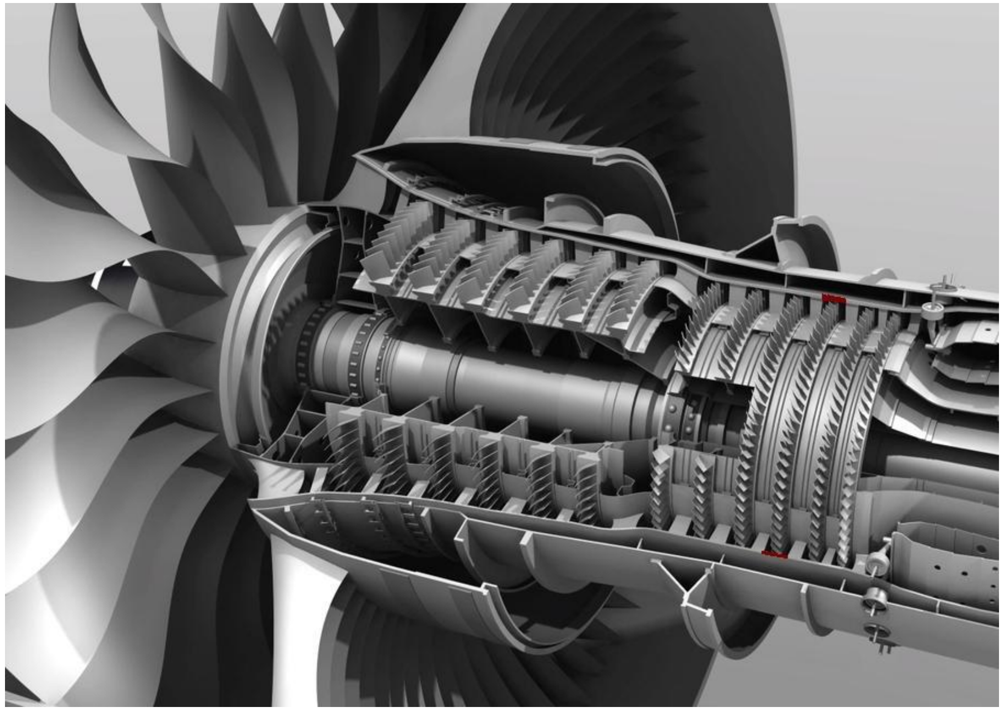
Dr.
Alessandra
Vizzaccaro

Model Order Reduction for Engineering Applications

Bio

- PhD @ Imperial College London:
  - ROM of Large Scale Simulation



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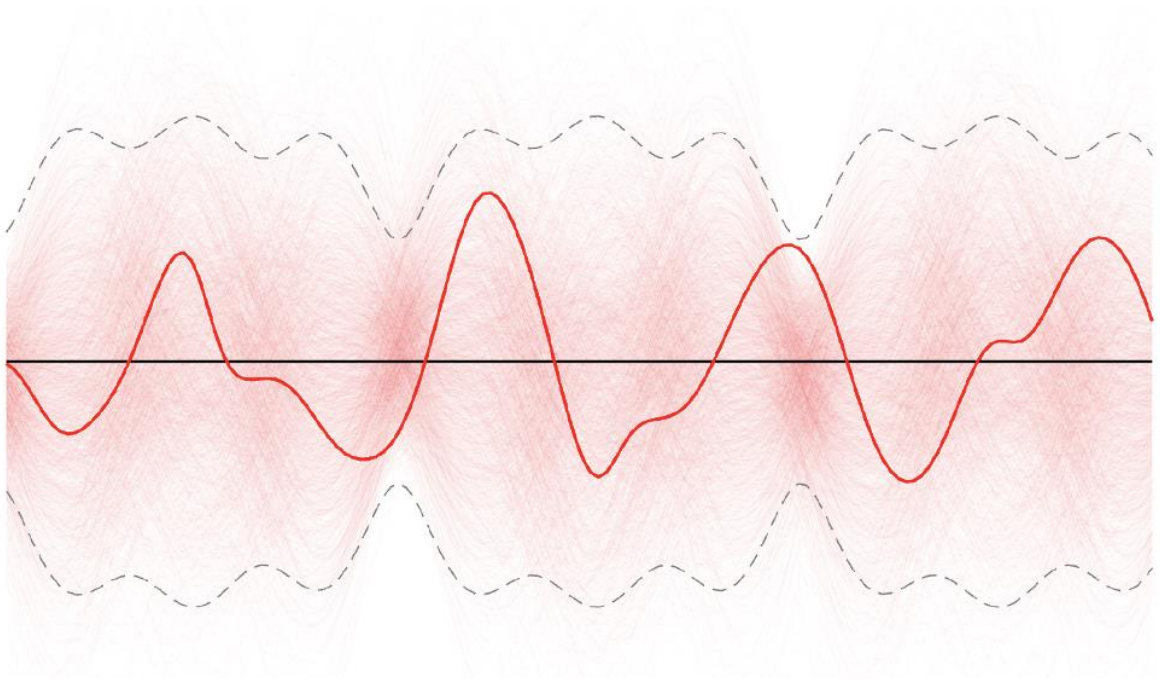
- PostDoc @ University of Bristol:
  - Real Time Hybrid testing

Bio

- PhD @ Imperial College London:
  - ROM of Large Scale Simulation
- PostDoc @ University of Bristol:
  - Real Time Hybrid testing
- Senior Lecturer @ University of Exeter:
  - UQ & ROM from data

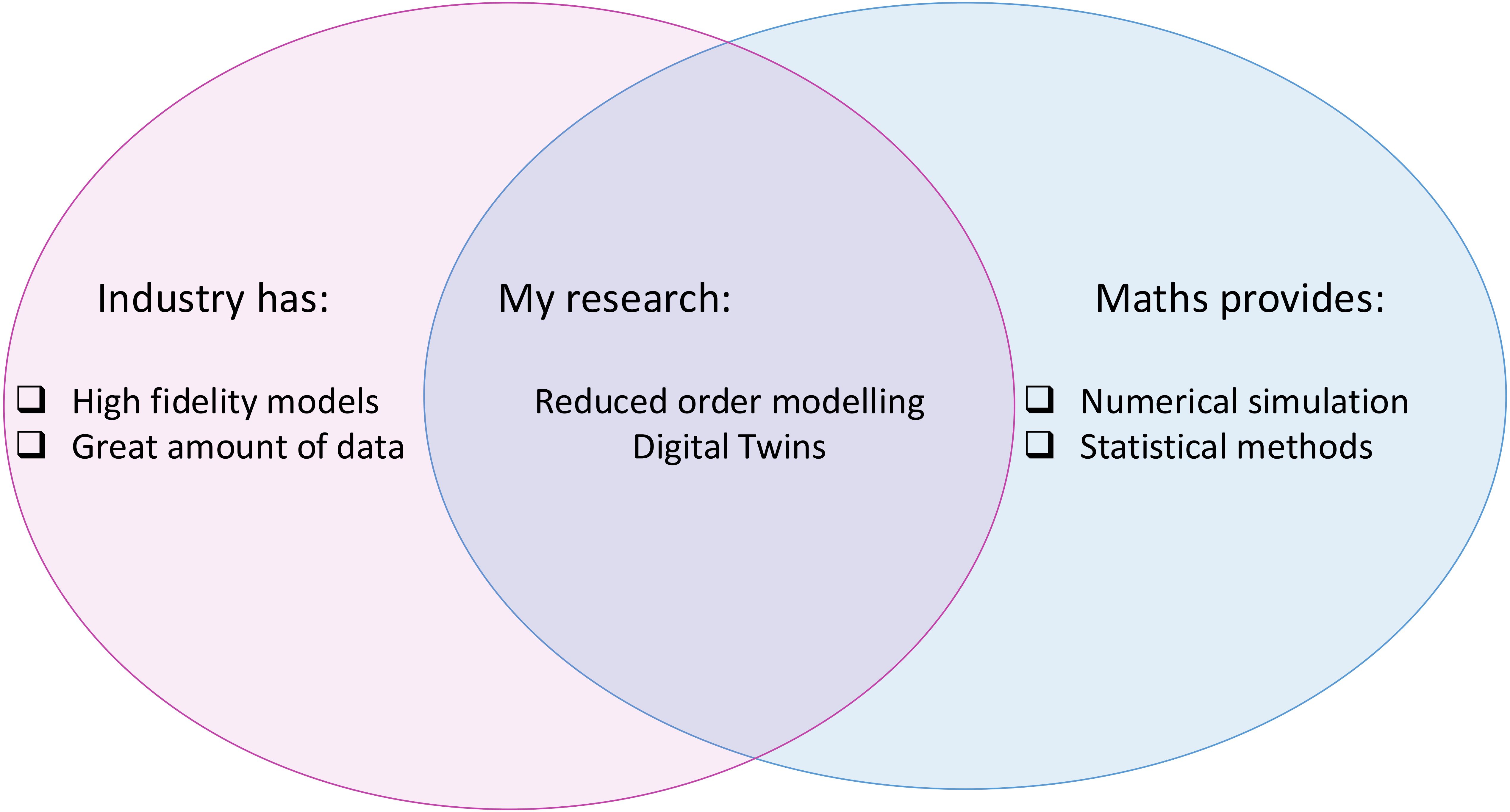


New era of engineering

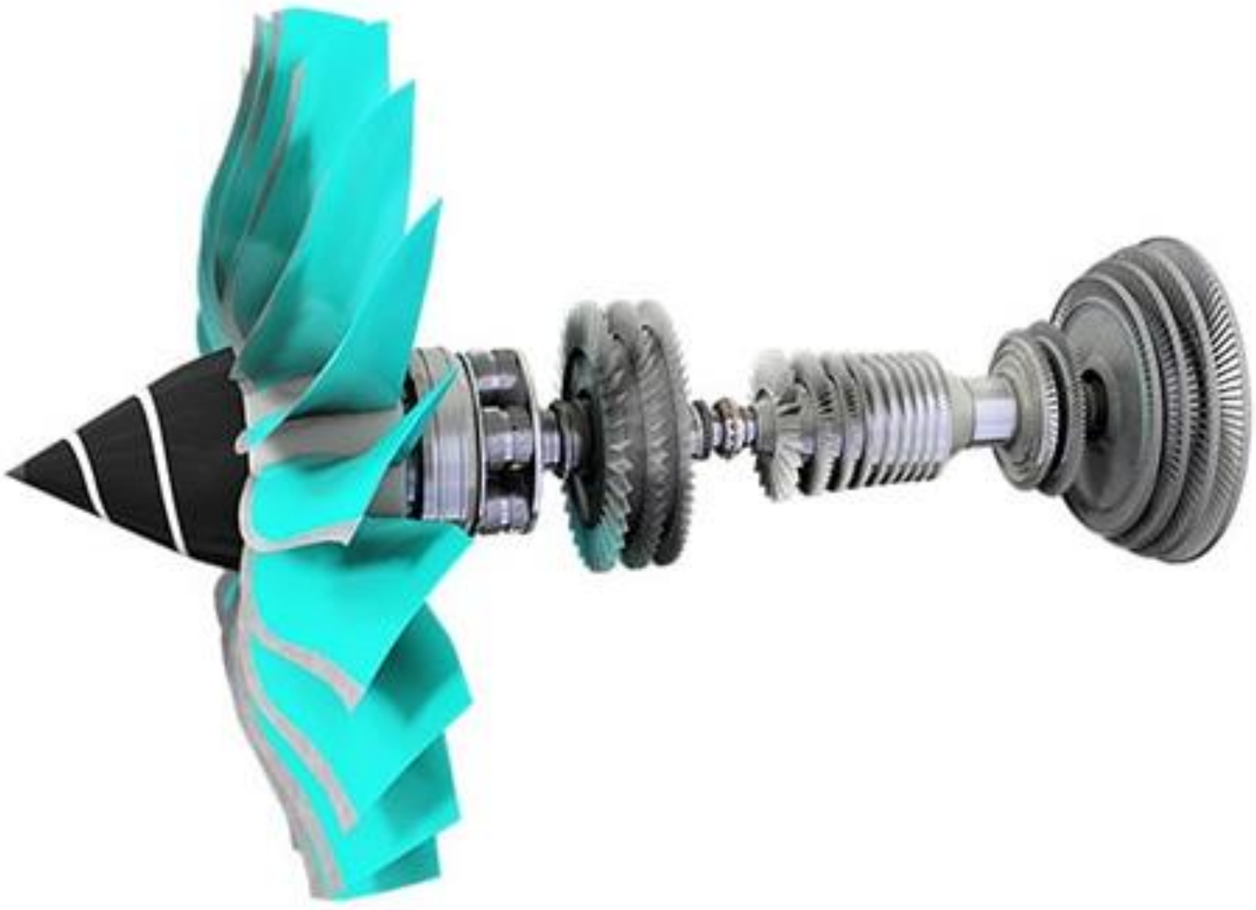
Unprecedented Computational Power

Better Sensing and monitoring

Artificial intelligence and decision making


Industry has:
High fidelity models
Great amount of data
My research:
Reduced order modelling
Digital Twins
Maths provides:
Numerical simulation
Statistical methods

Simulation-free ROMs



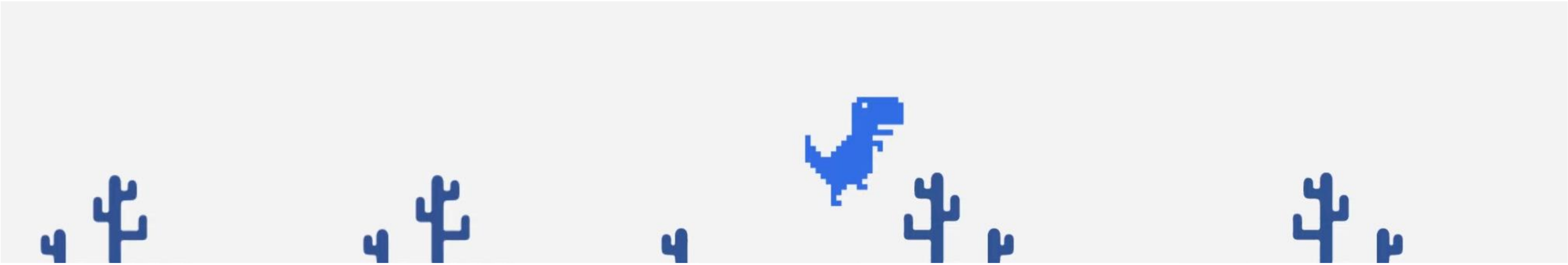
- ❑ Design stage
- ❑ Virtual testing
- ❑ Design Sensors
- ❑ Monitoring

Physics based models won't disappear, but we need better!

Equation based ROMs:

- ✓ Manageable
- ✓ Interpretable
- ✓ Coupled with data

When should we use ROMs?



High fidelity Models:
Large number of DOFs

Run **full** dynamic simulations:
Hours-days

Hard to accelerate computationally

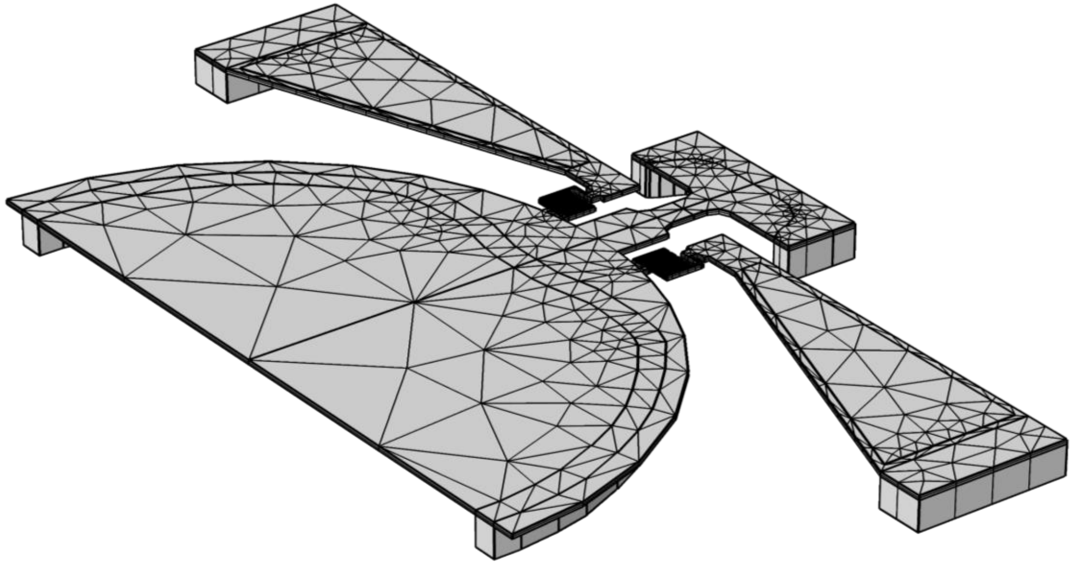
The dynamics live in a much lower dimensional space

With **ROM**:
Seconds-minutes

Accurate & fast surrogate

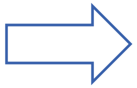
Main Idea

Large scale **3D** finite elements model



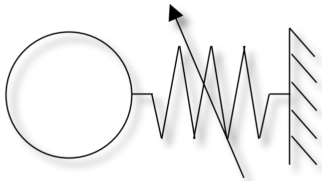
Nonlinear Dynamical system
with up to **Millions** of degrees of freedom

$$\dot{y} = F(y), \qquad y \in \mathbb{R}^N$$



Capture the **characteristic** dynamics
with **low dimensional** models

$$\dot{z} = f(z) \qquad z \in \mathbb{C}^n, \qquad n \ll N$$



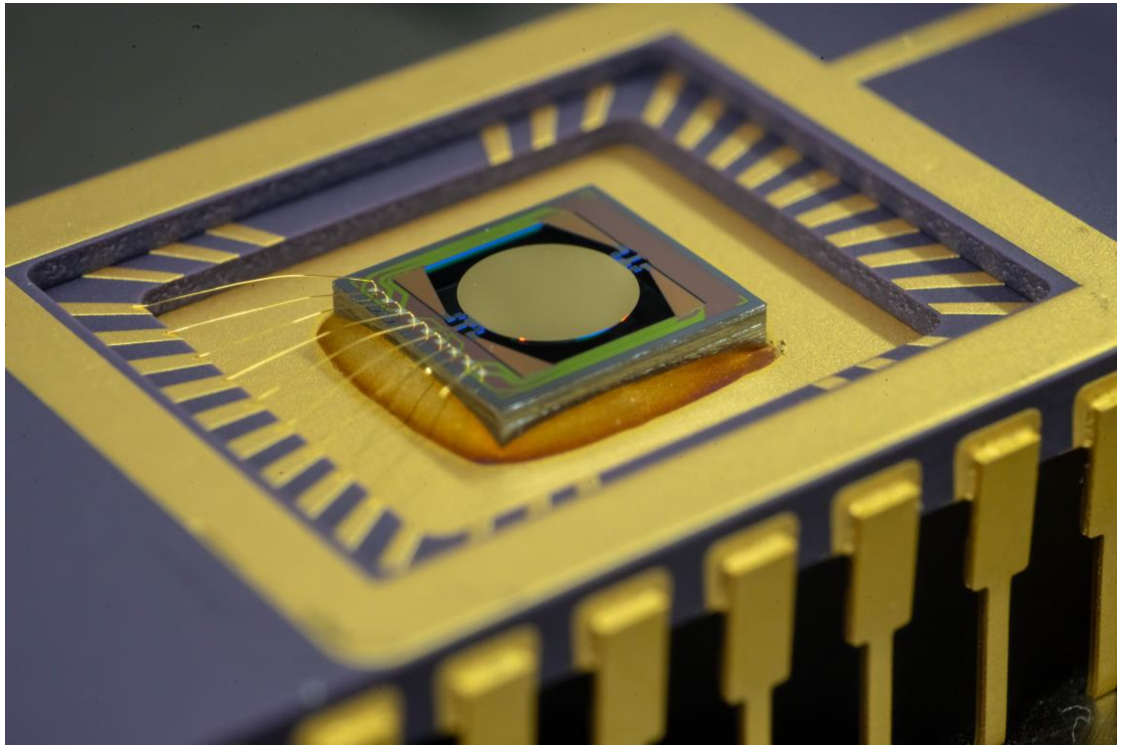
Characteristics of ROMs:

- **Small** number of parameters
- Physically **interpretable**
- Manageable
- **Fast** yet **accurate**

Example of Application: geometric nonlinearities

Slender structures in large vibrations

High precision devices in large rotation



Outline

Nonlinear ROM

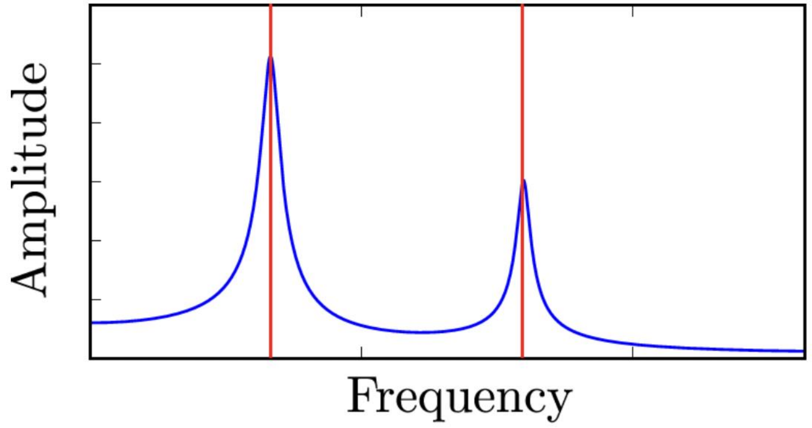
- Introduction
- Parametrisation Method
- Applications

The simplest ROM

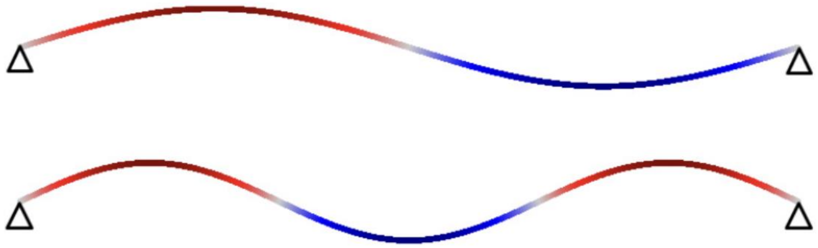
Large scale **3D** finite elements model with up to **millions** of degrees of freedom:

$$\mathbf{M\ddot{U} + KU = 0}$$

Natural frequencies $\boldsymbol{\omega_0}$



Mode shapes $\boldsymbol{\phi_0}$



Change of Coordinates	$\mathrm{U} = \boldsymbol{\phi_0}\mathrm{u}$
Reduced Dynamics	$\ddot{\mathrm{u}} + \boldsymbol{\omega_0}^2\,\mathrm{u} = 0$

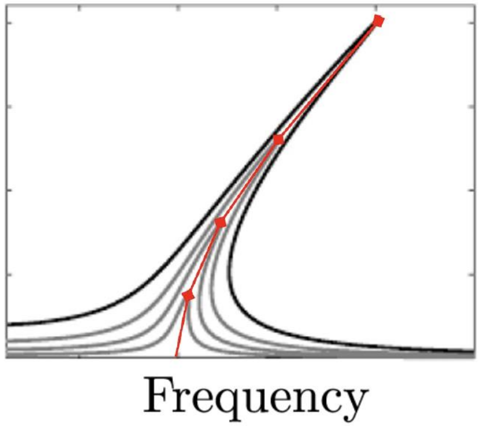
Linear Modal Analysis is the most trivial of ROMs

Nonlinear case

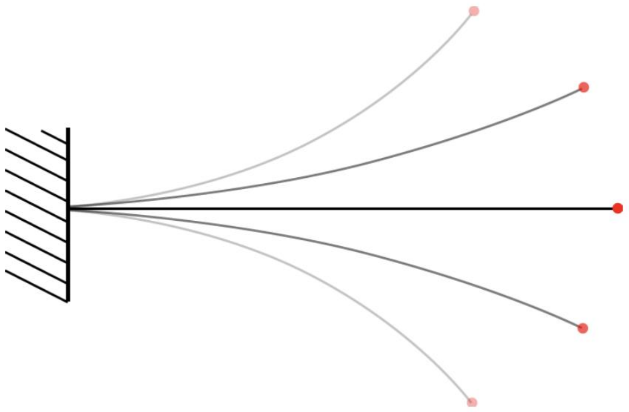
Large scale **3D** finite elements model with up to **millions** of degrees of freedom:

$$\mathbf{M\ddot{U}} + \mathbf{KU} + \mathbf{G(U,U)} + \mathbf{H(U,U,U)} = \mathbf{0}$$

Natural frequencies $\omega_{\mathrm{nl}} = \omega_{\mathrm{nl}}(a)$



Mode shapes $\phi_{\mathrm{nl}} = \phi_{\mathrm{nl}}(a)$



Change of Coordinates	$\mathrm{U} = \Psi(\mathrm{u}, \dot{\mathrm{u}})$
Reduced Dynamics	$\ddot{\mathrm{u}} + {\omega_{\mathrm{n}}}^2\, \mathrm{u} + f(\mathrm{u}, \dot{\mathrm{u}}) = 0$

The two main ingredients of every Nonlinear ROM

Outline

Nonlinear ROM

- Introduction
- Parametrisation Method
- Applications

Linear case

Nonlinear Dynamical system
with up to **Millions** of degrees of freedom

$$\dot{y} = F(y), \qquad y \in \mathbb{R}^N$$

Equilibrium

$$F(0) = 0$$

Linearisation at equilibrium

$$\nabla F(0) = A$$

Dissipative system: $\mathrm{Re}[\lambda_N] \leq \mathrm{Re}[\lambda_{N-1}] \leq \ \ldots \leq \mathrm{Re}[\lambda_1] < 0$

Linear case

$$\dot{y} = A\, y$$

Slow eigenvector

$$A\, V_1 = \lambda_1 V_1$$

Coordinates on subspace

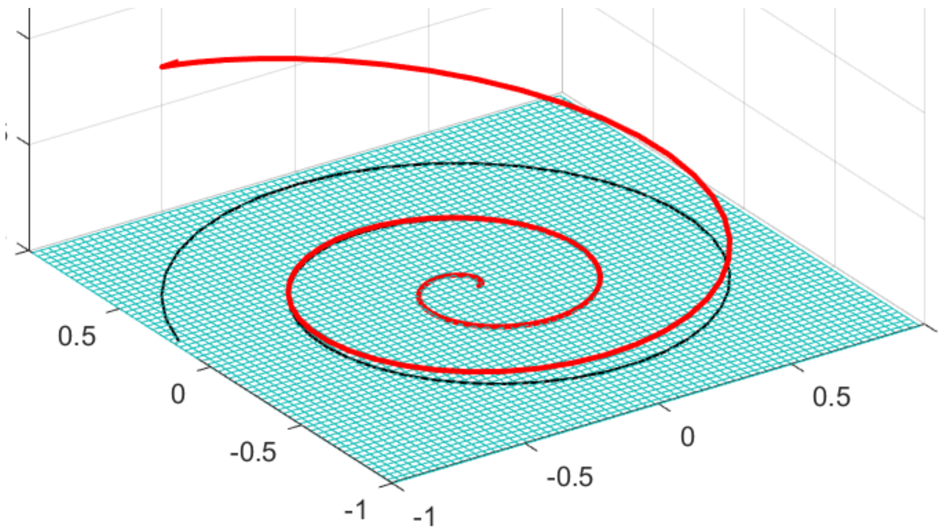
$$y = V_1\, x$$

Projection on subspace

$$\mathrm{V}_{L1}^T \mathrm{V}_1\, \dot{x} = V_{L1}^T A\, V_1\, x$$

Reduced dynamics

$$\dot{x} = \lambda_1 x$$

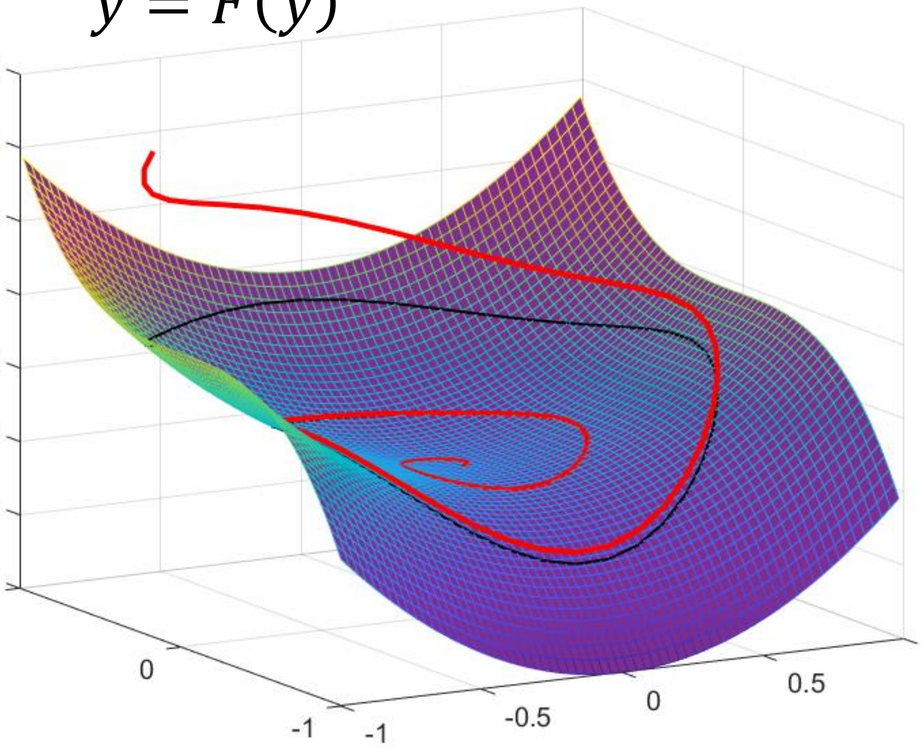


Parametrisation method for Slow Invariant manifold

Invariant manifold tangent to linear slow subspace

- ❑ Sound ROM

$$\dot{y} = F(y)$$

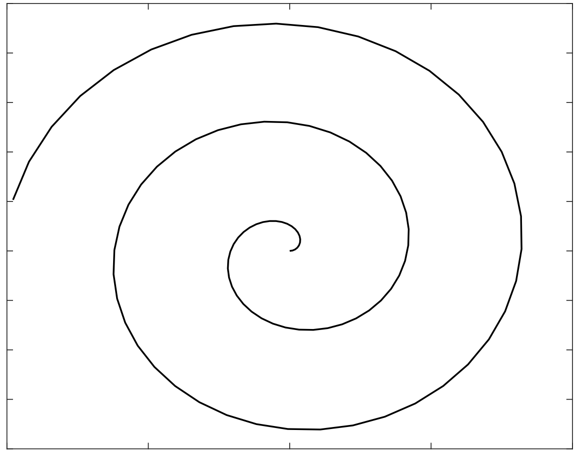


$$y = \Psi(z)$$

$$\longleftarrow$$

$$\nabla\Psi(z)\, f(z) = F(\Psi(z))$$

$$\dot{z} = f(z)$$



Parametrisation Method

$$\dot{x} = \check{F}(x)$$
$$\nabla\check{F}(0) = \mathrm{diag}[\lambda_1, \lambda_2, \dots, \lambda_N]$$

$$x = w(z)$$
$$\dot{z} = f(z)$$

$$\boldsymbol{y = V\,x}$$

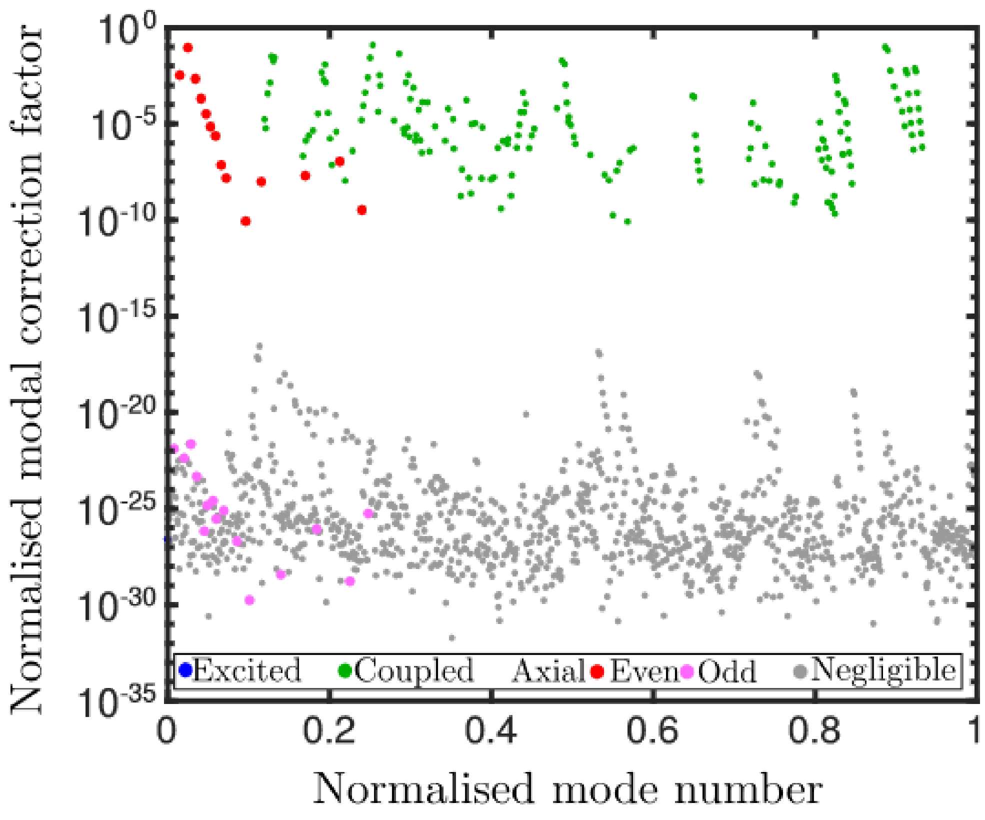
$$\dot{y} = F(y)$$
$$\nabla F(0) = A$$

$$y = Vw(z)$$
$$\dot{z} = f(z)$$

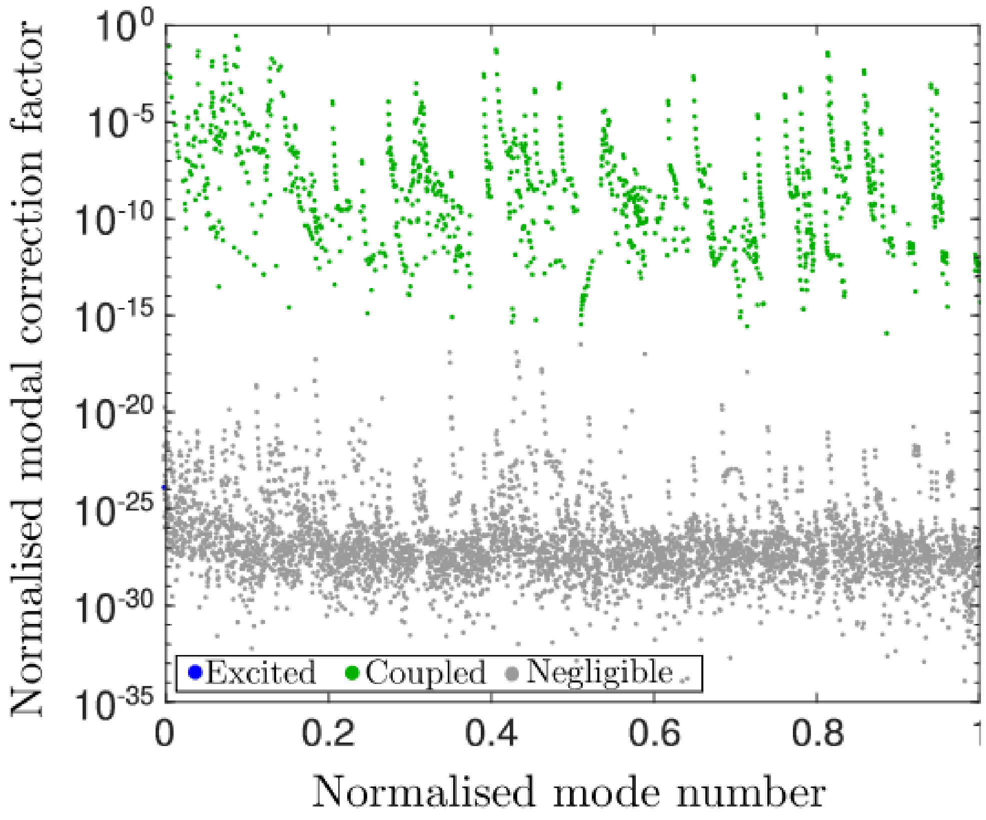
Limitations of Parametrisation Method for Large Scale FE models

1. Full eigenvector matrix computation
2. Sparsity of nonlinear tensor in physical coordinates

1. Limitation of Parametrisation Method



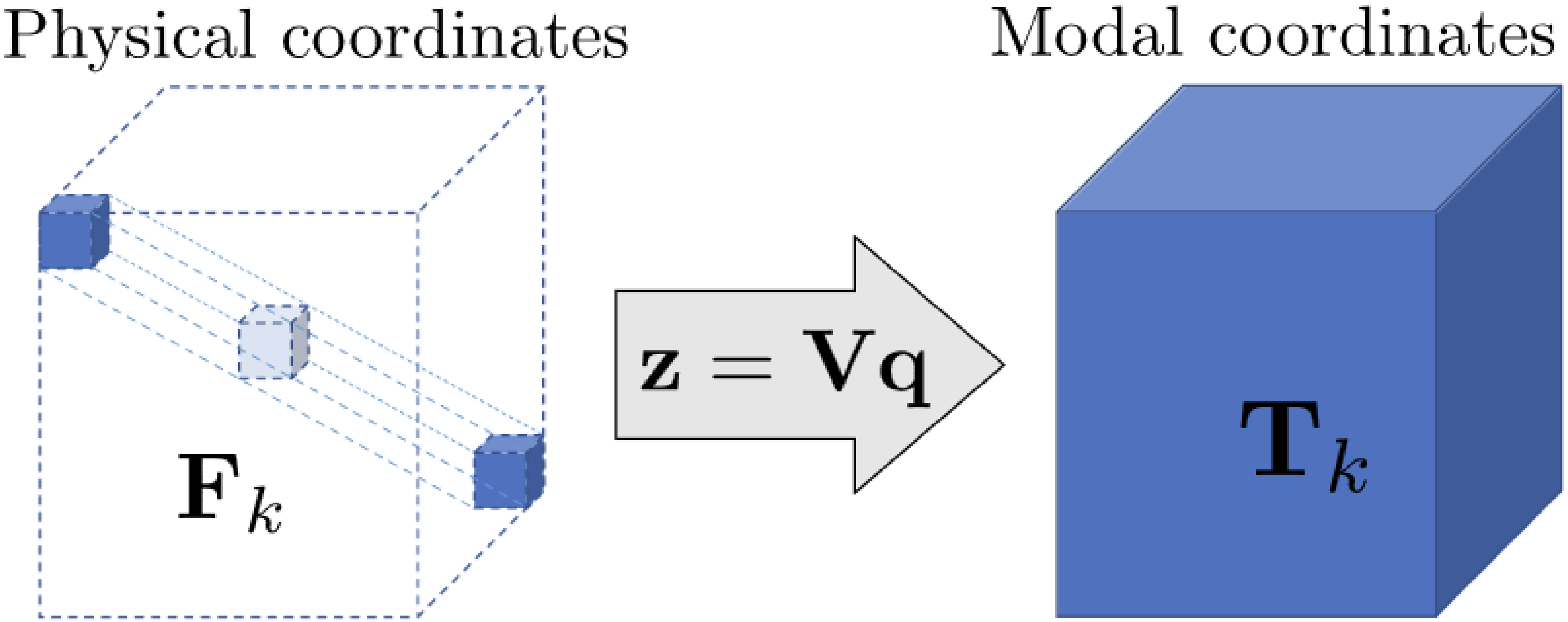
(a) Beam discretised with 1287 dofs



(b) Beam discretised with 5733 dofs

Non-intrusive reduced order modelling for the dynamics of geometrically nonlinear flat structures using three-dimensional finite elements. AV et al. ***Comput Mech, 2020***

2. Limitation of Parametrisation Method



$$\mathbf{F}(\mathbf{z}) = \sum_k \mathbf{F}_k \mathbf{z}^{\otimes k} \qquad \mathbf{T}(\mathbf{q}) = \sum_k \mathbf{T}_k \mathbf{q}^{\otimes k}$$

How to compute invariant manifolds and their reduced dynamics in high-dimensional finite element models. Jain and Haller. ***Nonlin Dyn, 2022***

Direct Parametrisation Method

$$\dot{x} = \check{F}(x)$$
$$\nabla\check{F}(0) = \mathrm{diag}[\lambda_1, \lambda_2, \dots, \lambda_N]$$

$$x = w(z)$$
$$\dot{z} = f(z)$$

$$\boldsymbol{y = V\,x}$$

Direct

$$\dot{y} = F(y)$$
$$\nabla F(0) = A$$

$$y = \Psi(z)$$
$$\dot{z} = f(z)$$

- ❑ Full eigenvector matrix computation
- ❑ Sparsity of nonlinear tensor in physical coordinates

Direct Parametrisation of Invariant Manifolds (DPIM)

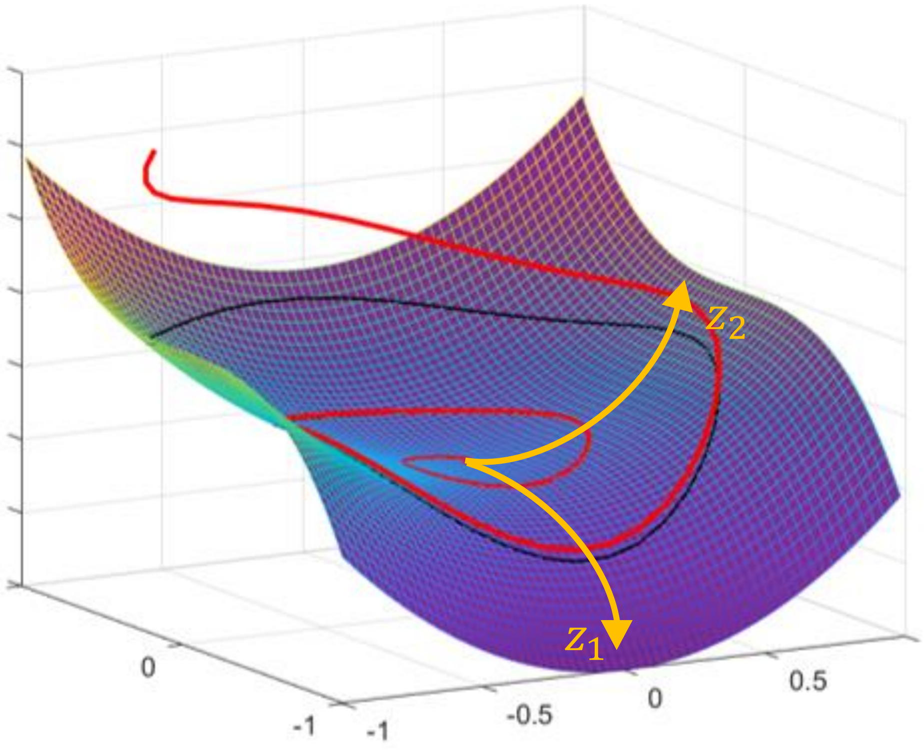
$$\dot{\boldsymbol{y}} = \boldsymbol{F}(\boldsymbol{y})$$

- **Parametrisation** of manifold as an embedding

$$\boldsymbol{y} = \boldsymbol{\Psi}(\mathbf{z})$$

- **Reduced dynamics** on it (at the same time)

$$\dot{\boldsymbol{z}} = \mathbf{f}(\mathbf{z})$$



- Both as polynomials

$$\boldsymbol{\Psi}(\mathbf{z}) = \boldsymbol{\Psi}^{(1)}_{i_1} z_{i_1} + \boldsymbol{\Psi}^{(2)}_{i_1 i_2} z_{i_1} z_{i_2} + \ldots + \boldsymbol{\Psi}^{(p)}_{i_1 i_2 i_3 .. i_p} z_{i_1} z_{i_2} z_{i_3} \ldots z_{i_p}$$

$$\mathbf{f}(\mathbf{z}) = \mathbf{f}^{(1)}_{i_1} z_{i_1} + \mathbf{f}^{(2)}_{i_1 i_2} z_{i_1} z_{i_2} + \ldots + \mathbf{f}^{(p)}_{i_1 i_2 i_3 .. i_p} z_{i_1} z_{i_2} z_{i_3} \ldots z_{i_p}$$

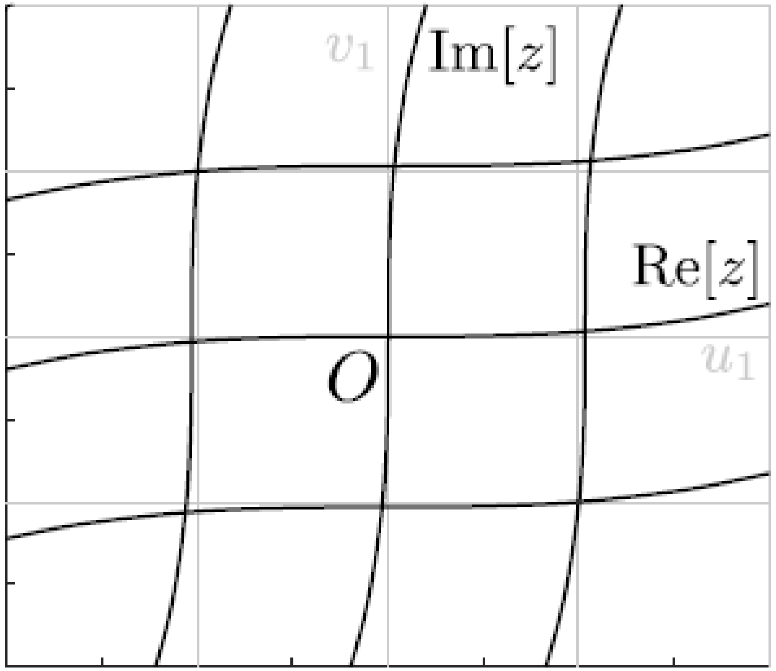
Linear monomials

$$\mathbf{\Psi}(\mathbf{z}) = \mathbf{\Psi}_{i_1}^{(1)} z_{i_1}$$

$$\dot{\mathbf{z}} \quad = \quad \mathbf{f}_{i_1}^{(1)} z_{i_1}$$

$$I = \{i_1\}$$

$$\left(A - \lambda_{i_1} \mathrm{Id}\right) \mathbf{\Phi}_{i_1} = \mathbf{0}$$



- **Linear** homological = eigenproblem
- Tangency to **master** subspace

Higher order monomials

$$\boldsymbol{\Psi}(\mathbf{z}) = \boldsymbol{\Phi}_{i_1} z_{i_1} + \boldsymbol{\Psi}^{(2)}_{i_1 i_2} z_{i_1} z_{i_2} + \ldots + \boldsymbol{\Psi}^{(p)}_{i_1 i_2 i_3 \ldots i_p} z_{i_1} z_{i_2} z_{i_3} \ldots z_{i_p}$$

$$\dot{\mathbf{z}} = \lambda_{i_1} z_{i_1} + \mathbf{f}^{(2)}_{\cancel{i_1} i_2} z_{i_1} z_{i_2} + \ldots + \mathbf{f}^{(p)}_{i_1 \cancel{i_2} i_3 \ldots i_p} z_{i_1} z_{i_2} z_{i_3} \ldots z_{i_p}$$

$$I = \{i_1 i_2 i_3 \ldots i_p\}$$

$$\sigma_I = \lambda_{i_1} + \lambda_{i_2} + \lambda_{i_2} + \ldots + \lambda_{i_p}$$

Invariance Equation:

$$\boldsymbol{\nabla \Psi}(\boldsymbol{z})\, \boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{F}(\boldsymbol{\Psi}(\boldsymbol{z}))$$

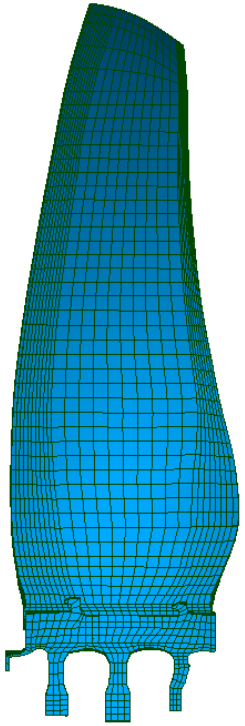
Homological Equation:

$$\boldsymbol{\Psi}^{(p)}_I \underline{\sigma_I = A\, \boldsymbol{\Psi}^{(p)}_I} + \boldsymbol{\Xi}^{(p)}_I$$

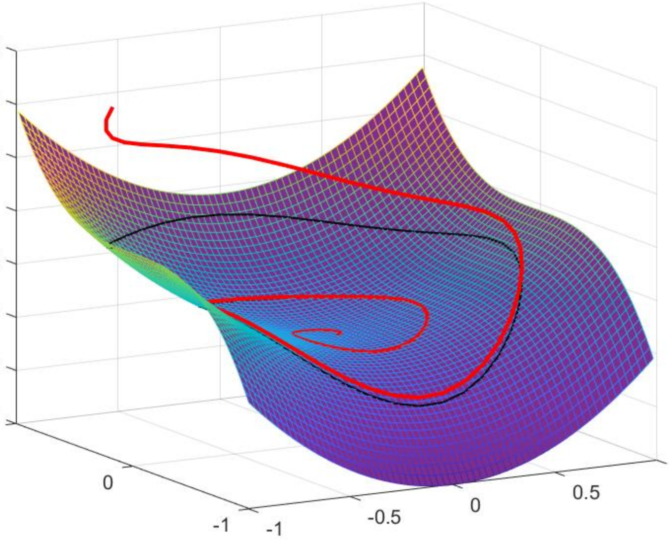
Singular if: $\sigma_I = \lambda_r$

Parametrisation method

A priori method for ROM of large-scale FE models

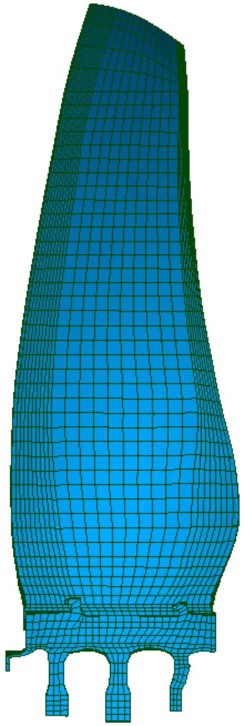


$$\dot{y} = F(y)$$



Invariant Manifold embedding through parametrisation

$$y = \Psi(z) \qquad \dot{z} = f(z)$$



Model Order Reduction in Julia

Imperial College London

ENSTA IP PARIS

POLITECNICO MILANO 1863

University of Exeter

AV et al., *CMAME* (2021)
Opreni et al., *Nonlin Dyn* (2021)
AV et al., *Nonlin Dyn* (2022)
Opreni et al., *Nonlin Dyn* (2022)
AV et al., *Nonlin Dyn* (2023)

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Nonlinear ROM

- Introduction
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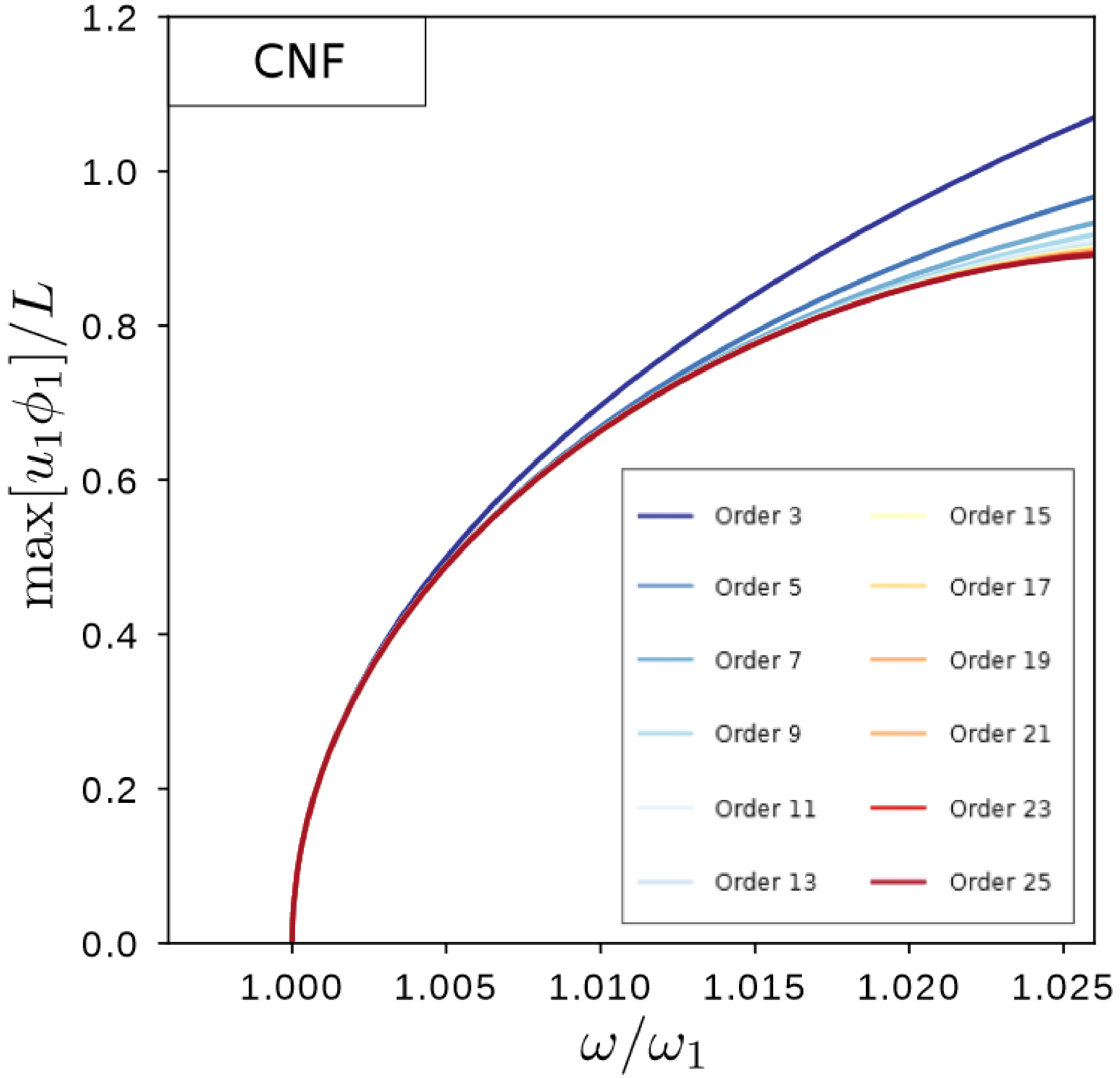
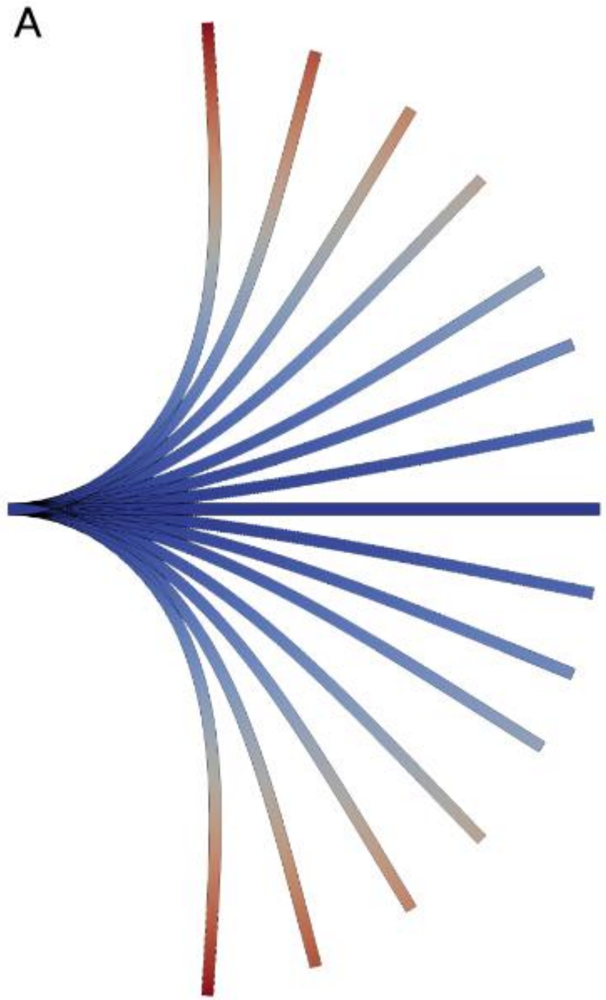
- Applications

Automated Higher Order Expansion

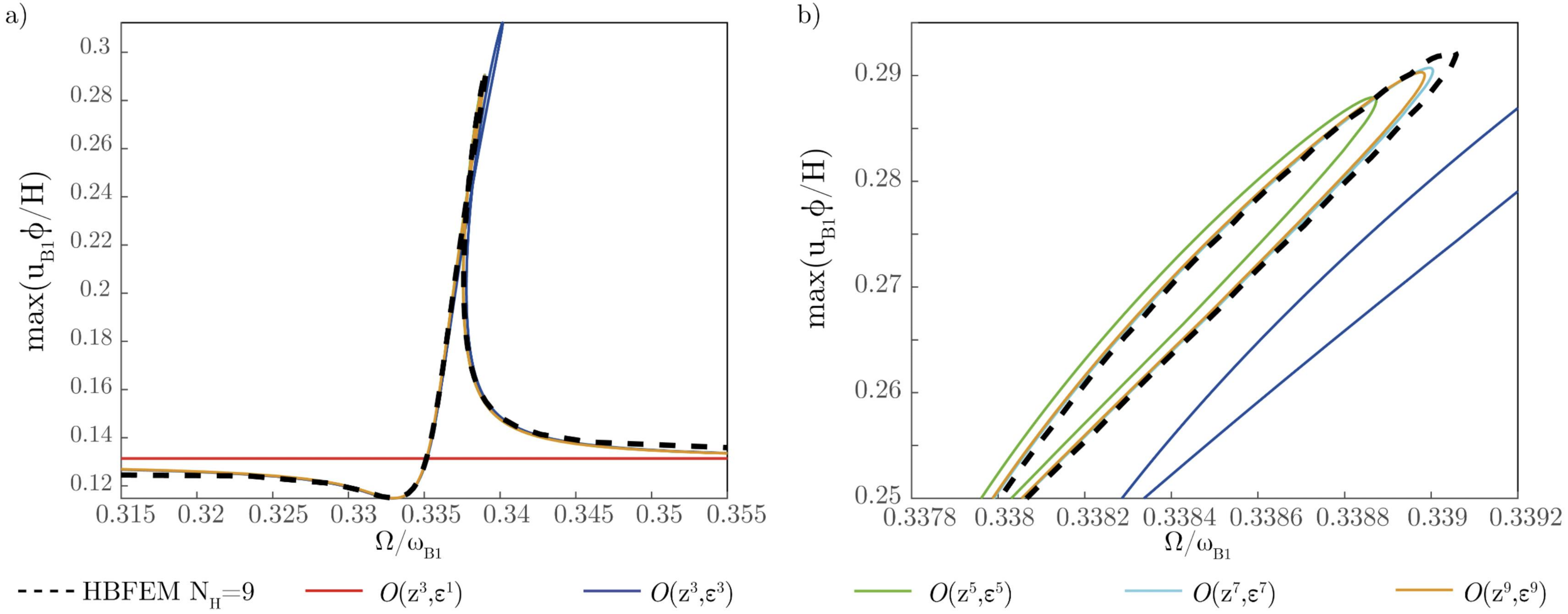
Cantilever Beam in Large amplitude Vibration

Convergence with the Order

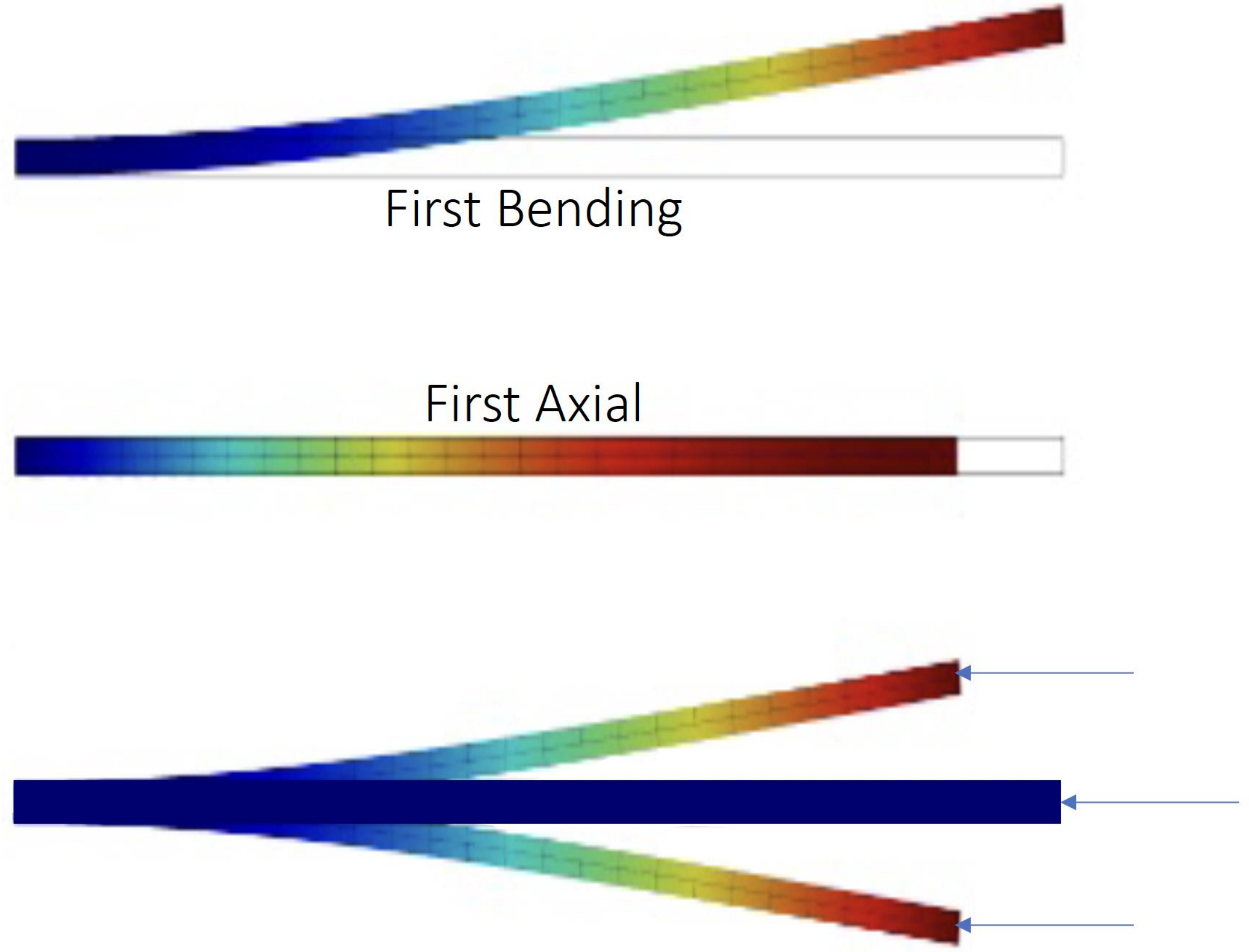


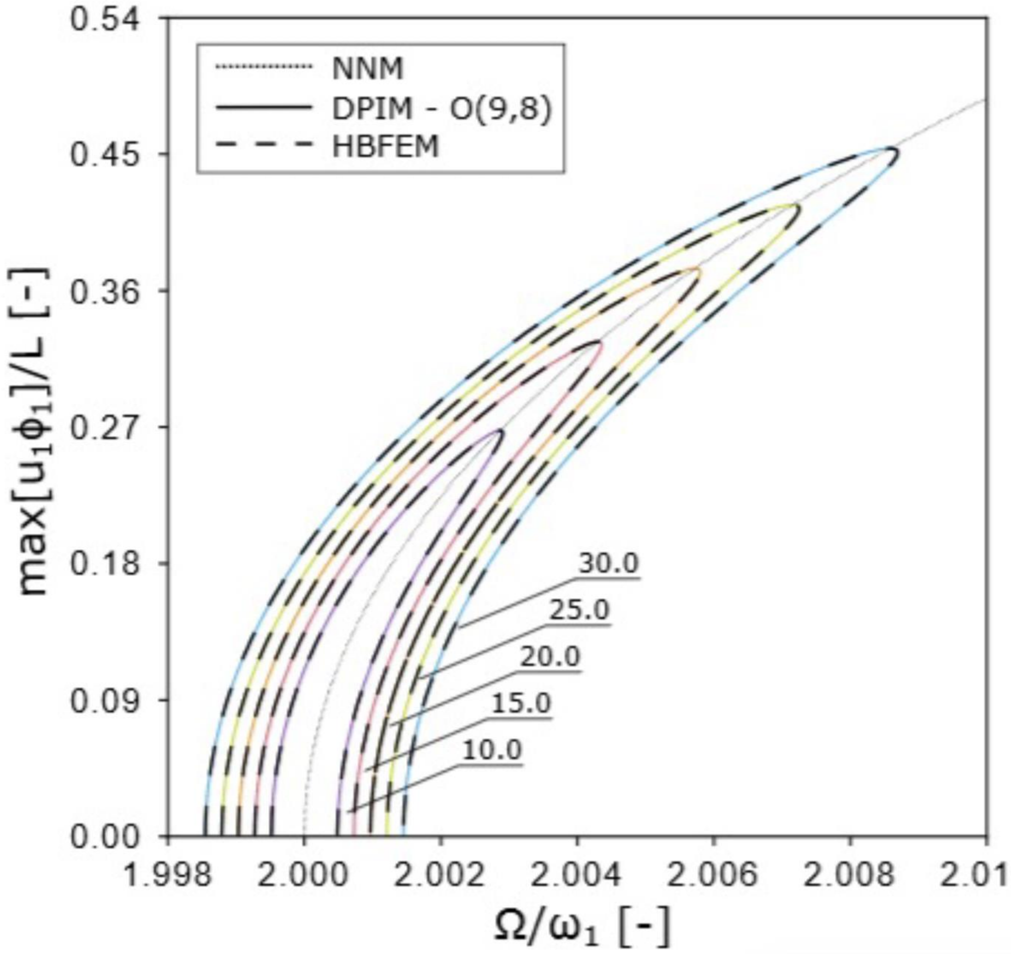


Application to super-harmonic resonance



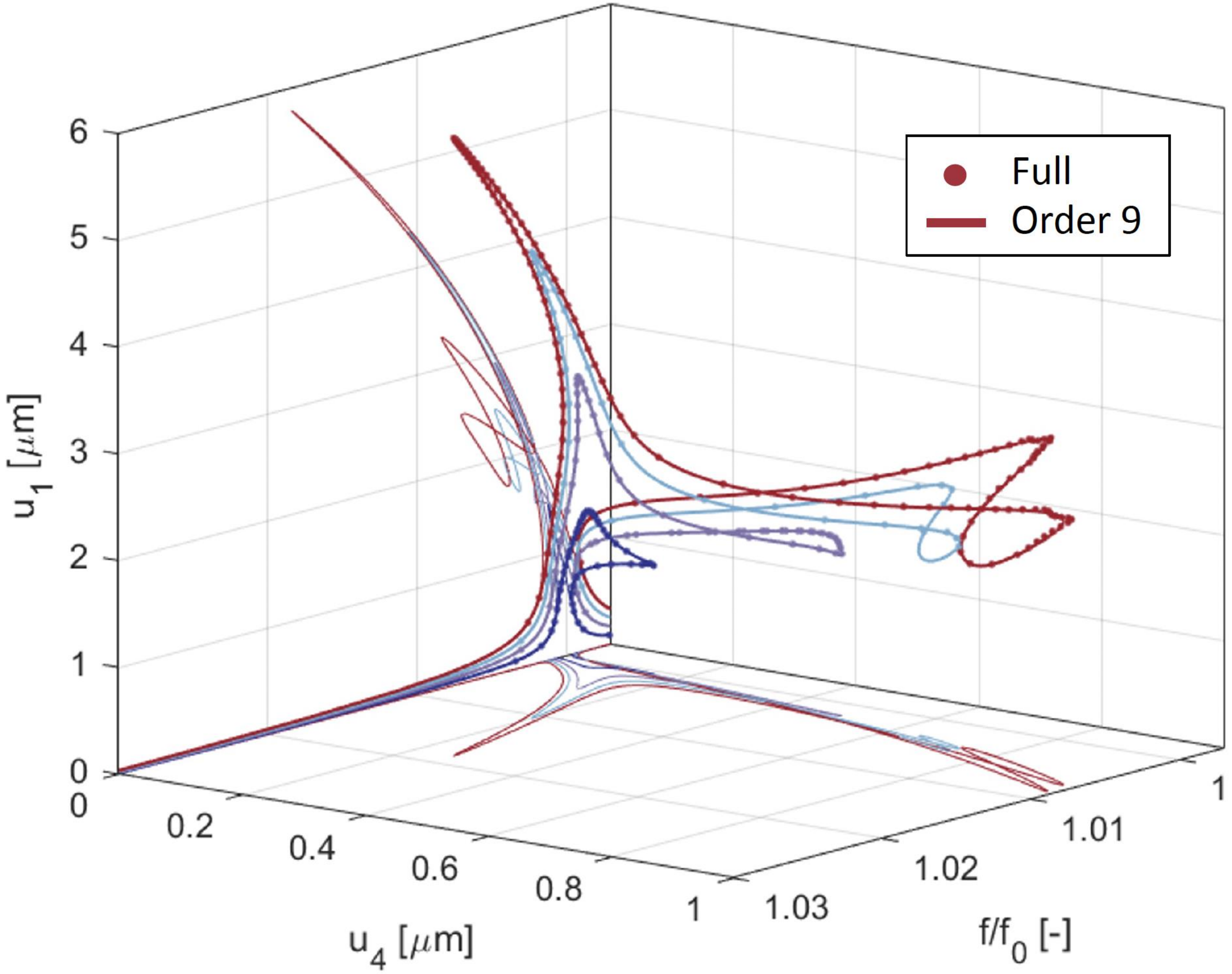
Application to parametric resonance





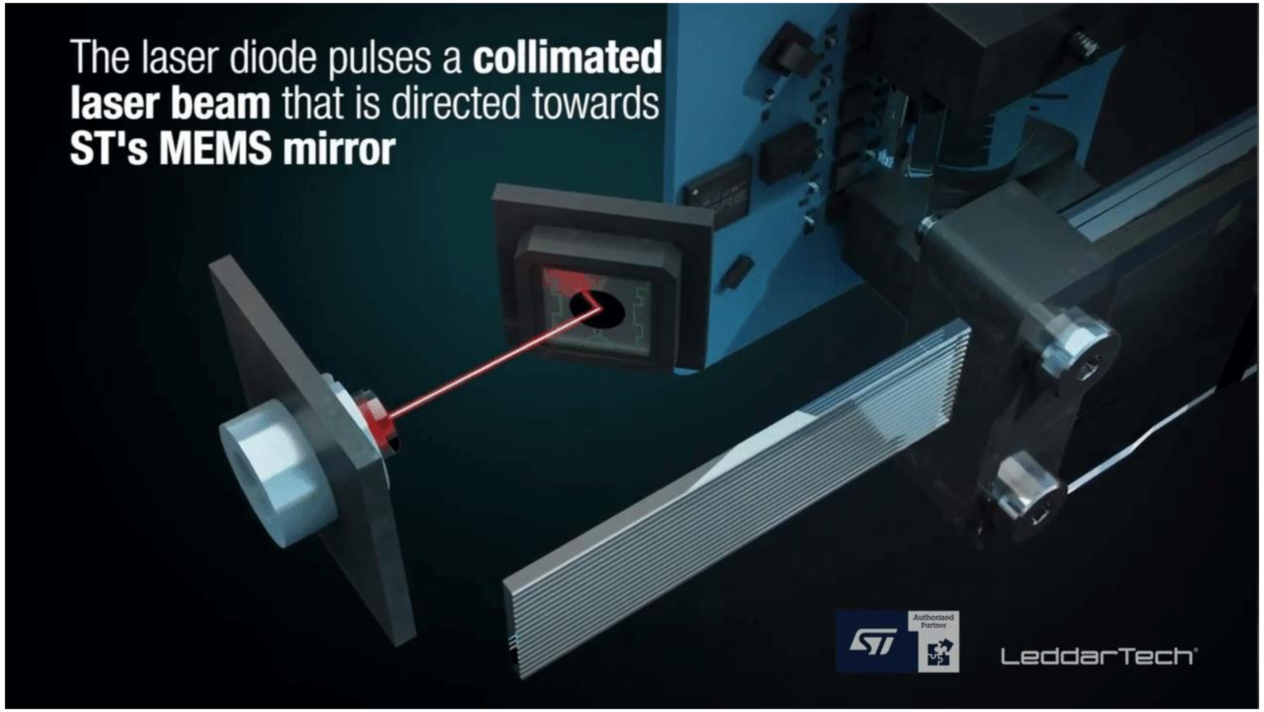
Application to internal resonance

1:3 internal resonance
between modes
captured
extremely accurately
by 2 oscillators ROM

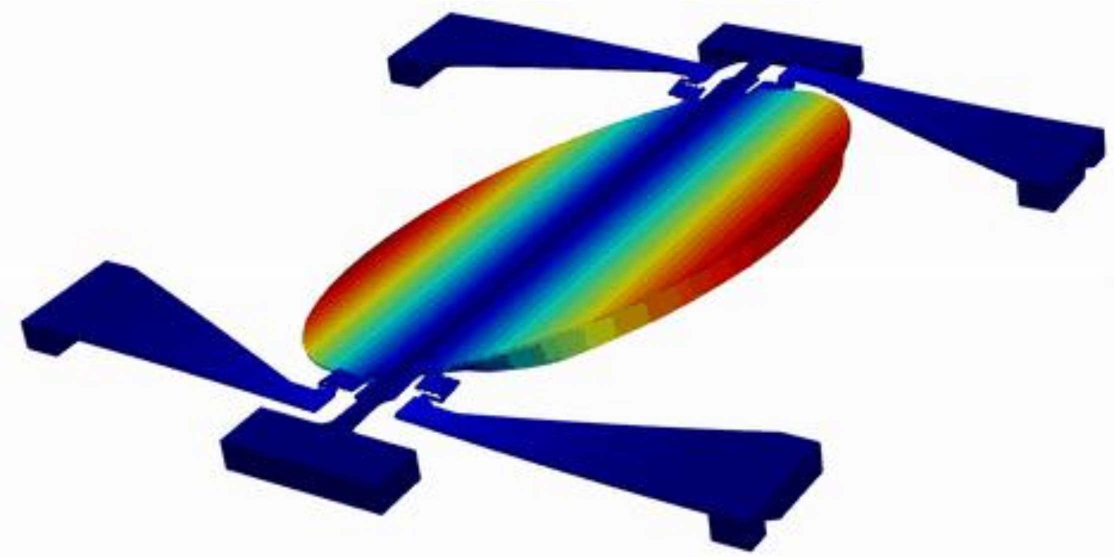


Industrial Application: MEMS micromirror

High precision device for LeddarTech

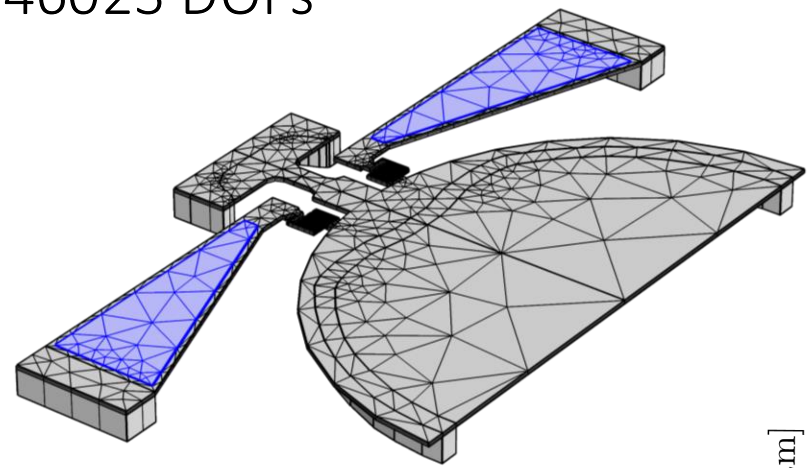


First Linear Mode at frequency ≈ 2 kHz



Industrial Application: MEMS micromirror

46023 DOFs

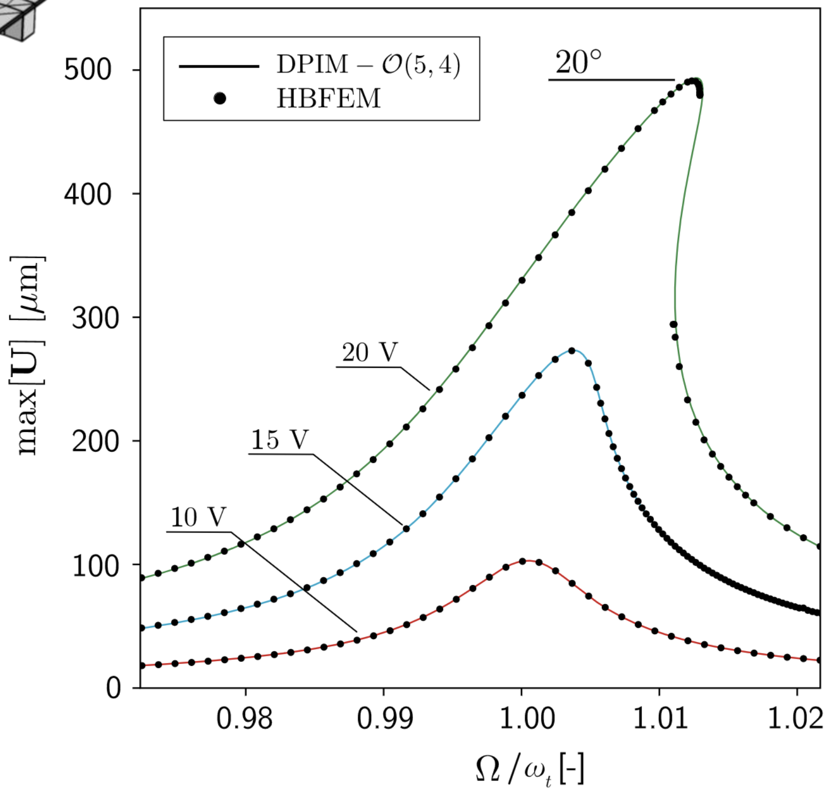


Time ROM:
1.5 minutes

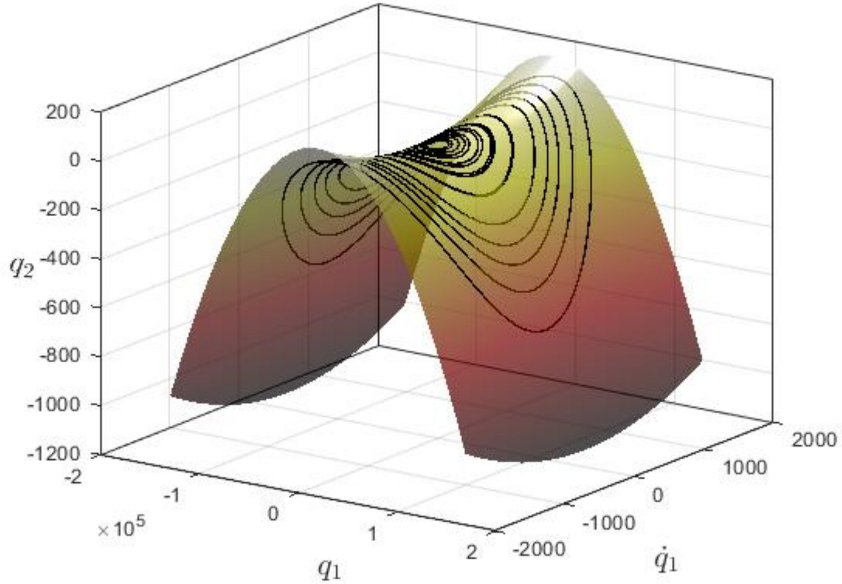
Time Full:
2 days

Dynamics of a single oscillator:

$$\ddot{R} + \zeta\omega_1\dot{R} + {\omega_1}^2 R + A\,R^3 + B\,R\dot{R}^2 = F$$

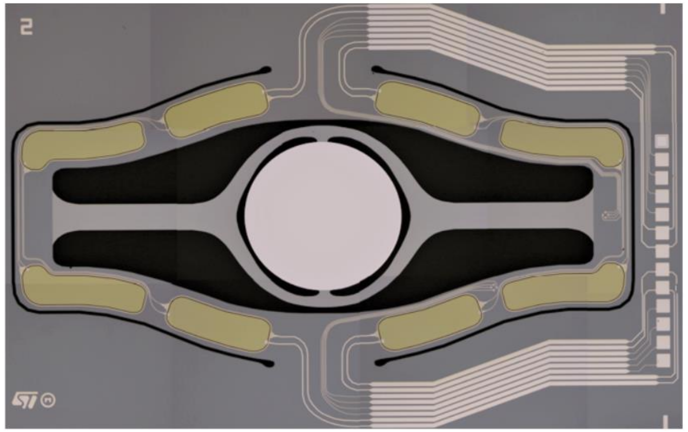


2-dimensional manifold

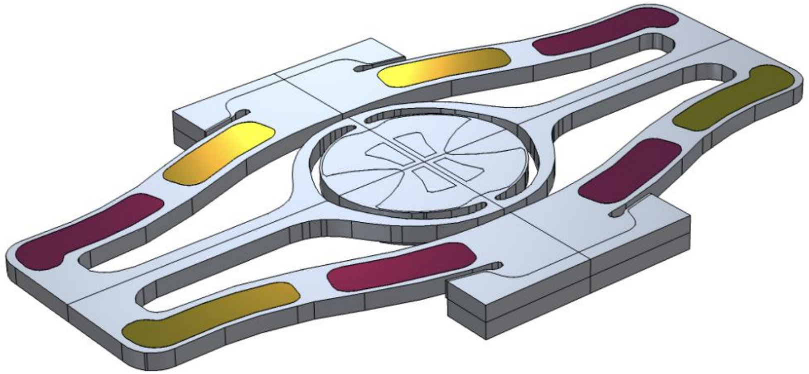


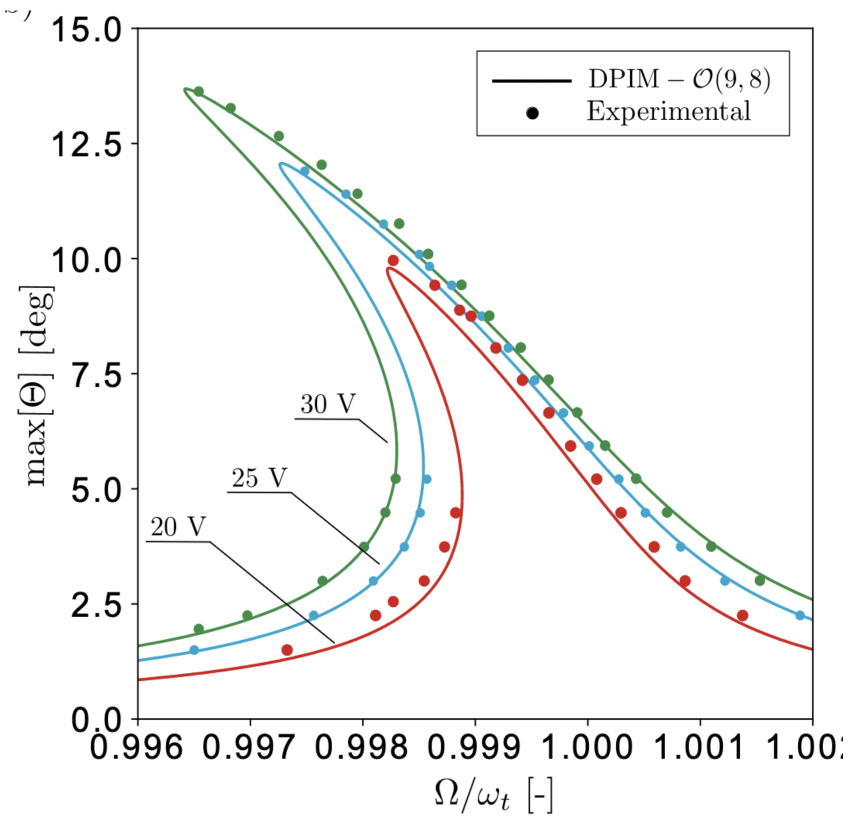
Industrial Application: MEMS micromirror

c)



d)



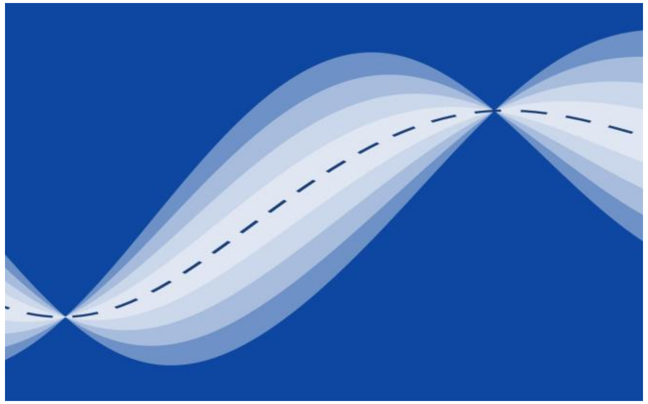


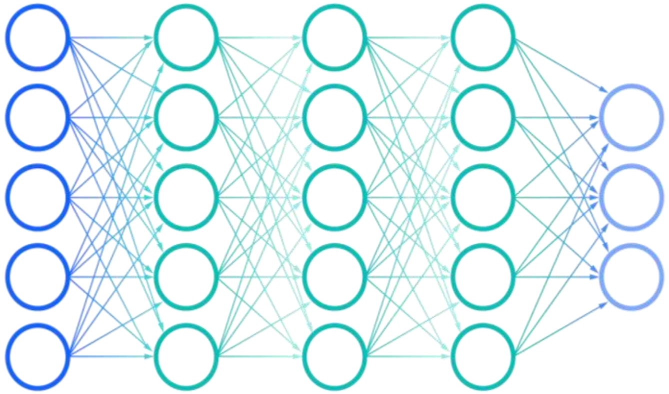
Still a lot to do

ROM used for:
- Parametric ROMs
- Sensitivity & UQ
- Design optimisation

Coupled with data:
- Physics informed
- Bayesian inference
- Better Data (DoE)

Whole Framework:
- Complex Systems
- Composability
- Digital Twins





Thank you for your attention