Inter-modal interactions in a chain of mass-in-mass non-linear oscillators

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Introduction

Systems for vibro-acoustical control

Linear systems (example: TMD[1]) :

- Efficient on a narrow band of frequencies
- Modify system characteristics

Design and introduction of non-linearity [2] (example: nonlinear energy sink [3]) :

- Efficient on wider bands of frequency

Design and exploit nonlinearities in metamaterials

[1] Frahm, 1911

[2] Roberson, 1952

[3] Vakakis et Gendelman, 2001

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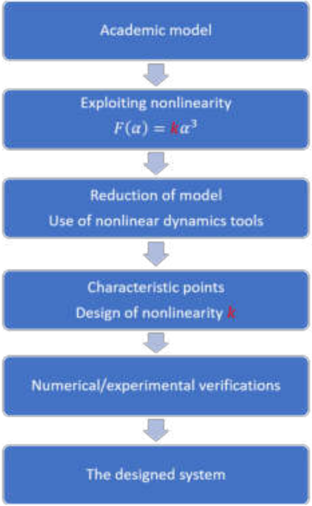
General methodology



Continuous approximation of the chain

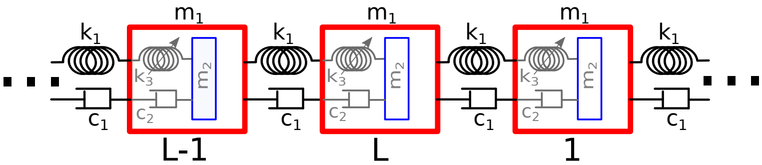


Figure: L-periodic chain composed by cubic nonlinear mass-in-mass cells

$$\begin{cases} \dfrac{\partial^2 U}{\partial \tau^2}(x,\tau) - \dfrac{\partial^2 U}{\partial x^2}(x,\tau) + \varepsilon \Lambda V^3(x,\tau) - \varepsilon \chi_1 \dfrac{\partial}{\partial \tau}\dfrac{\partial^2 U}{\partial x^2}(x,\tau) + \varepsilon \chi_2 \dfrac{\partial V}{\partial \tau}(x,\tau) = \varepsilon f(x,\tau) \\ \varepsilon \left(\dfrac{\partial^2 (U - V)}{d\tau^2}(x,\tau) - \Lambda V^3(x,\tau) - \chi_2 \dfrac{\partial V}{\partial \tau}(x,\tau) \right) = 0 \qquad \text{with } x \in [0 : L] \end{cases}$$

- $\varepsilon = \dfrac{m_2}{m_1} \ll 1$: mass ratio
- $\tau = \sqrt{k_1/m_1}\, t$: non-dimensional time variable
- $\varepsilon\Lambda = \dfrac{k_3}{k_1}$, $\varepsilon\chi_1 = \dfrac{c_1}{\sqrt{k_1 m_1}}$, $\varepsilon\chi_2 = \dfrac{c_2}{\sqrt{k_1 m_1}}$ and $\varepsilon f_j = \dfrac{F_j}{k_1}$

Dispersion equation

Linear conservative associated system:

$$\frac{\partial^2 U_l}{\partial \tau^2}(x,\tau) - \frac{\partial^2 U_l}{\partial x^2}(x,\tau) = 0$$

$$\frac{\partial^2 (U_l - V_l)}{\partial \tau^2}(x,\tau) = 0$$

Harmonic decomposition of periodic solutions:

$$U_l(x,\tau) = h_j(x) g_j(\tau)$$

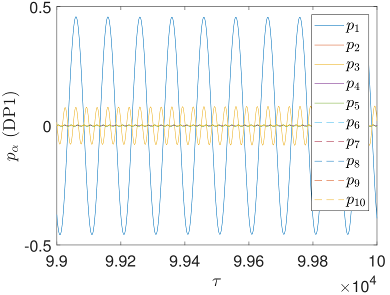
Expression of linear modes of the chain:

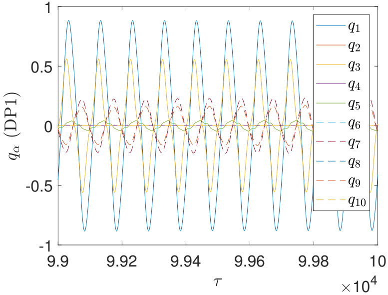
$$h_j(x) = \sqrt{\frac{2}{L}} \cos(\omega_j x + \theta_j)$$

$$\omega_j = \frac{2j\pi}{L}, \quad j = 1, \dots$$

Observation of modal exchanges due to non-linearity

Discrete system under first mode forcing associated to $f_1 = 0.004\sin(\omega_1\tau)$:





We investigate the $m : n$ inter-modal resonance ($m = 3n$) and define:

$$f(x,\tau) = f_n h_n(x)\sin(\nu_n\tau) + f_m h_m(x)\sin(\nu_m\tau), \quad \nu_j \approx \omega_j$$

$$U(x,\tau) = p_n(\tau)h_n(x) + p_m(\tau)h_m(x)$$

$$V(x,\tau) = q_n(\tau)h_n(x) + q_m(\tau)h_m(x)$$

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Complexification and harmonics selection

Projection of equations on internally resonant modes n and $m = kn$
Time decomposition in ε scale:

- fast time $\tau_0 = \tau$
- slow time $\tau_1 = \varepsilon\tau_0$

Introduction of Manevitch complex variables[4] ($i^2 = -1$) around ν_n and ν_m:

- $\phi_n e^{i\nu_n\tau} = N_n e^{i\delta_n} e^{i\nu_n\tau} = \dot{p}_n + i\nu_n p_n$
- $\phi_m e^{i\nu_m\tau} = N_m e^{i\delta_m} e^{i\nu_m\tau} = \dot{p}_m + i\nu_m p_m$
- $\psi_n e^{i\nu_n\tau} = M_n e^{i\gamma_n} e^{i\nu_n\tau} = \dot{q}_n + i\nu_n q_n$
- $\psi_m e^{i\nu_m\tau} = M_m e^{i\gamma_m} e^{i\nu_m\tau} = \dot{q}_m + i\nu_m q_m$

Frequencies of external solicitations:

- $\nu_n = \omega_n + \varepsilon\sigma_n$
- $\nu_m = \omega_m + \varepsilon\sigma_m = k\nu_n + \varsigma_k\varepsilon$

$$f = f_n h_n \sin(\nu_n\tau) + f_m h_m \sin(\nu_m\tau)$$

Implementation of Galerkin methodology (first harmonic ν_n and kth harmonic $k\nu_n$):

- $\frac{\nu_n}{2\pi}\int_0^{\frac{2\pi}{\nu_n}} s(\tau)e^{-i\nu_n\tau}d\tau$
- $\frac{\nu_n}{2\pi}\int_0^{\frac{2\pi}{\nu_n}} s(\tau)e^{-ik\nu_n\tau}d\tau$

[4] L. Manevitch, "The description of localized normal modes in a chain of nonlinear coupled oscillators using complex variables," Nonlinear Dynamics, vol. 25, pp. 95–109, 07 2001.

Complex equations

The system becomes:

$$\begin{cases} \dfrac{\partial \phi_n}{\partial \tau_0} + \varepsilon \left(\dfrac{\partial \phi_n}{\partial \tau_1} + \mathscr{E}_n(f_n, \sigma_n, \phi_n, \psi_n, \psi_n{}^*, \psi_m, \psi_m^*) \right) = 0 \\ \dfrac{\partial \phi_m}{\partial \tau_0} + \varepsilon \left(\dfrac{\partial \phi_m}{\partial \tau_1} + \mathscr{E}_m(f_m, \sigma_n, \varsigma_k, \phi_m, \psi_n, \psi_n{}^*, \psi_m, \psi_m^*) \right) = 0 \\ \dfrac{\partial \phi_n}{\partial \tau_0} - \dfrac{\partial \psi_n}{\partial \tau_0} + \mathscr{H}_n(\phi_n, \psi_n, \psi_n{}^*, \psi_m, \psi_m^*) = 0 \\ \dfrac{\partial \phi_m}{\partial \tau_0} - \dfrac{\partial \psi_m}{\partial \tau_0} + \mathscr{H}_m(\phi_m, \psi_n, \psi_n{}^*, \psi_m, \psi_m^*) = 0 \end{cases}$$

Fast time scale (ε^0):

- Detection of Slow Invariant Manifolds (SIMs)
- Stability of the SIM

Slow time scale (ε):

- Detection of singularities
- Detection of equilibrium points

Prediction of periodic and non-periodic behaviors

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Elements of the SIMs

System parameters: $\Lambda = 0.2$, $\chi_1 = 0.1$, $\chi_2 = 0.02$, $L = 100$

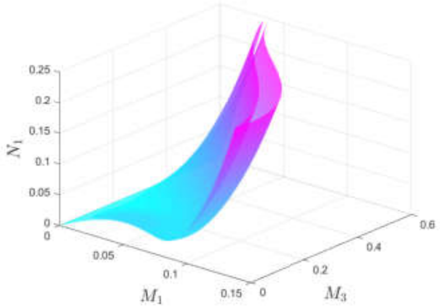

Figure: SIM envelopes around first mode ($\Gamma \in [0; 2\pi]$)

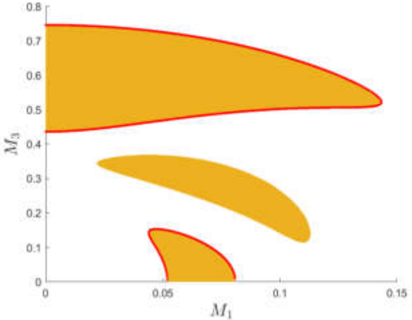

Figure: Unstable zone for $\Gamma = 3\gamma_1 - \gamma_3 = 0$

Free response of the system

Initial deformation on mode 1 $U(\tau = 0) = U_1 h_1(x)$
System parameters: $\varepsilon = 10^{-2}$, $\Lambda = 0.2$, $\chi_1 = 0.1$, $\chi_2 = 0.02$, $L = 100$

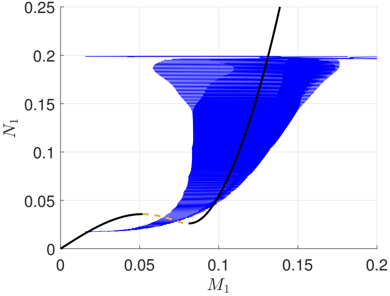

Figure: Free response on the 1 mode SIM

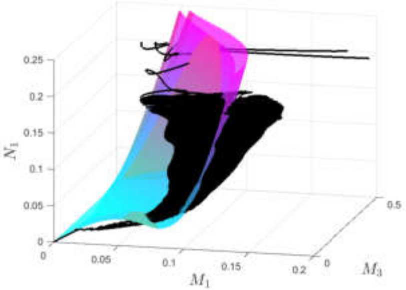

Figure: Free response on the 2 modes SIM

Frequency response curves

System parameters: $\varepsilon = 10^{-2}$, $\Lambda = 0.2$, $\chi_1 = 0.1$, $\chi_2 = 0.02$, $L = 100$, $f_1 = 0.004$, $f_3 = 0$, $\eta_3 = 0$

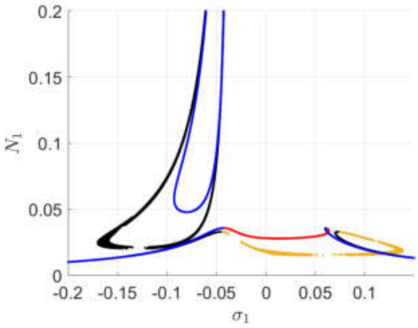

Figure: Frequency response curve according to mode 1

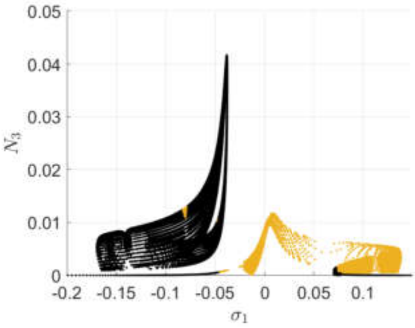

Figure: Frequency response curve according to mode 3

Frequency response curves

System specs: $\varepsilon = 10^{-2}$, $\Lambda = 0.2$, $\chi_1 = 0.1$, $\chi_2 = 0.02$, $L = 100$, $f_1 = 0.004$, $f_3 = 0$, $\sigma_3 = 0$, $\sigma_1 = 0$

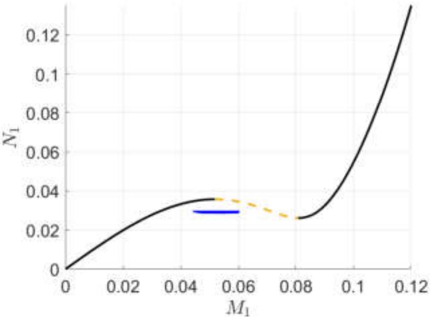

Figure: Frequency response curve according to mode 1

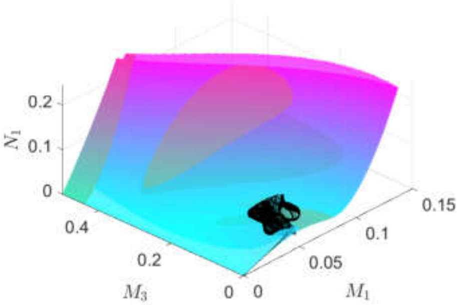

Figure: Frequency response curve according to mode 3

Example of application

System specs: $\varepsilon = 10^{-2}$, $\Lambda = 0.2$, $\chi_1 = 0.1$, $\chi_2 = 0.02$, $L = 100$, $f_1 = 0.004$, $f_3 = 0.024$, $\sigma_3 = -0.1$, $\Gamma \in [0; 2\pi]$

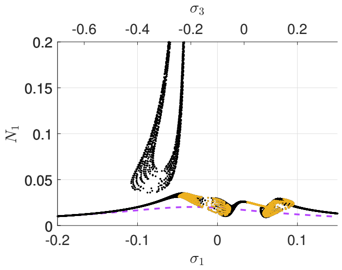

Figure: Frequency response curves according to mode 1

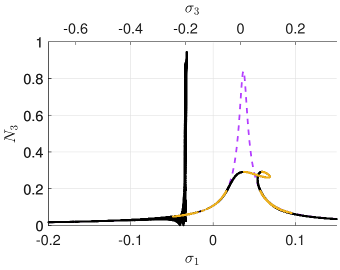

Figure: Frequency response curves according to mode 3

The experimental meta-cell

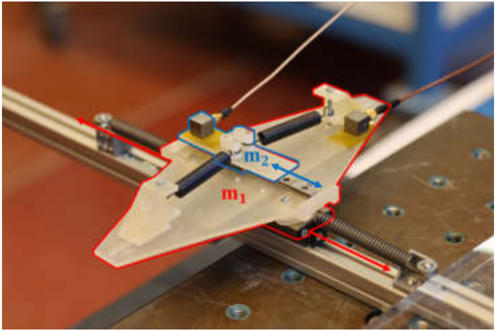

Figure: Experimental meta-cell

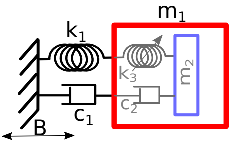

Figure: Theoretical meta-cell

specs: $k_1 = 2890$ N.m^{-1}, $k_3 = 1,83.10^5$ N.m^{-1}, $m_1 = 0.814$ kg, $m_2 = 0.026$ kg, $c_1 = 1.47$, $c_2 = 0.17$.

Experiment on one cell

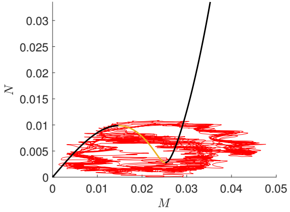

Figure: Experimental datas on the SIM for $B = 0.79$ mm and $f = 9.6$ Hz

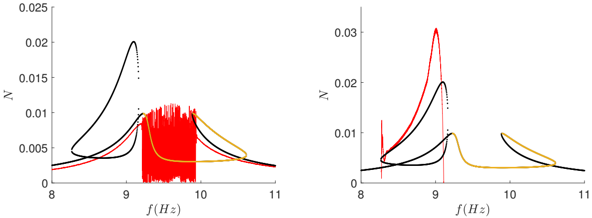

Figure: Experimental sweeps for $B = 0.79mm$ and $f \in [8:11]hz$

Ongoing experiment on the non-periodic chain of 6 cells

Figure: Experimental mass-in-mass chain

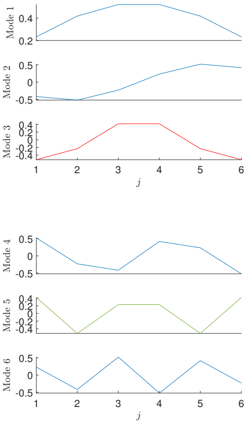

Figure: Modes of non-periodic chain

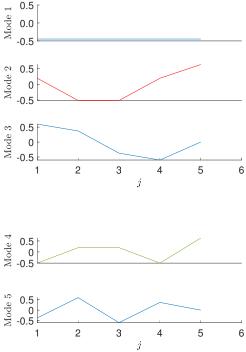

Figure: Modes of periodic chain

Ongoing experiment on the non-periodic chain of 6 cells

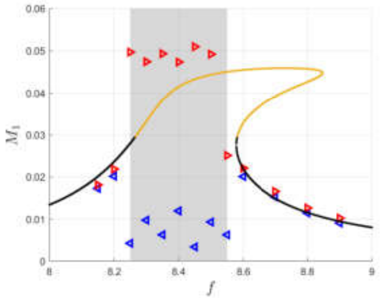

Figure: Experimental datas collected for $B = 1.5$ mm

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Conclusion

- Fast dynamics: determination of the SIMs + several possible unstable zones
- Slow dynamics: Equilibrium and singular points (periodic and non-periodic responses)

$\Downarrow$

Design tools for control or amplification of energy

Perspectives :

- Inclusion of more modal interactions (e.g. 3 modes)
- Experimental verification taking into consideration internal resonances