Computing Normal Forms of quadratic DAE

A. Grolet[a], A. Vizzacaro[b], M. Debeurre[a], O. Thomas[a]

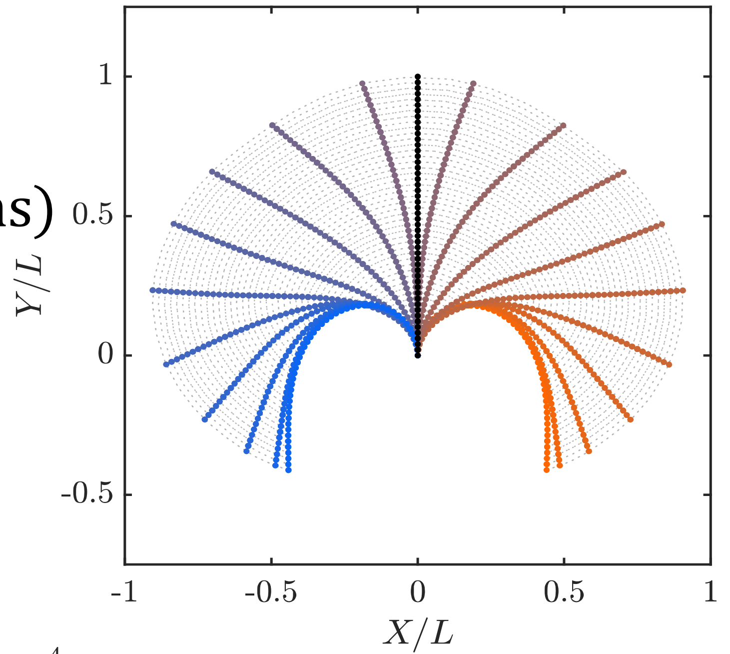
[a:]: Arts et Métiers, Lille, France

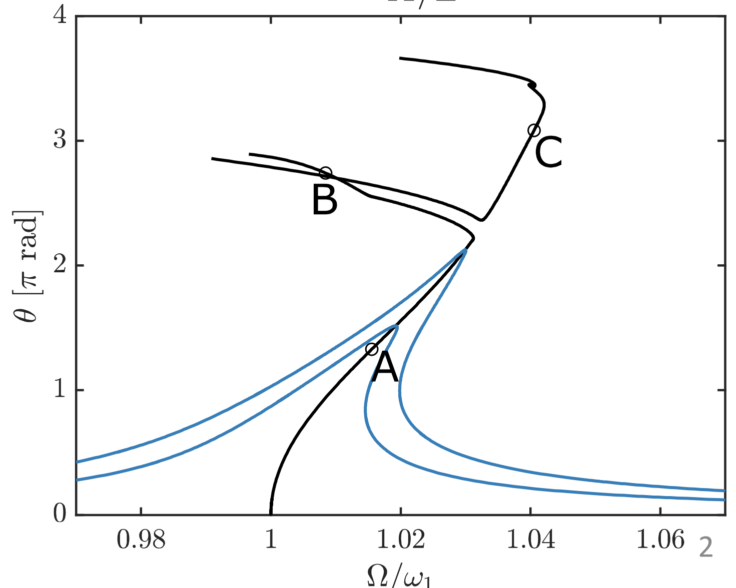
[b]: Univ. Of Bristol, Bristol, UK

09/11/2023 Femto-ST Besançon

Motivation

- Work on geometrically exact beam FEM
  - M. Debeurre PhD (Thread project)
  - Use of constraints to parameterize rotations (unit quaternions)
  - Model reduction
- Discussion with A.Vizzacaro:
  - Normal form theory on ODE (with C. Touzé)
  - Can we compute normal form of constrained system (DAE) ?
- Work with MANLAB (B.Cochelin, L .Guillot)
  - Computation and continuation of periodic solutions
  - Quadratic DAE as an input
- Can we compute normal form of quadratic DAE ?





Quadratic Differential Algebraic Equation

- Quadratic DAE (MANLAB input)

$$A\dot{y} = \mathrm{L}y + \mathrm{Q}(y, y)$$

  - y is the vector of unknowns (N variables)
  - A is the « mass » operator (NxN constant matrix, possibly singular)
  - L is the « stiffness » operator (NxN contant matrix)
  - Q is a bilinear operator (quadratic)

- A very general way of writting non linear dynamical equation:
  - Deals with constrained system (Lagrange multipliers)
  - Deals with any kind of nonlinearity, provided one finds a quadratic recast

Normal Form Theory

- Original dynamics (quadratic DAE)

$$\mathrm{A}\dot{y} = \mathrm{L}y + \mathrm{Q}(y, y)$$

- Normal form:
  - Find a polynomial change of variable

$$y = W(z) = P_d(z) + o\left(z^d\right)$$

  - Find the polynomial normal dynamics <u>under simplest form</u> (up to order d):

$$\dot{z} = f(z) + o(z^d)$$

  - Such that the original dynamics is exact (up to order d)

- In practice:
  - Derivatives (chain rule):

$$\dot{y} = \frac{\partial W}{\partial z}\dot{z} = \frac{\partial W}{\partial z} f$$

  - Homological equation:

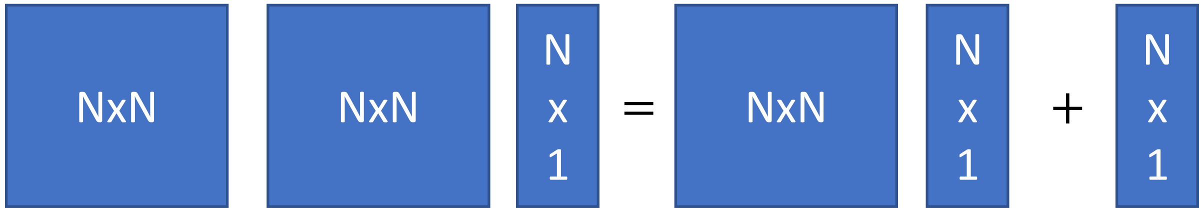
$$\mathrm{A}\frac{\partial W}{\partial z} f = \mathrm{L}W + \mathrm{Q}(W, W) + o(z^d)$$

  - Allows to compute W and f

Normal Form Theory

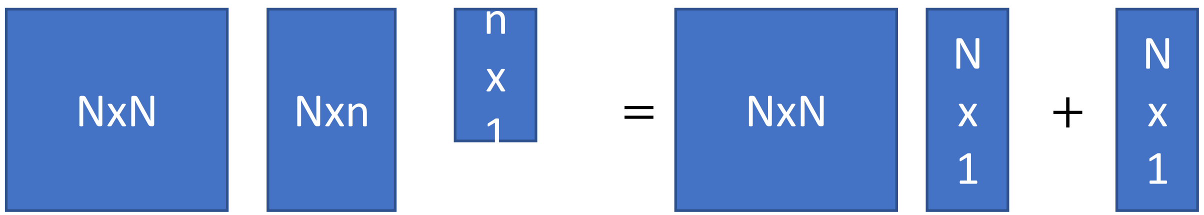
- Full normal form:
  - Change of variable:
    $$y = W(z)$$
  - z contains as many variables as y
  - N variables in total
  - Homological equation:

$$\mathrm{A}\frac{\partial W}{\partial z} f = \mathrm{L}W + \mathrm{Q}(W, W)$$



- <u>Reduced normal form</u>:
  - Change of variable:
    $$y = W(z)$$
  - z is much smaller in size than y
  - $n \ll N$ (model reduction)
  - Homological equation:

$$\boxed{\mathrm{A}\frac{\partial W}{\partial z} f = \mathrm{L}W + \mathrm{Q}(W, W)}$$



Polynomials representation and normal form

- Polynomial of deg. d in n variables:
  - Vector space of dimension M
  - Monomial Basis $B = B(z) = (b_1, \dots, b_M)$
  - Vectors of polynomials
- Change of coordinates:

$$y = W(z) = \sum_{1 \le m \le M} W_m \, b_m(z),$$

Coefficient : $W_m \in \mathbb{C}^N$

- Reduced normal dynamics :

$$\dot{z} = f(z) = \sum_{1 \le m \le M} f_m \, b_m(z),$$

Coefficient : $f_m \in \mathbb{C}^n$

- Homological equation:

$$\mathrm{A} \frac{\partial W}{\partial z} f = \mathrm{L} W + \mathrm{Q}(W, W)$$

- Substitute polynomial expression and balance monomials coefficients
- Needs mutiplication and derivative:
  - Linear operations
  - Use matrix representation (need the effect of multiplication and derivative on the basis elements)

Polynomials: Derivative and multiplication

- Consider the space of polynomial of degree d in n variables $z = (z_1, \ldots, z_n)$:
  - Vector space of dimension M
  - Basis $B = B(z) = (b_1, \ldots, b_M)$
- Derivatives:
  - There exists n « derivative matrices » ∇_k such that:

$$\frac{\partial B^T}{\partial z_k} = \nabla_k B^T$$

- Multiplication:
  - There exists M « multiplication matrices » Λ_r such that:

$$b_r B^T = \Lambda_r B^T$$

- Advantages of the vector space structure:
  - Multiplication and derivation matrices are sparse and can be computed prior to the normal form computation
  - Use linear algebra to write explicitly the normal form homological equation

Polynomials: application to normal form

- Polynomial of deg. d in n variables:
  - Basis $B = B(z) = (b_1, \dots, b_M)$
- Change of coordinates:

$$y = W(z) = \sum_{1 \le m \le M} W_m \, b_m(z),$$

Coefficient : $W_m \in \mathbb{C}^N$

- Reduced normal dynamics :

$$\dot{z} = f(z) = \sum_{1 \le m \le M} f_m \, b_m(z),$$

Coefficient : $f_m \in \mathbb{C}^n$

- Homological equation:

$$\mathrm{A}\frac{\partial W}{\partial z} f = \mathrm{L}W + \mathrm{Q}(W, W)$$

- Substitute polynomials expression :

$$AW \sum_{m=1}^{M-1} \sum_{j=1}^{n} f_m^j \nabla^j \Delta^m = LW + \sum_{m=1}^{M-1} \sum_{s=1}^{M-1} Q(W_m, W_s) b_m \Delta^s$$

- Equation for coeff. of monomial b_k:

$$(\sigma_{kk}\mathbf{A} - \mathrm{L})\,\mathbf{W}_k + \sum_{m \neq k} \sigma_{km}\mathbf{A}\mathbf{W}_m = \sum_{m=1}^{M} \sum_{l=1}^{M} \Lambda_{lk}^{m} \boldsymbol{Q}(\mathbf{W}_m, \mathbf{W}_l)$$

Solution to the homological equation

- Sequential resolution of the homological equation for each degree
- Order 0 (constant monomilal: $b_1 = 1$):
  - Static solution
- Order 1 (linear monomials : $b_2 = z_1 \dots, b_{n+1} = z_n$):
  - Linear modes of interest (complex mode shapes Y_k)
  - $W_2 = Y_k, \dots, W_{n+1} = Y_k,$
- Higher Orders:

$$(\sigma_{kk}\mathbf{A} - \text{Ł})\,\mathbf{W}_k + \sum_{m \neq k} \sigma_{km}\mathbf{A}\mathbf{W}_m = \sum_{m=1}^{M}\sum_{l=1}^{M} \Lambda_{lk}^{m}\boldsymbol{Q}(\mathbf{W}_m, \mathbf{W}_l)$$

  - After close inspection of the terms:

$$\begin{bmatrix} \sigma_{kk}\mathbf{A} - \text{Ł} & \mathbf{A}\mathbf{Y}_{r\in\mathcal{R}^{(k)}} & 0 \\ \mathbf{X}^{\mathrm{T}}_{r\in\mathcal{R}^{(k)}}\mathbf{A} & 0 & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{W}_k \\ \mathbf{f}_{k,s\in\mathcal{R}^{(k)}} \\ \mathbf{f}_{k,s\notin\mathcal{R}^{(k)}} \end{bmatrix} = \begin{bmatrix} RHS_k \\ 0 \\ 0 \end{bmatrix}$$

Normal form of DAE sum up

- Original dynamics (quadratic DAE)

$$\mathrm{A}\dot{y} = \mathrm{Ly} + \mathrm{Q(y, y)}$$

- Normal form:
  - change of variable

$$y = W(z) = \sum_{1 \le m \le M} W_m \, b_m(z),$$

  - normal dynamics under simplest form :

$$\dot{z} = f(z) = \sum_{1 \le m \le M} f_m \, b_m(z)$$

- Compute derivation (∇_r) and multiplication (Λ_s) matrices

- Homological equation (monom. b_k):

$$\begin{bmatrix} \sigma_{kk}\mathbf{A} - \mathrm{Ł} & \mathbf{AY}_{r \in \mathcal{R}^{(k)}} & \mathbf{0} \\ \mathbf{X}^{\mathrm{T}}_{r \in \mathcal{R}^{(k)}}\mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{W}_k \\ \mathbf{f}_{k, s \in \mathcal{R}^{(k)}} \\ \mathbf{f}_{k, s \notin \mathcal{R}^{(k)}} \end{bmatrix} = \begin{bmatrix} RHS_k \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

- Solve for W_k and f_k sequentially

- Use reduced dynamics to compute solutions,

- Go back to physical variables using the change of variables

Example 1: Duffing

- Duffing Oscillator

$$\ddot{u} + u + u^3 = 0$$

- Quadratic Equation of motion

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ r \end{pmatrix} + \begin{pmatrix} 0 \\ -ur \\ -u^2 \end{pmatrix}$$

$$A\dot{y} = Ly + Q(y, y)$$

Example 1: Duffing

- Linear modes

$$Y_1 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}, \; Y_2 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}, Y_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \lambda_1 = i, \lambda_2 = -i, \lambda_3 = \infty$$

- Normal form (ordre 1)
  - Linear change of variable

$$W_1 = Y_1 \text{ and } W_2 = Y_2 \text{ and } f_{11} = i, f_{22} = -i \text{ and } f_{12} = f_{21} = 0.$$

  - Uncoupled linear dynamics

$$\dot{z} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = f(z) = \begin{pmatrix} iz_1 \\ -iz_2 \end{pmatrix}$$

Example 1: Duffing

- Normal form (ordre 2)
  - Coeff z_1^2 ($2\,\lambda_1 = 2i$) : $f_3 = 0$

$$W_3 = (2iA - L)^{-1} Q(Y_1, Y_1)$$

$$W_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

  - Coeff $z_1 z_2$($\lambda_1 + \lambda_2 = 0$) : $f_4 = 0$

$$W_4 = -L^{-1}\left[Q(Y_1, Y_2) + Q(Y_1, Y_2)\right]$$

$$W_4 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

  - Coeff z_2^2 ($2\,\lambda_2 = -2i$) : $f_5 = 0$

$$W_5 = (-2iA - L)^{-1} Q(Y_2, Y_2)$$

$$W_5 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

At this point the reduced dynamics is still linear !

Example 1: Duffing

- Normal form (ordre 3, non resonant terms)

  - Coeff z_1^3 $(3\lambda_1 = 3\,i) : f_6 = 0$

$$W_6 = (3iA - L)^{-1}(Q(Y_1, W_3) + Q(W_3, Y_1))$$

$$W_6 = \begin{pmatrix} \frac{i}{4} \\ \frac{3}{4} \\ 0 \end{pmatrix}$$

  - Coeff z_2^3 $(3\lambda_2 = -3\,i) : f_9 = 0$

$$W_9 = (-3iA - L)^{-1}(Q(Y_2, W_5) + Q(W_5, Y_2))$$

$$W_9 = \begin{pmatrix} \frac{i}{4} \\ \frac{3}{4} \\ 0 \end{pmatrix}$$

Example 1: Duffing

- Normal form (ordre 3, resonant terms)

  - Coeff $z_1^2 z_2$ $(2\,\lambda_1 + \lambda_2 = \lambda_1 = i)$

$$(iA - L)W_7 + f_{17}AY_1 = Q(Y_1, W_4) + Q(W_3, Y_2) + Q(W_4, Y_1) + Q(Y_2, W_3)$$

$$X_1^T W_7 = 0$$

$$W_7 = \begin{pmatrix} \frac{3i}{4} \\ \frac{3}{4} \\ 0 \end{pmatrix}$$

  - Coeff $z_1 z_2^2$ $(\lambda_1 + 2\,\lambda_2 = \lambda_2 = -i)$

$$(-iA - L)W_8 + f_{28}AY_2 = Q(Y_1, W_5) + Q(W_5, Y_1) + Q(W_4, Y_2) + Q(Y_2, W_4)$$

$$X_2^T W_8 = 0$$

$$W_8 = \begin{pmatrix} -\frac{3i}{4} \\ \frac{3}{4} \\ 0 \end{pmatrix}$$

-> Addition of 2 nonlinear terms in the reduced dynamics

Example 1: Duffing

- Normal form : Sum up
  - Change of coordiantes

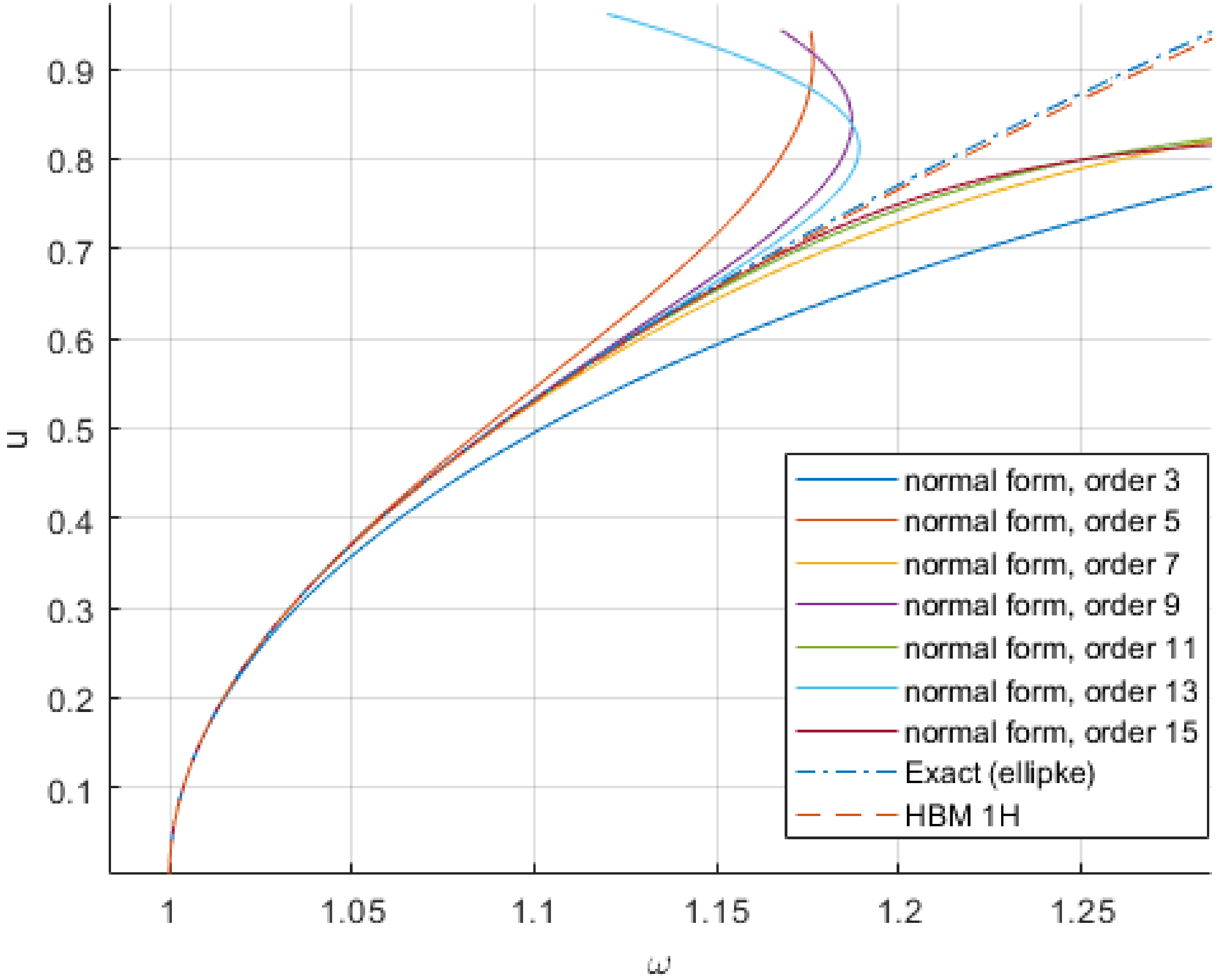
$$y = \begin{pmatrix} u \\ v \\ r \end{pmatrix} = W(z) = \begin{pmatrix} i(-z_1 + z_2) + \frac{i}{4} z_1^3 + \frac{3i}{4} z_1^2 z_2 - \frac{3i}{4} z_1 z_2^2 + \frac{i}{4} z_2^3 \\ z_1 + z_2 + \frac{3}{4} z_1^3 + \frac{3}{4} z_1^2 z_2 + \frac{3}{4} z_1 z_2^2 + \frac{3}{4} z_2^3 \\ -z_1^2 + 2 z_1 z_2 + z_2^2 \end{pmatrix}$$

  - Reduced dynamics:

$$\dot{z} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = f(z) = \begin{pmatrix} i z_1 + \frac{3i}{4} z_1^2 z_2 \\ -i z_2 - \frac{3i}{4} z_1 z_2^2 \end{pmatrix} \quad \xrightarrow{z_1 = \rho e^{i\theta}} \quad \begin{aligned} \dot{\rho} &= 0 + o(\rho^3) \\ \dot{\theta} &= 1 + \frac{3}{2}\rho^2 + o(\rho^3) \end{aligned}$$

Example 1: Duffing

- Backbone curve :

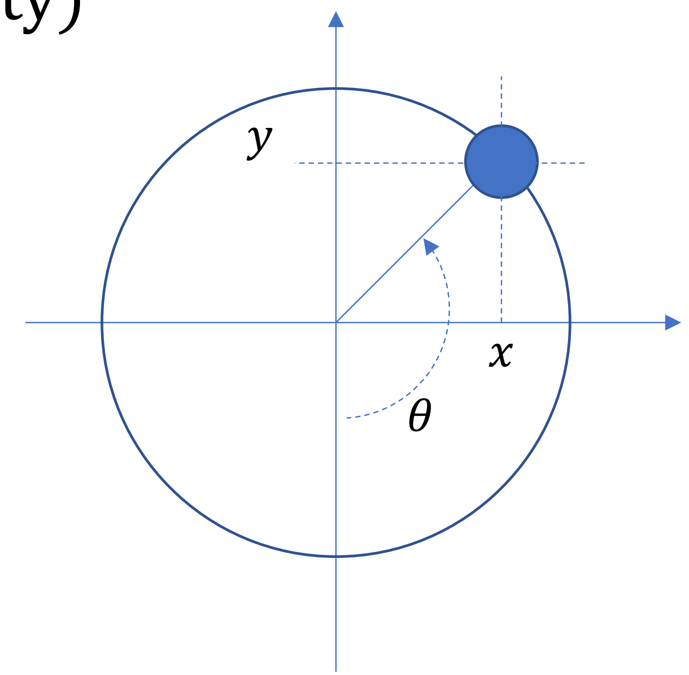


$$\dot{\rho} = 0 + o(\rho^3)$$

$$\dot{\theta} = 1 + \frac{3}{2}\rho^2 + o(\rho^3)$$

Example 2: simple pendulum

- Simple pendulum (Unit lenght, mass and gravity)
  - Parametrization ($n \in N$):
    - $p_0 = \cos\frac{\theta}{n}, p_3 = \sin\frac{\theta}{n}$
  - Kinetic Energy:
    - $T = \frac{n^2}{2}\left(\dot{p_0}^2 + \dot{p_3}^2\right)$
  - Potential Energy:
    - $U = -T_n(p_0)$
  - Constraint:
    - $p_0^2 + p_3^2 - 1 = 0$



- Equation of motion

$$n^2\ddot{p}_0 - \frac{\partial T_n}{\partial p_0} = 2\lambda p_0$$

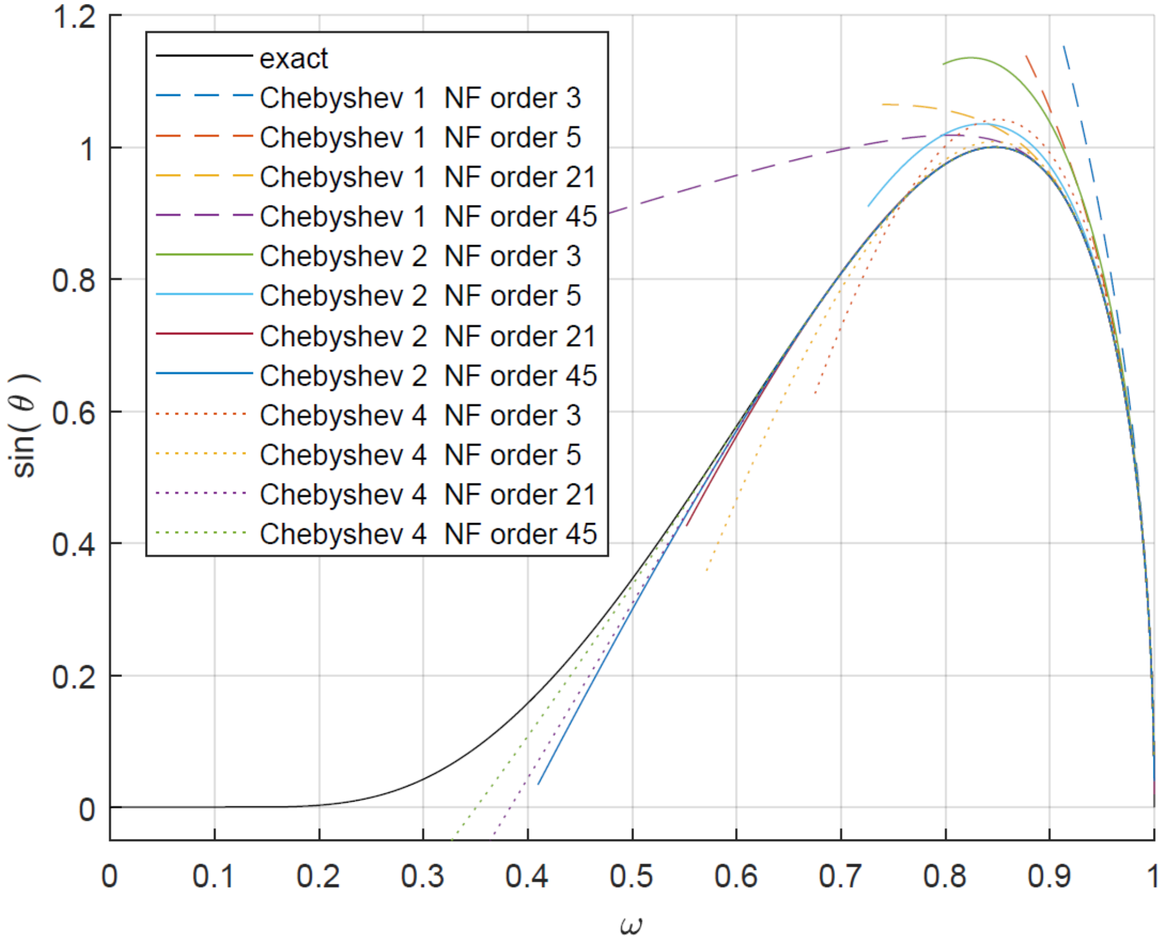
$$n^2\ddot{p}_3 = 2\lambda p_3$$

$$p_0^2 + p_3^2 - 1 = 0$$

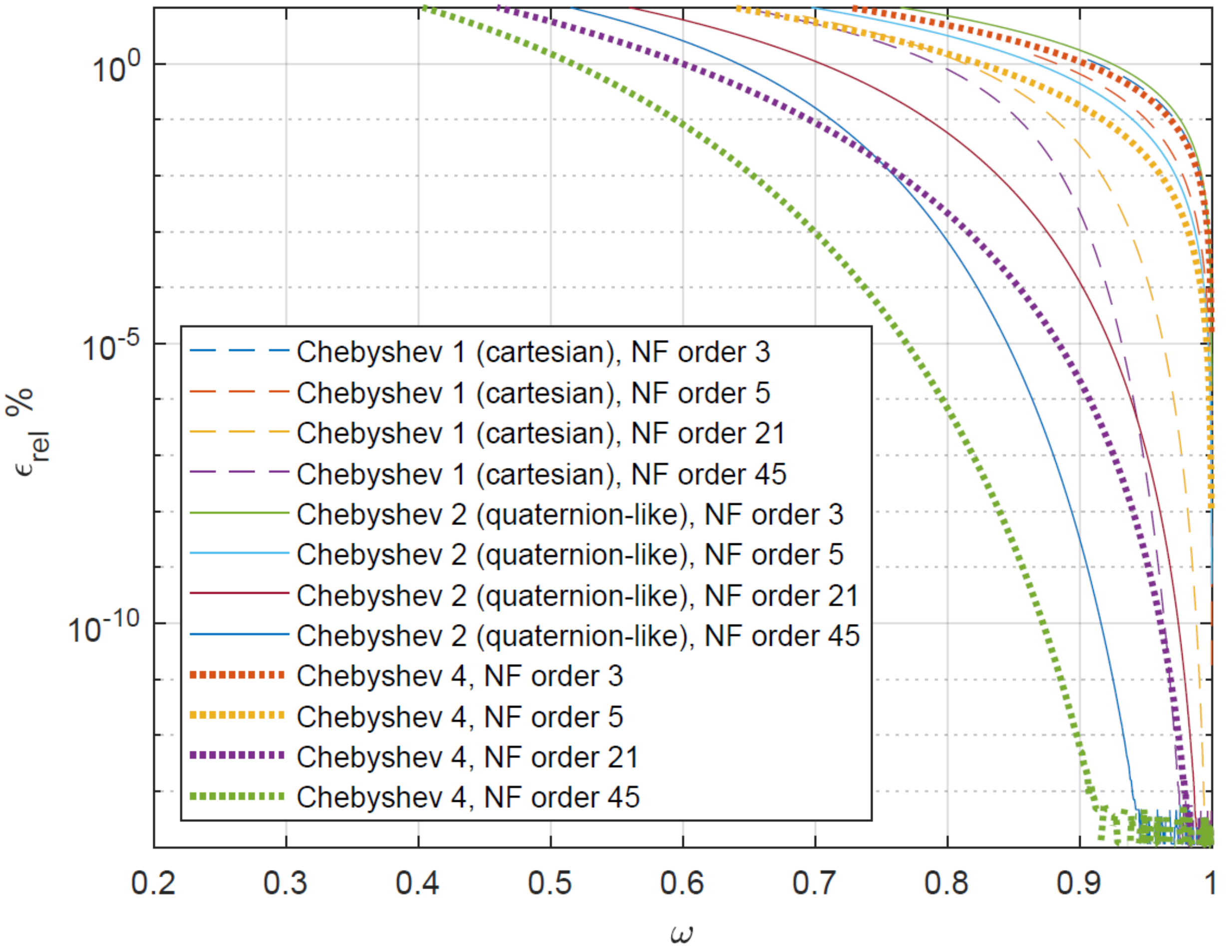
- T_n: n − th Chebichev polynomials
- $n = 1$:catesian parametrisation
- $n = 2$: quaternion like parametrisation

Example 2: simple pendulum

- Results

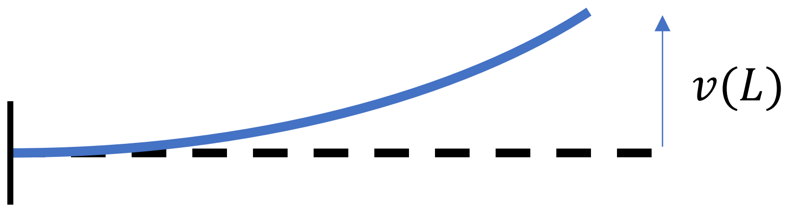


Natural angular freq.

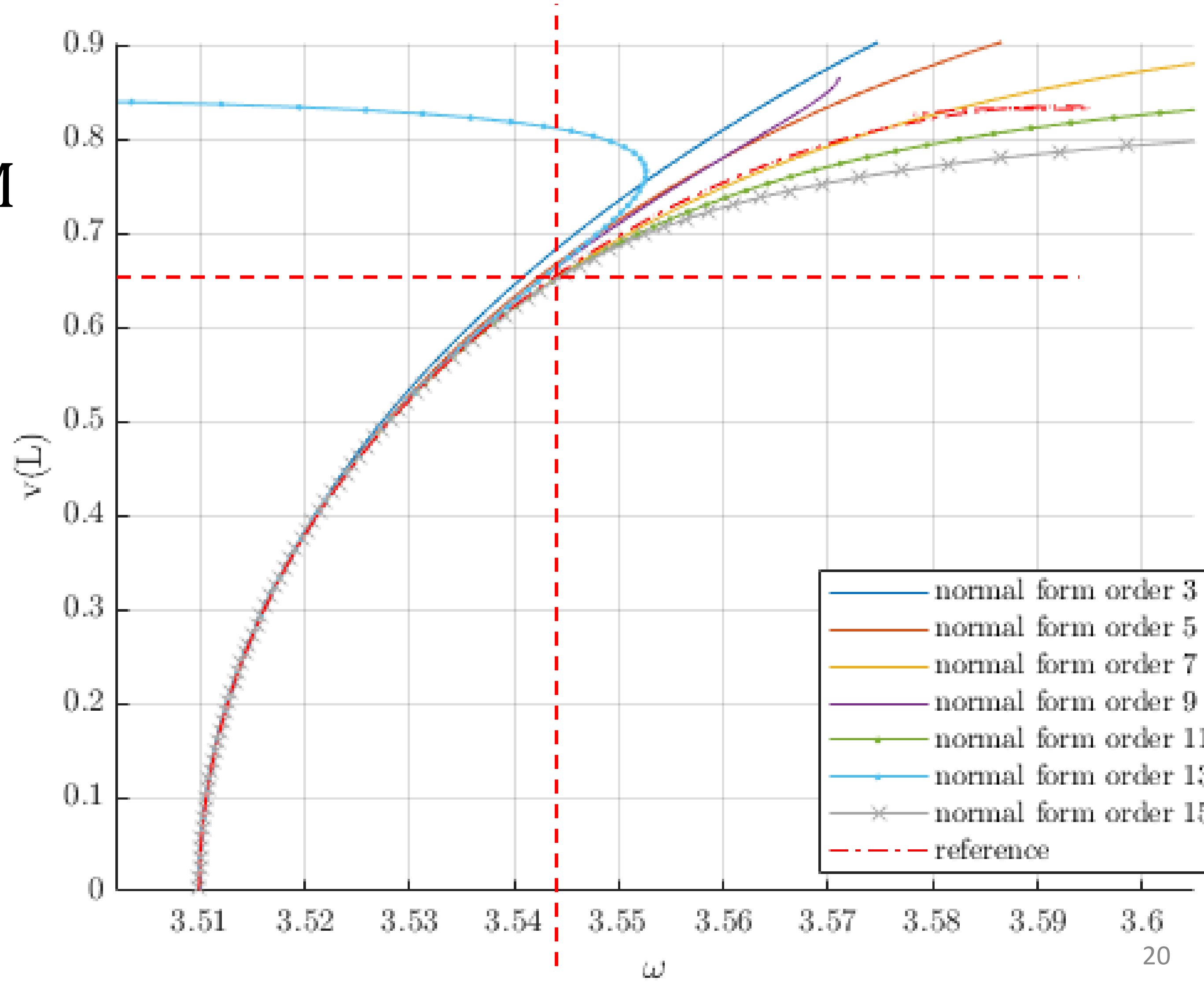


Relative error to analytical solution

Example 3: geometrically exact beam FE



- 2D geometrically exact beam FEM
  - Parametrisation of rotation using « quaternions »
  - Reweritten under a quadratic DAE
- Cantilever straight beam
  - FEM: 30 quadratic elements
  - DAE: ~ 500 variables
- Reference solution:
  - HBM: 20 Harmonics
- Normal form
  - 2 variables



Quick word on Validity range

- Backbone

$$\omega(\rho) = \dot{\theta} = \sum a_n \, \rho^n$$

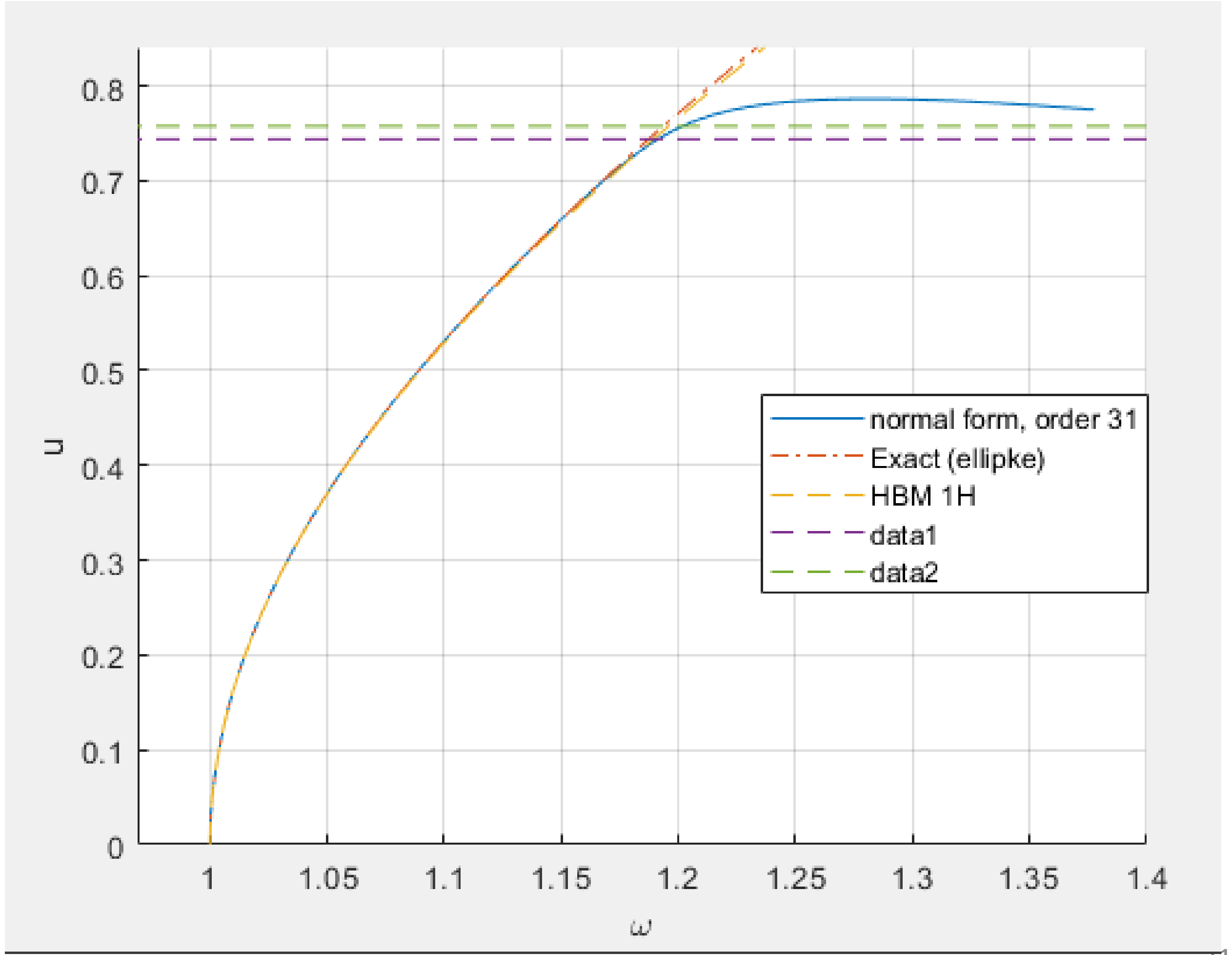
- Radius of convergence of a series
  - D'Alembert Criteria

$$\frac{1}{\rho^*} = \lim \frac{a_{n+1}}{a_n}$$

  - Cauchy Criteria

$$\frac{1}{\rho^*} = \lim (a_n)^{1/n}$$

Duffing example



Conclusion

- Normal form on quadratic DAE
  - Very general framework for non linear dynamics
  - Use of vector space structure to write the equation explicitly
  - Allows for reduced order model construction
  - Application on geometrically exact FEM (with unit quaternions)
  - Convergence radius (a posteriori validity bound, using Cauchy or D'Alembert criteria)

- Future work
  - Start from the hamiltonian directly (Canonical transformation, Birkoff normal form)
  - Link with the asymptotic numerical method (MANLAB)
  - Link with the Koopman theory

Computing Normal Forms of quadratic DAE

A. Grolet[a], A. Vizzacaro[b], M. Debeurre[a], O. Thomas[a]

[a]: Arts et Métiers, Lille, France

[b]: Univ. Of Bristol, Bristol, UK

09/11/2023 Femto-ST Besançon

Polynomials and multiplication

- Consider for example polynomial of degree $d = 2$ in $n = 2$ variables (z_1, z_2) :

$$p(z) = p_0 + (p_1 z_1 + p_2 z_2) + (p_3 z_1^2 + p_4 z_1 z_2 + p_5 z_2^2) = p_0 + p_1 b_1 + p_2 b_2 + p_3 b_3 + p_4 b_4 + p_5 b_5$$

- Vector space of dimension $M = 6$
- Basis : $B = B(z) = (1 \quad z_1 \quad z_2 \quad z_1^2 \quad z_1 z_2 \quad z_2^2\,)$
- Multiplication:

$$p(z)q(z) = \sum_m p_m b_m \sum_s q_s b_s = \sum_{ms} p_m q_s b_m b_s$$

$$p(z)q(z) = \sum_r \Big(\sum_{ms} p_m q_s \Lambda_{ms}^r\Big) b_r$$

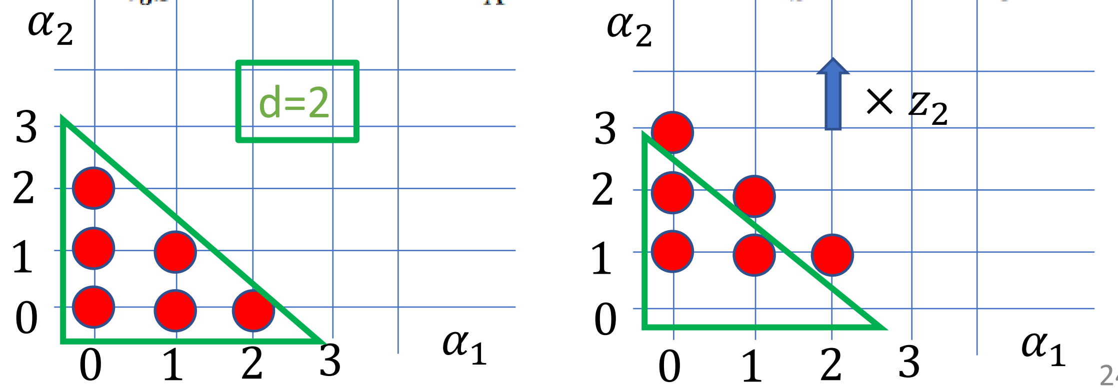
$$p(z)q(z) = \sum_r (pq)_r\, b_r$$

- One only needs to compute the products $b_m b_s$ and express it relative to the basis:

$$b_m b_s = \Lambda_{ms}^r\, b_r$$

- Example : $b_3 B^T = \Lambda_3 B^T$

$$\underbrace{\begin{pmatrix} z_2 \\ z_1 z_2 \\ z_2^2 \\ z_1^2 z_2 \\ z_1 z_2^2 \\ z_2^3 \end{pmatrix}}_{b_3 \mathcal{B}} = \underbrace{\left(\begin{array}{c|cc|ccc} 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)}_{\Lambda^3} \underbrace{\begin{pmatrix} 1 \\ z_1 \\ z_2 \\ z_1^2 \\ z_1 z_2 \\ z_2^2 \end{pmatrix}}_{\mathcal{B}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ z_1^2 z_2 \\ z_1 z_2^2 \\ z_2^3 \end{pmatrix}}_{H.O.T}$$



Polynomials and derivatives

- Polynomial of degree $d = 2$ in $n = 2$ variables (z_1, z_2) :

$$p(z) = \sum p_m b_m$$

- Basis : $B = B(z) = (1 \quad z_1 \quad z_2 \quad z_1^2 \quad z_1 z_2 \quad z_2^2\ )$
- Derivative:

$$\frac{\partial p}{\partial z_k}(z) = \sum_m p_m \frac{\partial b_m}{\partial z_k} = \sum_m p_m \sum_r \nabla_{mk}^r\, b_r$$

$$\frac{\partial p}{\partial z_k}(z) = \sum_r (\sum_r p_m \nabla_{mk}^r\ )\, b_r$$

$$\frac{\partial p}{\partial z_k}(z) = \sum_r \left(\frac{\partial p}{\partial z_k}\right)_r \quad b_r$$

- One only needs to compute the derivative of the basis monomials

$$\frac{\partial b_m}{\partial z_k} = \nabla_{mk}^r\, b_r$$

- Example : $\frac{\partial B^T}{\partial z_2} = \nabla_2 B^T$:

$$\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ z_1 \\ 2z_2 \end{pmatrix}}_{\frac{\partial \mathcal{B}}{\partial z_2}} = \underbrace{\left(\begin{array}{c|cc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{array}\right)}_{\nabla^2} \underbrace{\begin{pmatrix} 1 \\ z_1 \\ z_2 \\ z_1^2 \\ z_1 z_2 \\ z_2^2 \end{pmatrix}}_{\mathcal{B}}$$