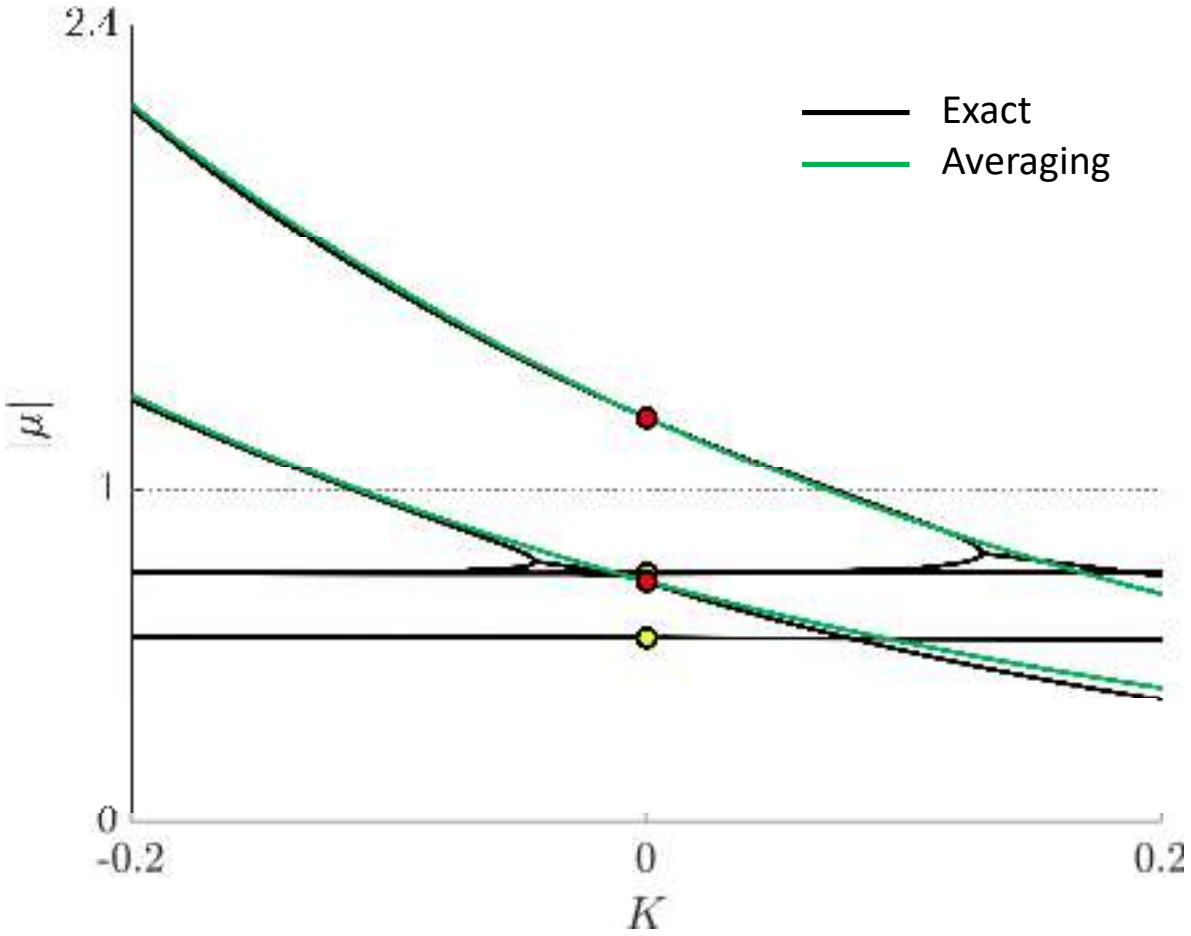
$$\ddot{x} + x + h(x, \dot{x}) = K(\boldsymbol{Q}^T \boldsymbol{R} - \dot{x})$$

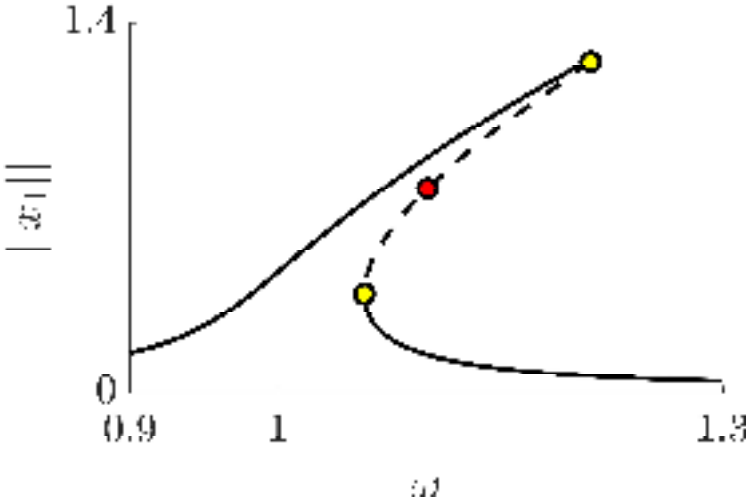
$$\dot{\boldsymbol{Y}} = \mu \boldsymbol{Q}(\dot{x} - \boldsymbol{Q}^T \boldsymbol{Y})$$

- … Averaging … Stability analysis

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_u - \frac{K}{2}\mathbb{I}^2 & \mathbb{O}^{2\times(2N_h+1)} \\ \boldsymbol{C} & -\frac{\mu}{2}\mathbb{I}^{2N_h+1} \end{pmatrix} \xrightarrow{eigenvalues} \begin{cases} \lambda_{1,2} = \lambda_u - \frac{K}{2} \\ \lambda_j = -\frac{\mu}{2} \end{cases}$$

EIGENVALUE TRAJECTORY VS ODE





OBSERVER-BASED FEEDBACK

- Practically, how to implement **velocity feedback from displacement measurement**?

- Can we use the synthesized signal from the adaptive filter as feedback signal?

DIRECT VS OBSERVER FEEDBACK

- Direct feedback

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}K(\boldsymbol{Q}^T\boldsymbol{R}(\boldsymbol{Y}) - \boldsymbol{c}\boldsymbol{x}) \\ \dot{\boldsymbol{Y}} = \mu\boldsymbol{Q}(\boldsymbol{c}\boldsymbol{x} - \boldsymbol{Q}^T\boldsymbol{Y}) \end{cases}$$

- Linearized system

$$\begin{pmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{W}} \end{pmatrix} = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{b}\boldsymbol{c}K & \boldsymbol{b}K\boldsymbol{Q}^T \left.\dfrac{\partial \boldsymbol{R}}{\partial \boldsymbol{Y}}\right|_{\widetilde{\boldsymbol{Y}}} \\ \mu\boldsymbol{Q}\boldsymbol{c} & -\mu\boldsymbol{Q}\boldsymbol{Q}^T \end{bmatrix} \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{W} \end{pmatrix}$$

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- Observer feedback

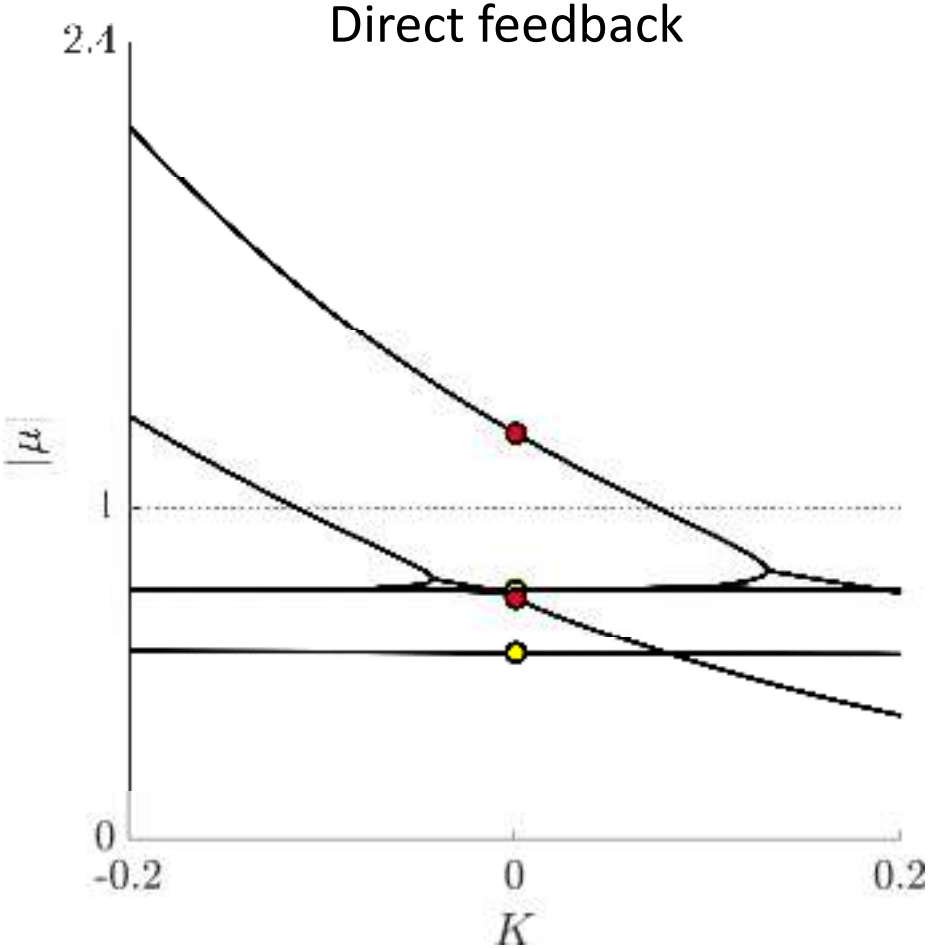
$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}K\boldsymbol{Q}^T(\boldsymbol{R}(\boldsymbol{Y}) - \boldsymbol{Y}) \\ \dot{\boldsymbol{Y}} = \mu\boldsymbol{Q}(\boldsymbol{c}\boldsymbol{x} - \boldsymbol{Q}^T\boldsymbol{Y}) \end{cases}$$

- Linearized system

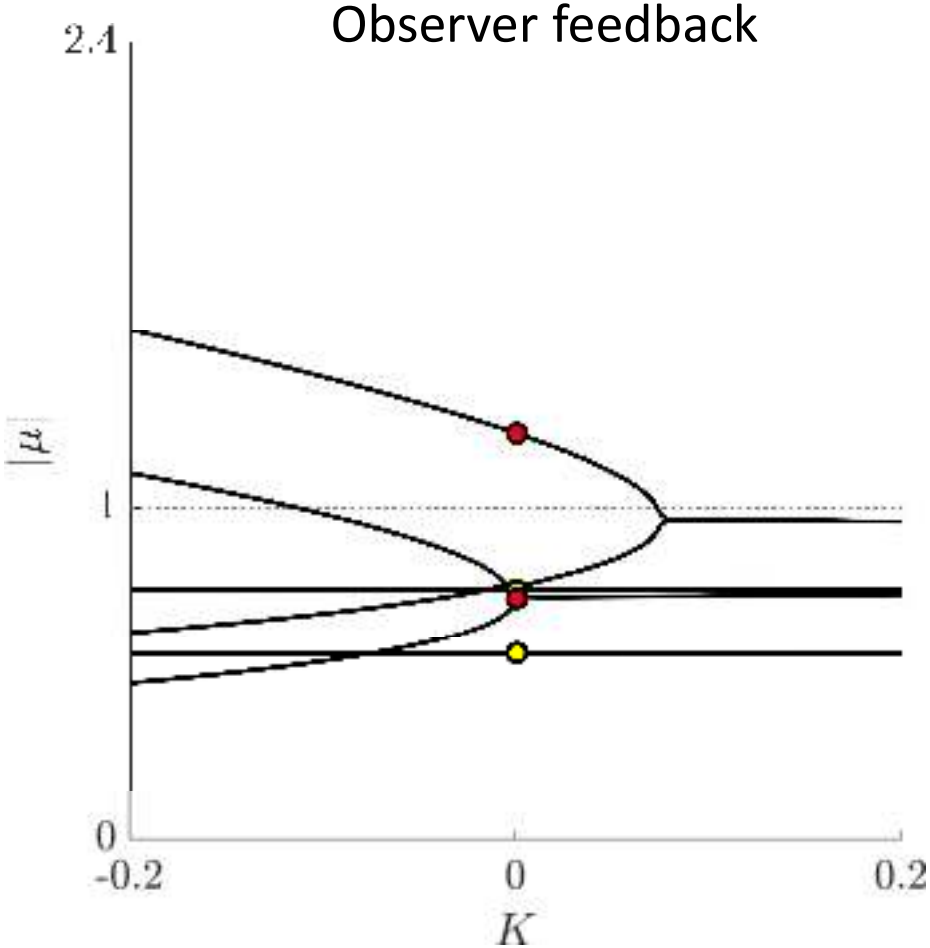
$$\begin{pmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{W}} \end{pmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b}K\boldsymbol{Q}^T \left(\left.\dfrac{\partial \boldsymbol{R}}{\partial \boldsymbol{Y}}\right|_{\widetilde{\boldsymbol{Y}}} - \mathbb{I}^{2N_h+1} \right) \\ \mu\boldsymbol{Q}\boldsymbol{c} & -\mu\boldsymbol{Q}\boldsymbol{Q}^T \end{bmatrix} \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{W} \end{pmatrix}$$

INFLUENCE ON STABILITY

Direct feedback



Observer feedback



STRONG COUPLING WITH THE ADAPTIVE FILTER

- Averaging analysis

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_u & \boldsymbol{\mathcal{B}} \\ \boldsymbol{\mathcal{C}} & -\frac{\mu}{2}\mathbb{I}^{2N_h+1} \end{pmatrix}$$

- Eigenvalue computation

$$\det\left[\boldsymbol{J}_u - \lambda\mathbb{I} + \left(\frac{\mu}{2}+\lambda\right)^{-1}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{C}}\right] = 0$$

- Remarkably $\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{C}} = -\frac{\mu K}{4}\mathbb{I}$, such that

$$\lambda + \frac{1}{2}\frac{\mu K}{\mu + 2\lambda} = \lambda_u$$

STRONG COUPLING WITH THE ADAPTIVE FILTER

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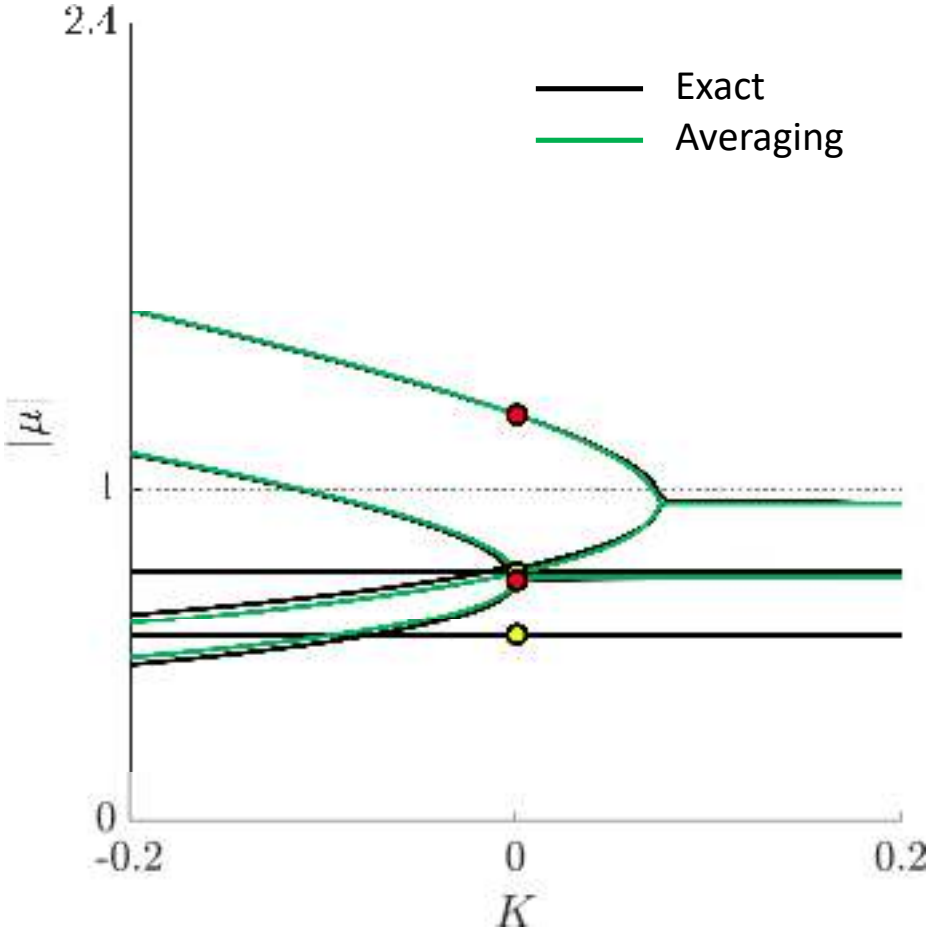
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$$\lambda + \frac{1}{2}\frac{\mu K}{\mu + 2\lambda} = \lambda_u$$



OBSERVER VELOCITY FEEDBACK

- Measured displacement and velocity feedback

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}K\dfrac{d}{dt}\left(\boldsymbol{Q}^T(\boldsymbol{R}(\boldsymbol{Y}) - \boldsymbol{Y})\right) \\ \dot{\boldsymbol{Y}} = \mu\boldsymbol{Q}(\boldsymbol{c}\boldsymbol{x} - \boldsymbol{Q}^T\boldsymbol{Y}) \end{cases}$$

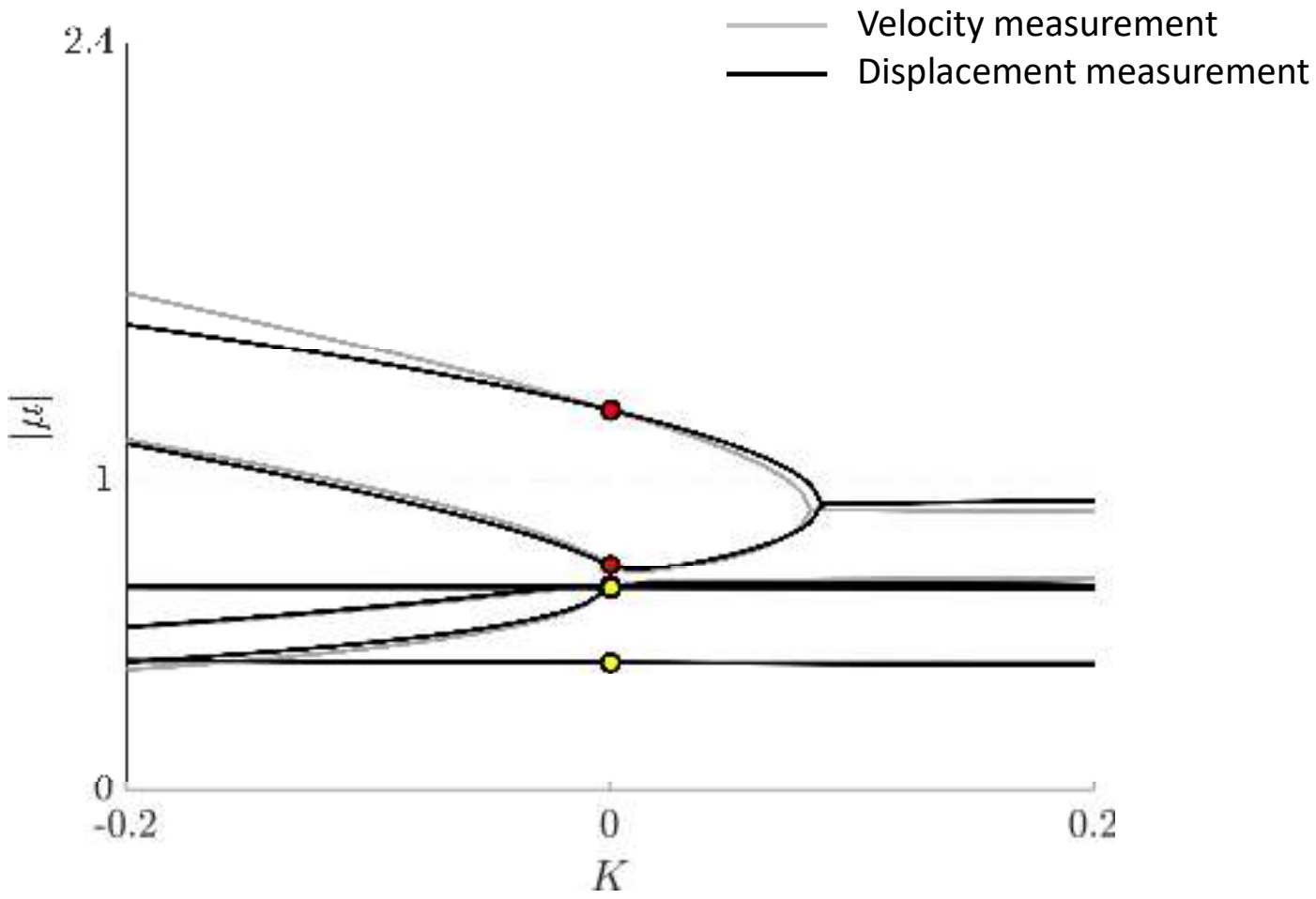
with

$$\frac{d}{dt}\left(\boldsymbol{Q}^T(\boldsymbol{R}(\boldsymbol{Y}) - \boldsymbol{Y})\right) = \dot{\boldsymbol{Q}}^T(\boldsymbol{R}(\boldsymbol{Y}) - \boldsymbol{Y}) + \underbrace{\boldsymbol{Q}^T\left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{Y}} - \mathbb{I}\right)\dot{\boldsymbol{Y}}}_{\mathcal{O}(\mu)}$$

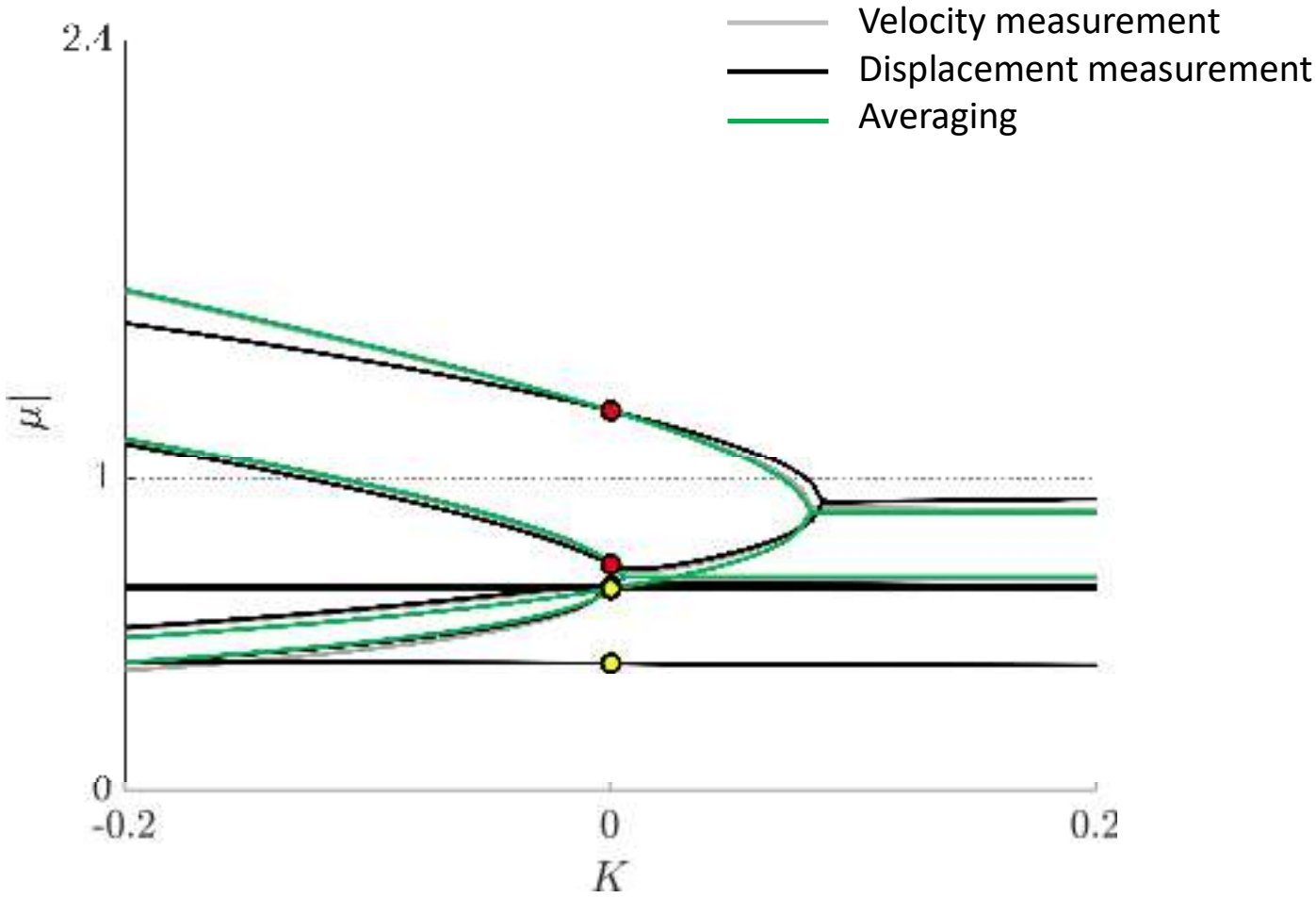
- The averaging analysis shows the **same behavior** as velocity measurement

$$\lambda + \frac{1}{2}\frac{\mu K}{\mu + 2\lambda} = \lambda_u$$

OBSERVER VELOCITY FEEDBACK



OBSERVER VELOCITY FEEDBACK



1. MOTIVATIONS

2. TWO CONTROL STRATEGIES

   2.1. PHASE LOCKED LOOP

   2.2. CONTROL BASED CONTINUATION

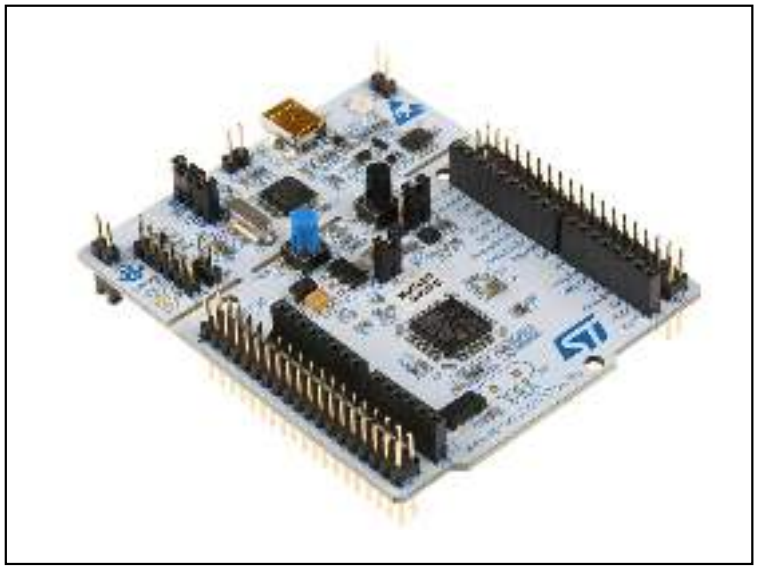
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REAL TIME CONTROLLER





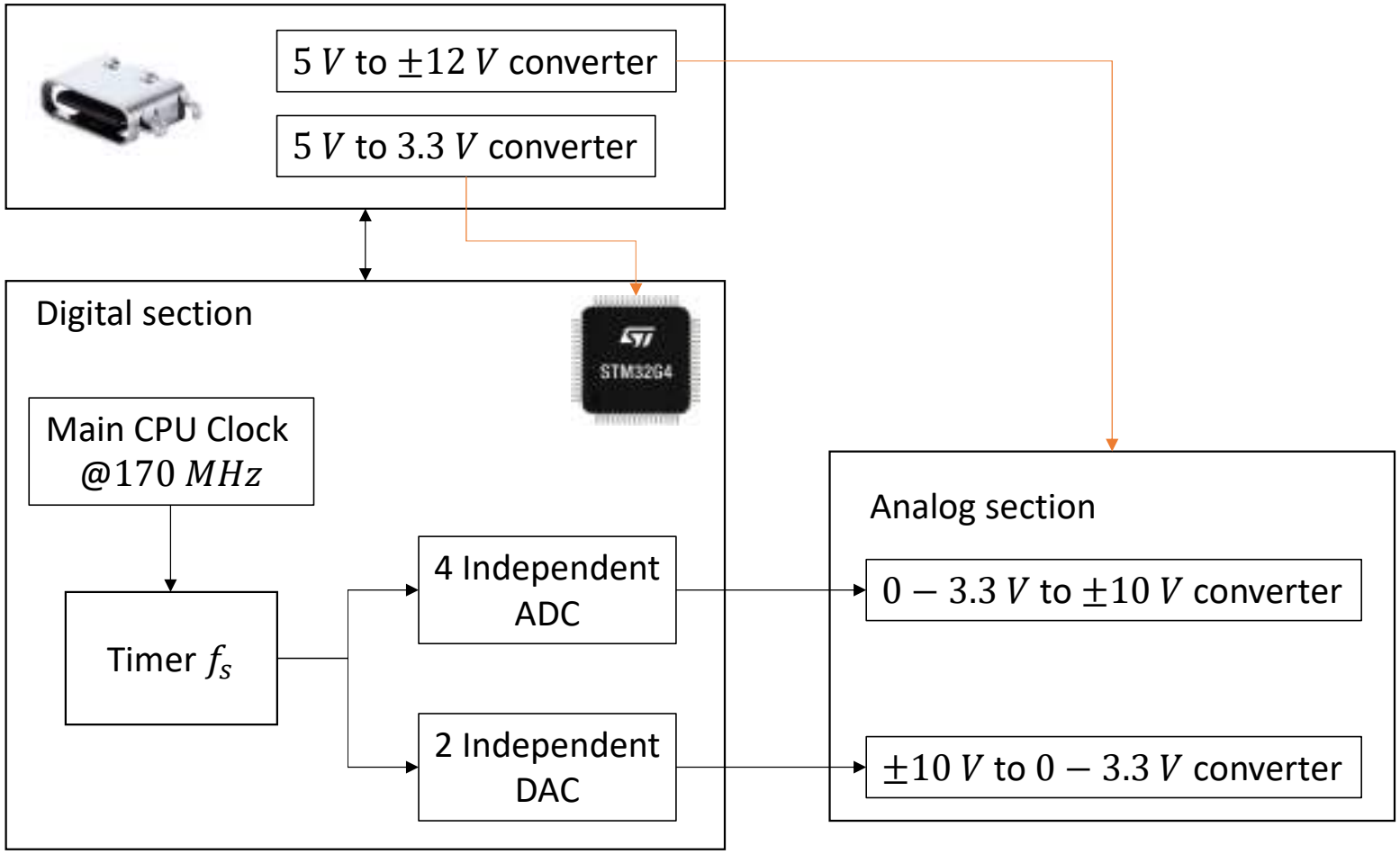
DIY CONTROLLER

- Design constrains
  - Sample control
  - Data I/O streaming
  - Robust implementation

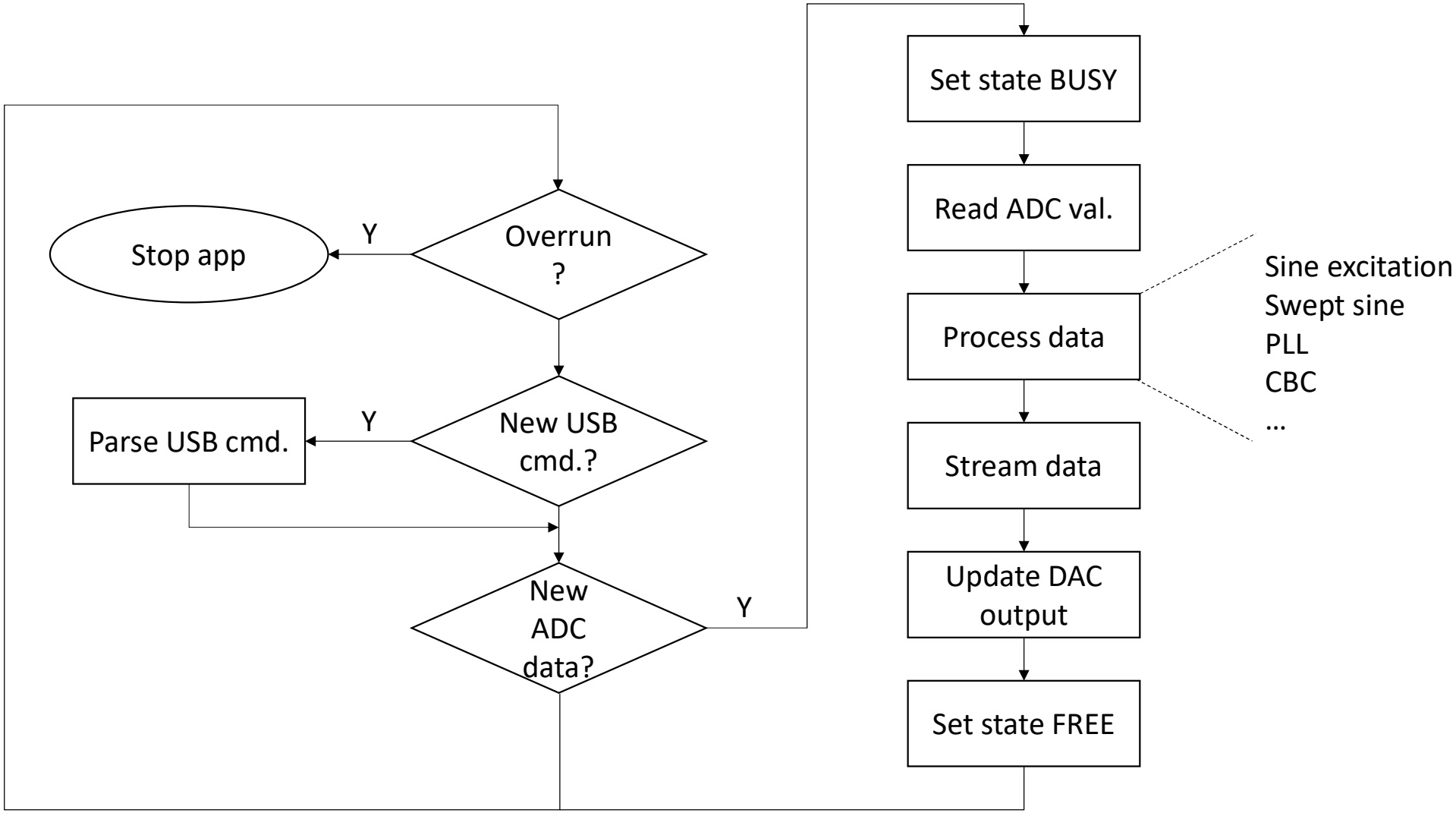


- 4 Inputs $\pm 10V$, 12 bits ADC
- 2 Outputs $\pm 10V$, 12 bits DAC
- USB power supply and data transfer

HARDWARE CONFIGURATION



A FAIRLY SIMPLE IMPLEMENTATION



DISCRÉTISATION TEMPORELLE

- Système en temps continu

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}K(\boldsymbol{Q}^T\boldsymbol{R} - y)$$

$$\dot{\boldsymbol{Y}} = \mu\boldsymbol{Q}(y - \boldsymbol{Q}^T\boldsymbol{Y})$$

$$y = \boldsymbol{c}\boldsymbol{x}$$

- Conversion en temps discret (ZOH)

$$\dot{\boldsymbol{Y}}(t) \approx \frac{\boldsymbol{Y}_{k+1} - \boldsymbol{Y}_k}{t_s}$$

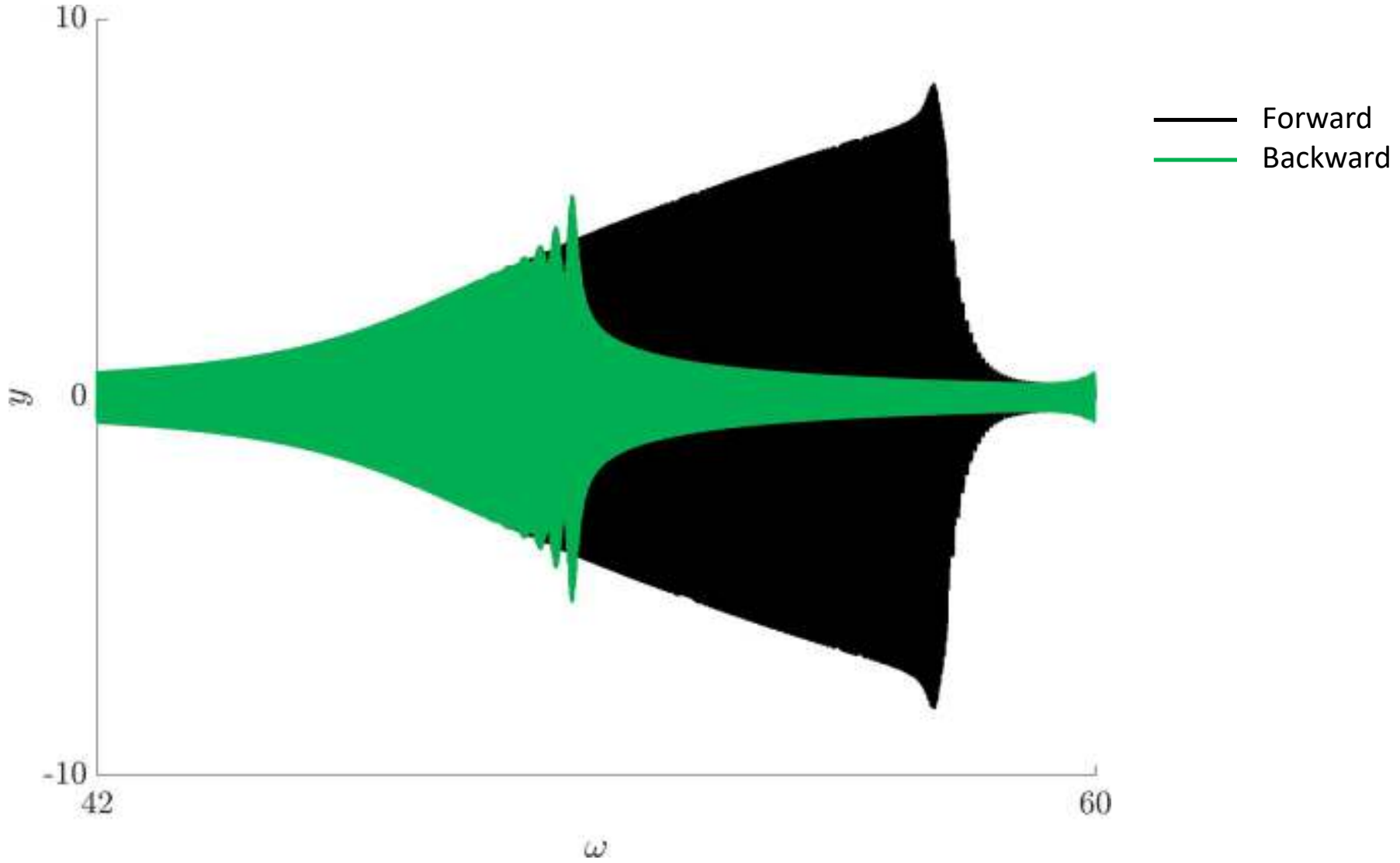
- Système en temps discret

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + t_s\boldsymbol{f}(\boldsymbol{x}_k) + \boldsymbol{b}Kt_s\left(\boldsymbol{Q}_k^T\boldsymbol{R} - y_k\right)$$

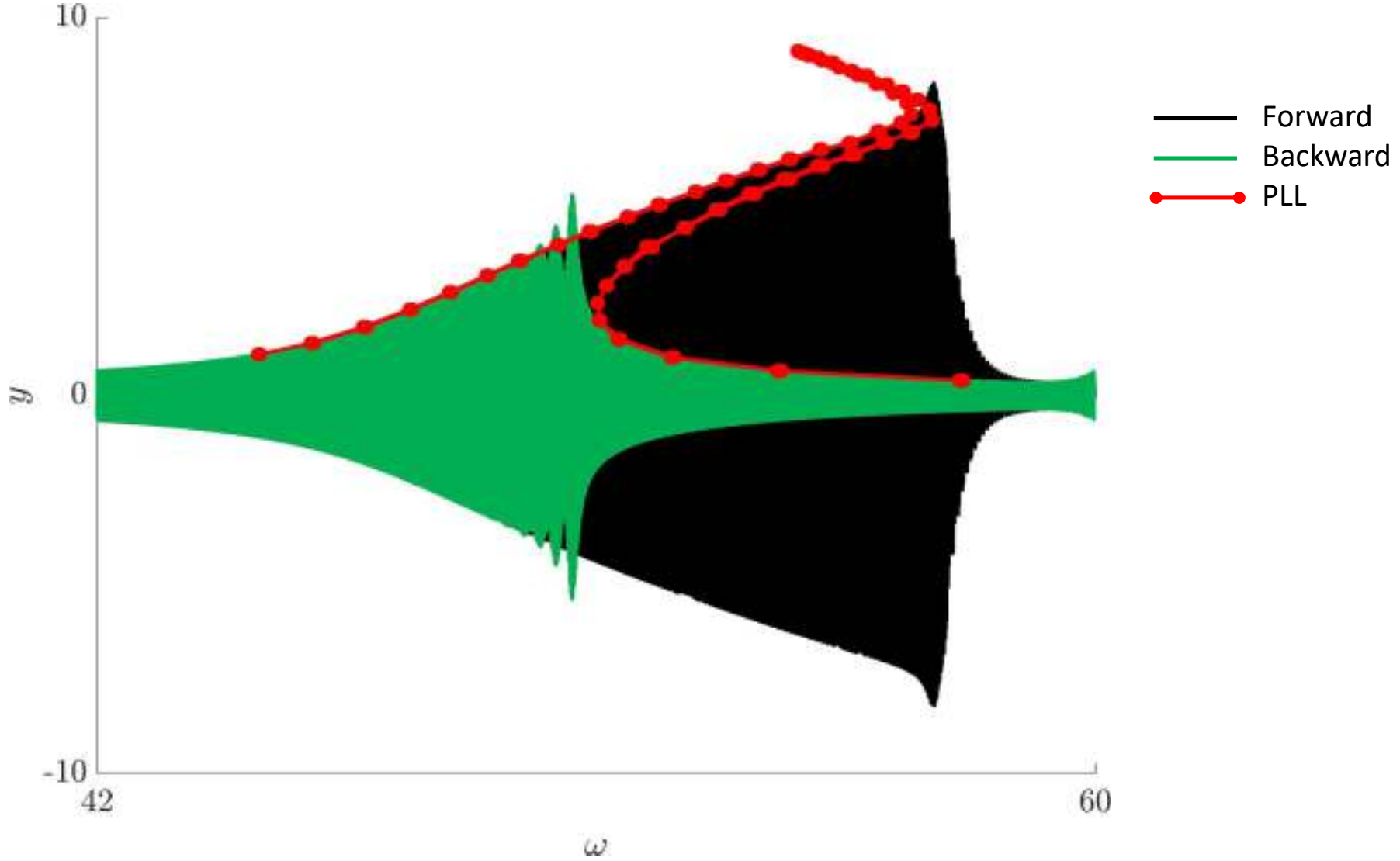
$$\boldsymbol{Y}_{k+1} = \boldsymbol{Y}_k + \mu t_s\boldsymbol{Q}_k\left(y_k - \boldsymbol{Q}_k^T\boldsymbol{Y}_k\right)$$

DEMO: ANALOG DUFFING OSCILLATOR

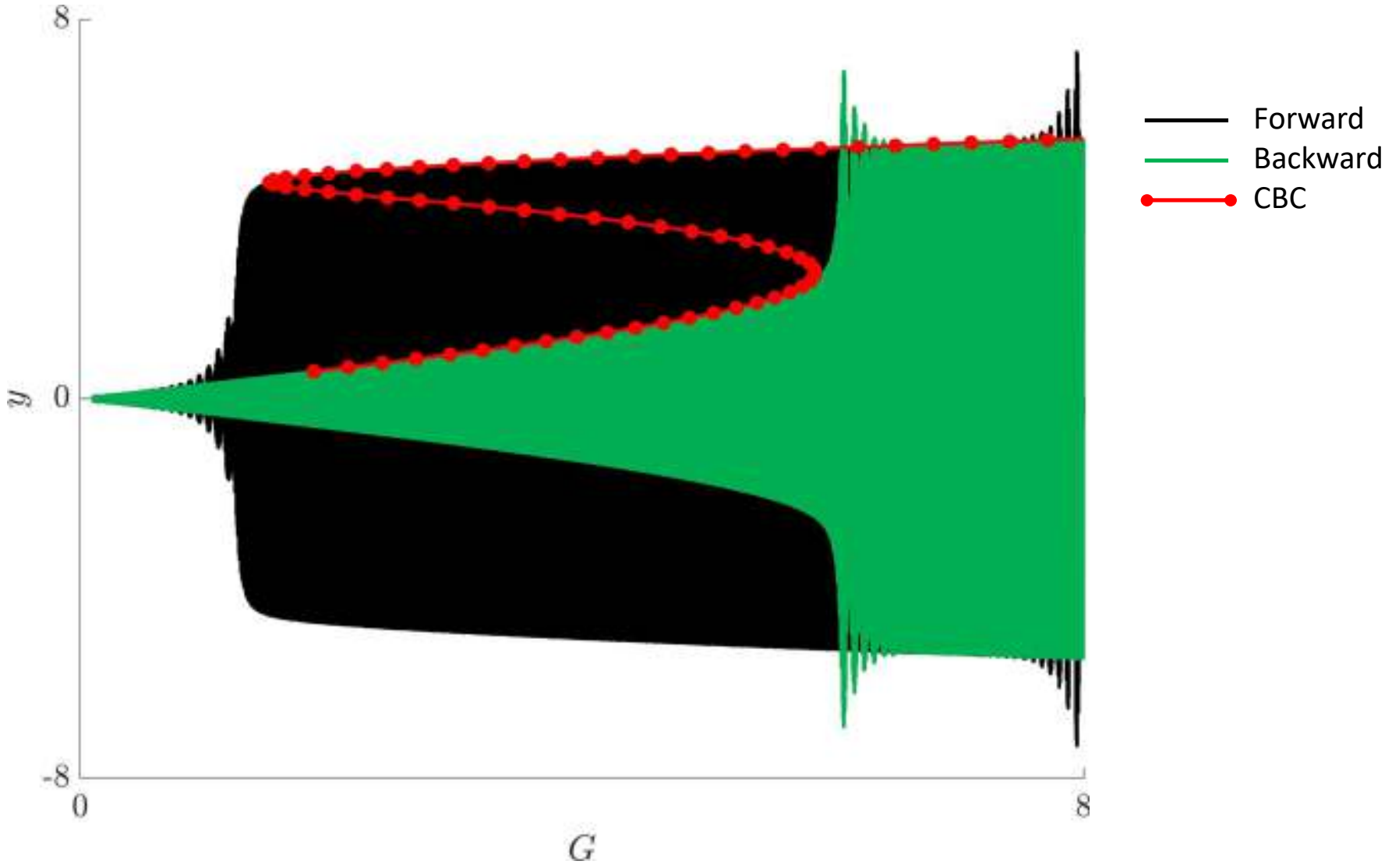
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CONCLUSION

PLL / Adaptive filter

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t)\big) + \boldsymbol{b}G\cos\big(\theta(t)\big)$$

$$\dot{\boldsymbol{Y}}(t) = \mu\boldsymbol{Q}\big(\boldsymbol{c}\boldsymbol{x}(t) - \boldsymbol{Q}^T\boldsymbol{Y}(t)\big)$$

$$\dot{\theta}(t) = \omega_0 + K\int\big(\alpha_1^*\big(\boldsymbol{Y}(t)\big) - \phi\big)dt$$

$$\alpha_1^*\big(\boldsymbol{Y}(t)\big) = \tan^{-1}\left(\frac{Y_{s,1}(t)}{Y_{c,1}(t)}\right)$$

CBC

$$\boldsymbol{R}(\boldsymbol{Y}) = \begin{pmatrix} Y_0 & R_{c,1} & 0 & Y_{c,2} & \cdots & Y_{s,N_h}\end{pmatrix}^T$$

Direct feedback

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}K(\boldsymbol{Q}^T\boldsymbol{R}(\boldsymbol{Y}) - \boldsymbol{c}\boldsymbol{x})$$

$$\dot{\boldsymbol{Y}} = \mu\boldsymbol{Q}(\boldsymbol{c}\boldsymbol{x} - \boldsymbol{Q}^T\boldsymbol{Y})$$

Observer feedback

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}K\boldsymbol{Q}^T(\boldsymbol{R}(\boldsymbol{Y}) - \boldsymbol{Y})$$

$$\dot{\boldsymbol{Y}} = \mu\boldsymbol{Q}(\boldsymbol{c}\boldsymbol{x} - \boldsymbol{Q}^T\boldsymbol{Y})$$

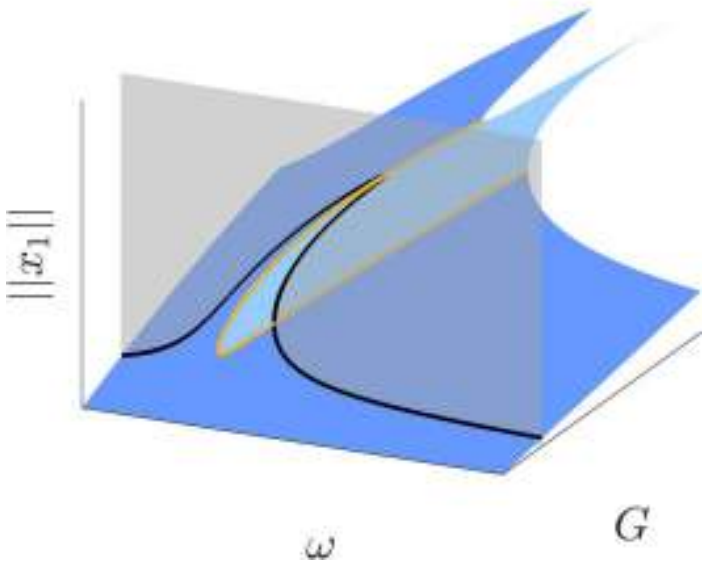
Observer feedback, velocity synthesis

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}K\dot{\boldsymbol{Q}}^T(\boldsymbol{R}(\boldsymbol{Y}) - \boldsymbol{Y})$$

$$\dot{\boldsymbol{Y}} = \mu\boldsymbol{Q}(\boldsymbol{c}\boldsymbol{x} - \boldsymbol{Q}^T\boldsymbol{Y})$$

CONCLUSION

- PLL : continuation in the *classical* **Frequency-Amplitude** plane
  - Tuning rules for the gains
  - Application to nonlinear damping identification [4]
  - Extension to secondary resonance [5]
  - Application to full scale aircraft [6, 7]





[4] Debeurre, *et Al*. (2024). Phase resonance testing of highly flexible structures: Measurement of conservative nonlinear modes and nonlinear damping identification. *MSSP*

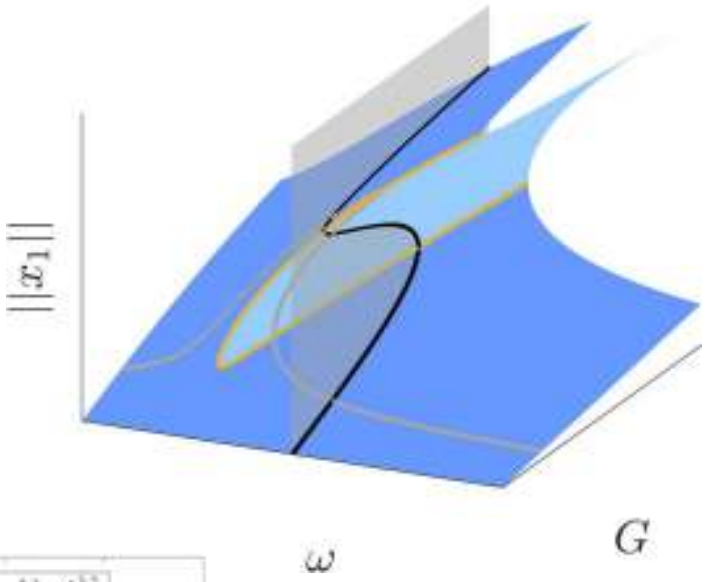
[5] Zhou, T., & Kerschen, G. (2024). Identification of secondary resonances of nonlinear systems using phase-locked loop testing. *JSV*

[6] Zhou, T., *et Al*. (2025). Experimental nonlinear modal analysis of an F-16 aircraft using phase-locked loop control. *Journal of Aircraft*

[7] Chukwu, A. *et Al.*. (2026). Modal experimental continuation of an aircraft during ground vibration test. In *AIAA AVIATION 2026 Forum*

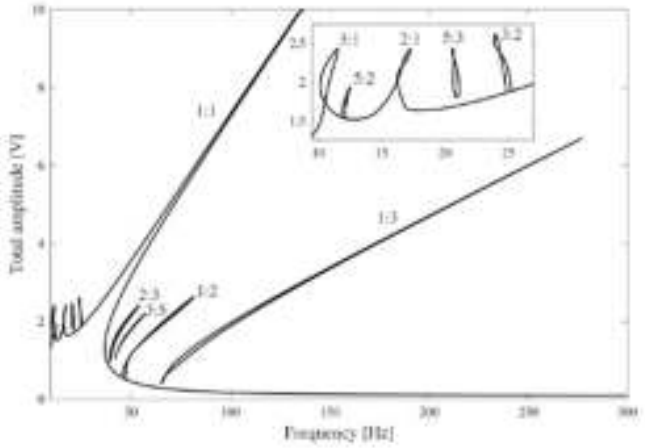
CONCLUSION

- *Simple* CBC : continuation in the **Forcing Amplitude-Response Amplitude** plane



  - Several variations
    - Pseudo arclength embedding [8]
    - Autonomous systems [9]
    - Fold continuation [10]

  - Original application
    - Biology, Collective human behavior…



[8] Spits, A., & Raze, G. (2026). Measuring Nonlinear Resonances using Extended Arclength Control-Based Continuation. *JSV*
[9] Caron, R., *et Al.* (2026). Control-Based Continuation for self oscillating systems: application to flow induced vibrations. *MSSP*
[10] Renson, L., *et Al*. (2017). Experimental tracking of limit-point bifurcations and backbone curves using control-based continuation. *IJBC*

PERSPECTIVE

- Automatic gain tuning
- Stability estimation, branching behavior
- MDOF system
- …

SUGGESTED READING



Denis, V., Jossic, M., Giraud-Audine, C., Chomette, B., Renault, A., & Thomas, O. (2018). Identification of nonlinear modes using phase-locked-loop experimental continuation and normal form. *Mechanical Systems and Signal Processing*



Raze, G., Abeloos, G., & Kerschen, G. (2025). Experimental continuation in nonlinear dynamics: recent advances and future challenges: G. Raze et al. *Nonlinear Dynamics*

CONTINUATION EXPÉRIMENTALE
de la théorie à la pratique



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