Reduction methods in nonlinear dynamics:

center manifolds and normal forms

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femto-st SCIENCES & TECHNOLOGIES

EIPHI-BFC GRADUATE SCHOOL CROSS DISCIPLINARY SCIENCE AND TECHNOLOGY

Motivation

- **study nonlinear waves and patterns**

Nonlinear Waves on Water

Water wave
[David Sanger Photography]

Solitary wave
Lagoon of Molokai, Hawaii
[photo: R.I. Odom]

Roll wave
Channel in Lions Bay, Canada
[website of N. Balmforth]

Mascaret de St Pardon
Dordogne river

Tsunami in Asia

Rogue wave
Chemical tanker ship Stolt Surf
[photo: K. Petersen]

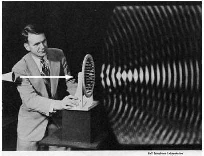
Other Nonlinear Waves

Kelvin-Helmholtz clouds
Mount Duval, Australia
[English Wikipedia: GRAHAMUK]

Fire rainbow
Northern Idaho

Morning Glory cloud
near Burketown, Australia
[author: Mick Petrov]

Hurricane



Sound wave
Bell Telephone Laboratories
[book by David C. Knight]

Patterns in Nature

Sand patterns
[photo: R. Niebrugge]

The Mathematics of ... Nonlinear Waves and Patterns

- **observed in nature, experiments, numerical simulations**
- **particular solutions of PDEs or ODEs**
  - *well-defined temporal and spatial structure*
  - *e.g., traveling waves*
- **play a key role in the dynamics of the underlying system**

The Mathematics of ... Nonlinear Waves and Patterns

Questions

- **existence** – *spatial and temporal properties*
- **stability** – *spatial and temporal behavior*
- **interactions**
- ...
- **role in the dynamics of the system**

The Mathematics of ... Nonlinear Waves and Patterns

Answers

- ... many different methods ...
- ... not enough ...
  - numerical
  - analytical

Two reduction methods

- **center manifolds**
- **normal forms**

Some applications [1]

- **bifurcations of nonlinear waves and patterns**

[1] Focus on results not on equations!

Water waves

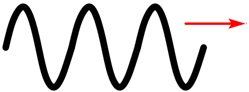
Water waves

Water waves

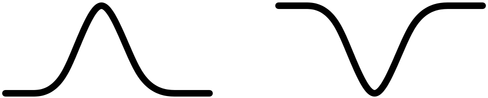
Water-wave problem

- **gravity-capillary water waves**
  - *three-dimensional inviscid fluid layer*
  - *constant density* ρ
  - *gravity and surface tension*
  - *(ir)rotational flow*

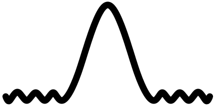
2D Traveling Waves



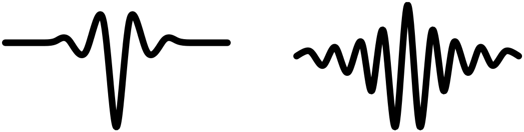
periodic wave



solitary waves



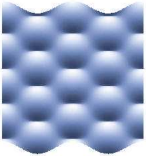
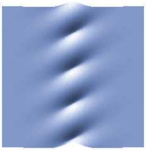
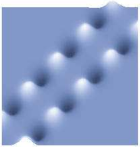
generalized solitary waves

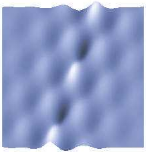
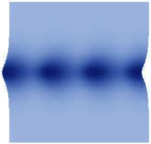


solitary waves

**[Nekrasov, Levi-Civita, Struik, Lavrentiev, Friedrichs & Hyers, . . .
Amick, Kirchgässner, Iooss, Buffoni, Groves, Toland, Lombardi, Sun, . . .]**

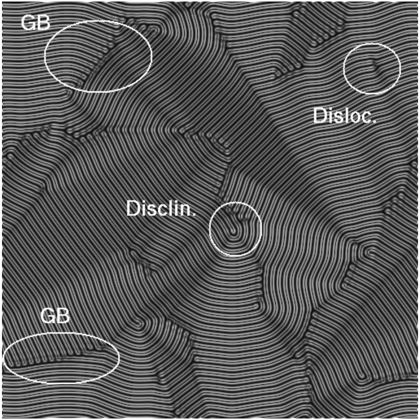
3D Traveling Waves

[Groves, Mielke, Craig, Nicholls, H., Kirchgässner, Deng, Sun, Sandstede, Iooss, Plotnikov, Wahlén, . . .]

Defects in striped patterns



- **grain boundaries**
- **dislocations**
- **disclinations**

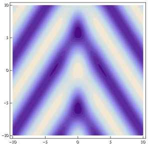
[D. Boyer, J. Viñals]

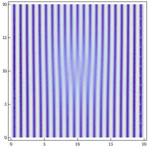
Defects in Striped Patterns

- **Occur in a wide range of systems**
  - Rayleigh-Bénard convection experiment
  - crystal patterns in material science
  - chemical reactions
  - biology
  - ......

Existence of defects

- **grain boundaries and dislocations**
  (Rayleigh-Bénard convection, Swift-Hohenberg equation)

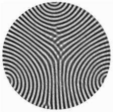


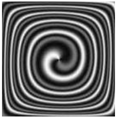


[H., Scheel, Wu, Iooss, Buffoni, Lloyd, . . .]

More defects

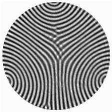
- **Some may be treatable by related methods . . .**

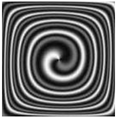




More defects

- Some may be treatable by related methods . . .

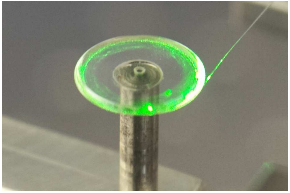




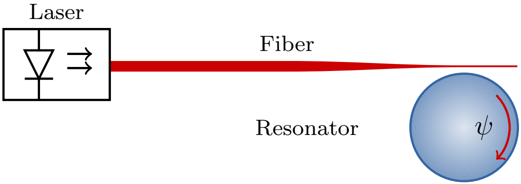
- Some cannot be treated by any of these methods . . .


A Problem from Optics

- **Home-made whispering gallery modes resonators**

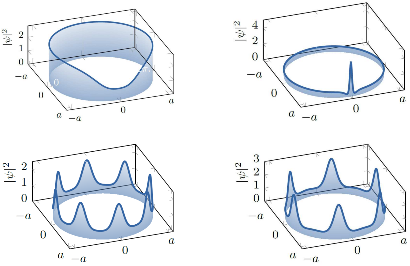


[Yanne Chembo, Rémi Henriet, Aurelien Coillet]



Frequency combs

- **optical signals:** *superposition of modes with equally spaced frequencies + stationary in suitable reference frame*



- *analytical results are in very good agreement with numerical and experimental results*

[Chembo, Godey, H., Delcey, Reichel, Mandel, . . .]

Center manifolds

- **reduce dimensions**

Center Manifolds

Dynamical system *(infinite-dimensional)*

$$\boxed{\frac{\mathbf{d}}{\mathbf{dt}}\mathbf{U} = \mathbf{F}(\mathbf{U}), \quad \boldsymbol{U(t)} \in \boldsymbol{\mathcal{X}}}$$

- start with particular solution: equilibrium

$$\boxed{\mathbf{U}(t) = \mathbf{U}_*}$$

(often $\mathbf{U}_* = 0$*, but not always)*

- **Question:** *local dynamics?*

Example

Swift-Hohenberg equation:

$$\frac{\partial u}{\partial t} = -\left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u + \mu u - u^3$$

- $u(x,t)$ 2π-periodic in x, parameter $\mu \in \mathbb{R}$

Example

- *Swift-Hohenberg equation:*

$$\frac{\partial u}{\partial t} = -\left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u + \mu u - u^3$$

  - $u(x,t)$ 2π-periodic in x, parameter $\mu \in \mathbb{R}$

- *Dynamical system:*

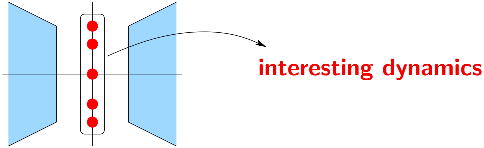
$$\boxed{\frac{\mathbf{d}}{\mathbf{dt}} \mathbf{U} = \mathbf{F}(\mathbf{U}, \mu), \quad \boldsymbol{U(t)} \in \boldsymbol{\mathcal{X}}}$$

  - $\mathbf{U} = u, \quad \mathbf{F}(\mathbf{U}, \mu) = -\left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u + \mu u - u^3$
  - $\mathcal{X} = L^2_{per}(0, 2\pi)$
  - particular solution: $\mathbf{U}_* = 0$

Linearized equation

$$\frac{d}{dt}\mathbf{U} = \mathcal{A}_*\mathbf{U}, \qquad \mathcal{A}_*\mathbf{U} = D_{\mathbf{U}}\mathcal{F}(\mathbf{U}_*)$$

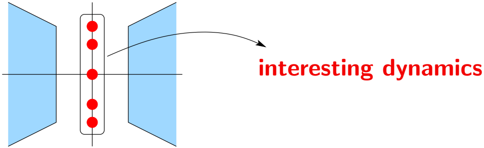
- spectrum of $\mathcal{A}_*$:



Linearized equation

$$\frac{d}{dt}U = \mathcal{A}_* U, \qquad \mathcal{A}_* U = D_U \mathcal{F}(U_*)$$

- **spectrum of $\mathcal{A}_*$:**



- **center space X_c:** *sum of the generalized eigenspaces associated with purely imaginary eigenvalues*
- **X_c contains all bounded solutions**

Nonlinear dynamics

System

$$\boxed{\frac{d}{dt}\mathbf{U} = \mathcal{A}_*\mathbf{U} + \mathcal{R}(\mathbf{U})}$$

Center manifold

- *analogue of the center space* $\mathbf{X}_c$ *for the nonlinear equation*
- contains *all* small bounded solutions of the dynamical system
- *exist in finite dimensions*
- **infinite dimensions:** *three main hypotheses*

Center Manifolds

Hypothesis

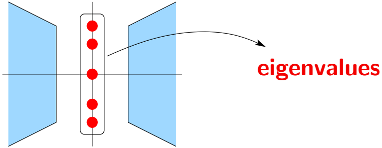
1. $\mathcal{A}_*$ **is a closed operator** *in a Hilbert (Banach) space* $\mathcal{X}$ *with dense domain* $\mathcal{Y} \subset \mathcal{X}$*;* $\mathcal{R} : \mathcal{Y} \to \mathcal{Z}$ **is well defined.**

Center Manifolds

Hypothesis

1. $\boldsymbol{\mathcal{A}_*}$ **is a closed operator** *in a Hilbert (Banach) space* $\mathcal{X}$ *with dense domain* $\mathcal{Y} \subset \mathcal{X}$; $\boldsymbol{\mathcal{R}}$ **:** $\boldsymbol{\mathcal{Y} \to \mathcal{Z}}$ **is well defined.**

2. **spectrum of $\mathcal{A}_*$: finite number of purely imaginary eigenvalues**



Center Manifolds

Hypothesis

1. **$\mathcal{A}_*$ is a closed operator** *in a Hilbert (Banach) space $\mathcal{X}$ with dense domain $\mathcal{Y} \subset \mathcal{X}$;* **$\mathcal{R} : \mathcal{Y} \to \mathcal{Z}$ is well defined.**
2. **spectrum of $\mathcal{A}_*$: finite number of purely imaginary eigenvalues**
3. **resolvent estimates:** *(see also [H. & Iooss, 2011])*

$$\|(\mathcal{A}_* - i\omega)^{-1}\|_{\mathcal{X}\to\mathcal{X}} \leq \frac{C}{|\omega|}, \quad \|(\mathcal{A}_* - i\omega)^{-1}\|_{\mathcal{Z}\to\mathcal{Y}} \leq \frac{C}{|\omega|^{1-\alpha}}$$

for $|\omega| \geq \omega_*$, and some $\alpha \in [0,1)$.

Center Manifolds

Theorem

The dynamical system $\boxed{\dfrac{\mathrm{d}\mathsf{U}}{\mathrm{d}t} = \mathcal{A}_* + \mathcal{R}(\mathsf{U})}$ *possesses a locally invariant manifold*

$$\boxed{\mathcal{M}_c = \{\mathsf{U} = \mathsf{U}_0 + \Phi(\mathsf{U}_0)\ ;\ \mathsf{U}_0 \in X_c\}}$$

Center Manifolds

Theorem

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$$\boxed{\mathcal{M}_c = \{\mathbf{U} = \mathbf{U}_0 + \mathbf{\Phi}(\mathbf{U}_0)\ ;\ \mathbf{U}_0 \in \mathbf{X}_c\}}$$

- $\mathbf{X}_c$ *is the (finite-dimensional) center space of the linearized equation*

$$X_c = \bigoplus_{i\kappa \in \sigma(\mathcal{A}_*)} E_{i\kappa};$$

- $\mathbf{\Phi}$ *is a map of class* C^k*;*
- $\mathcal{M}_c$ *contains all bounded solutions of the system.*

Center Manifolds

Reduced dynamics:

- *solutions of the infinite-dimensional dynamical system which belong to the center manifold*

$$\boxed{\mathbf{U}(t) = \mathbf{U_0}(t) + \mathbf{\Phi}(\mathbf{U_0}(t)), \quad \mathbf{U_0}(t) \in \mathbf{X_c}}$$

- $\mathbf{U_0}(t)$ *solves the reduced system*

$$\boxed{\frac{\mathbf{dU_0}}{\mathbf{d}t} = \mathcal{A}_0 \mathbf{U}_0 + \mathcal{R}_0(\mathbf{U_0})}$$

(the dimension of this system is often small)

- $\mathcal{A}_0 = \boldsymbol{\mathcal{A}}_* \big|_{\mathbf{X_c}}$ **and the Taylor expansion of** $\mathcal{R}_0(\mathbf{U_0})$ **can be computed . . .**

Center manifolds

- *locally invariant manifolds tangent to X_c at 0;*
  - *their dimension is equal to the dimension of X_c*
- *not unique, but the Taylor expansion of $\mathbf{\Phi}$ is unique . . .*
- *manifolds of class C^k for C^k vector fields, but not analytic . . .*

Further properties

- *parameter-dependent version*
- *symmetries of the original system are inherited by the reduced system, e.g.,*
  - *equivariance*
  - *reversibility*
  - *Hamiltonian structure*

Center manifolds

- *open questions . . .*
  - *infinite-dimensional center manifolds?*
  - *quasilinear dynamical systems in Banach spaces?*
- *used in the study of local bifurcations (existence of nonlinear waves) / stability problems*
- *together with stable/unstable manifolds provide a description of the nonlinear dynamics*
- *together with normal forms provide a rigorous justification of amplitude equations*

Normal forms

- **simplify nonlinear vector fields**

Normal forms

Dynamical system *(finite-dimensional)*

$$\boxed{\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u)} \qquad (1)$$

Hypotheses

1. $\mathbf{L} \in \mathcal{L}(\mathbb{R}^n)$;
2. for $k \geq 2$, there exists a neighborhood $\mathcal{V} \subset \mathbb{R}^n$ of 0 such that $\mathbf{R} \in \mathcal{C}^k(\mathcal{V}, \mathbb{R}^n)$ and

$$\boxed{\mathbf{R}(0) = 0, \quad D\mathbf{R}(0) = 0.}$$

Theorem

There exists a change of variables $\boxed{u = v + \mathbf{\Phi}(v)}$, with $\mathbf{\Phi} : \mathbb{R}^n \to \mathbb{R}^n$ a polynomial of degree p, which transforms the system (1) in its **"normal form"**

$$\boxed{\frac{dv}{dt} = \mathbf{L}v + \mathbf{N}(v) + \boldsymbol{\rho}(v)}$$

with the following properties:

Theorem

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$$\boxed{\frac{dv}{dt} = \mathbf{L}v + \mathbf{N}(v) + \boldsymbol{\rho}(v)}$$

with the following properties:

- $\boldsymbol{\rho}$ is of class $\mathcal{C}^k$ in a neighborhood of 0, and $\boldsymbol{\rho}(v) = o(\|v\|^p)$;
- $\mathbf{N} : \mathbb{R}^n \to \mathbb{R}^n$ **is a polynomial of degre** p;

Properties of $\mathbf{N}$

1. $\mathbf{N} : \mathbb{R}^n \to \mathbb{R}^n$ polynomial of degre p, $\boxed{\mathbf{N}(0) = 0,\ D\mathbf{N}(0) = 0}$
2. $\boxed{\mathbf{N}(e^{t\mathbf{L}^*} v) = e^{t\mathbf{L}^*}\mathbf{N}(v)}$ for all $(t, v) \in \mathbb{R} \times \mathbb{R}^n$
3. $\boxed{D\mathbf{N}(v)\mathbf{L}^* v = \mathbf{L}^*\mathbf{N}(v)}$ for all $v \in \mathbb{R}^n$
   - we use the identity

$$\frac{d}{dt}\left(e^{-t\mathbf{L}^*}\mathbf{N}(e^{t\mathbf{L}^*} v)\right) = e^{-t\mathbf{L}^*}\left(-\mathbf{L}^*\mathbf{N}(e^{t\mathbf{L}^*} v) + D\mathbf{N}(e^{t\mathbf{L}^*} v)\mathbf{L}^* e^{t\mathbf{L}^*} v\right)$$

[C. Elphick et al.]

Proof

- $\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u)$, $\boxed{u = v + \mathbf{\Phi}(v)}$, $\frac{dv}{dt} = \mathbf{L}v + \mathbf{N}(v) + \boldsymbol{\rho}(v)$

$$\boxed{(\mathbb{I} + D\mathbf{\Phi}(v))\,(\mathbf{L}v + \mathbf{N}(v) + \boldsymbol{\rho}(v)) = \mathbf{L}(v + \mathbf{\Phi}(v)) + \mathbf{R}(v + \mathbf{\Phi}(v))}$$

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- $\mathbf{R}(u) = \sum_{2\leq q\leq p} \mathbf{R}_q(u^{(q)}) + o(\|u\|^p)$,
  $\mathbf{\Phi}(v) = \sum_{2\leq q\leq p} \mathbf{\Phi}_q(v^{(q)})$, $\mathbf{N}(v) = \sum_{2\leq q\leq p} \mathbf{N}_q(v^{(q)})$

$$\boxed{D\mathbf{\Phi}_2(v^{(2)})\mathbf{L}v - \mathbf{L}\mathbf{\Phi}_2(v^{(2)}) = \mathbf{R}_2(v^{(2)}) - \mathbf{N}_2(v^{(2)})}$$

$$\boxed{D\mathbf{\Phi}_q(v^{(q)})\mathbf{L}v - \mathbf{L}\mathbf{\Phi}_q(v^{(q)}) = \mathbf{Q}_q(v^{(q)}) - \mathbf{N}_q(v^{(q)})}$$

Proof

- $\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u)$, $\boxed{u = v + \mathbf{\Phi}(v)}$, $\frac{dv}{dt} = \mathbf{L}v + \mathbf{N}(v) + \rho(v)$

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- **Linear equation:** $\boxed{\mathcal{A}_{\mathbf{L}}\mathbf{\Phi}_q = \mathbf{Q}_q - \mathbf{N}_q}$

Proof

- **Solve the linear equation:**

$$\boxed{\mathcal{A}_{\mathbf{L}}\mathbf{\Phi}_q = \mathbf{Q}_q - \mathbf{N}_q}$$

  - $\mathcal{A}_{\mathbf{L}^*}$ adjoint of $\mathcal{A}_{\mathbf{L}}$
  - $\mathbf{Q}_q - \mathbf{N}_q \in \ker(\mathcal{A}_{\mathbf{L}^*})^{\perp} = \mathrm{im}(\mathcal{A}_{\mathbf{L}})$
  - $\mathbf{P}_{\ker(\mathcal{A}_{\mathbf{L}^*})}(\mathbf{Q}_q - \mathbf{N}_q) = 0$
  - $\boxed{\mathbf{N}_q = \mathbf{P}_{\ker(\mathcal{A}_{\mathbf{L}^*})}\mathbf{Q}_q \in \ker(\mathcal{A}_{\mathbf{L}^*})}$

Remark: *the normal form is not unique*

- **0^2 normal form** *(Takens–Bogdanov bifurcation)*

$$\mathbf{L} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} A \\ B \end{pmatrix} \in \mathbb{R}^2, \quad \boxed{\mathbf{N}(u) = \begin{pmatrix} AP(A) \\ BP(A) + Q(A) \end{pmatrix}}$$

$$P(0) = Q(0) = Q'(0) = 0$$

Examples

- **0^2 normal form** *(Takens–Bogdanov bifurcation)*

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$$P(0) = Q(0) = Q'(0) = 0$$

- ***$i\omega$* normal form** *(Hopf bifurcation)*

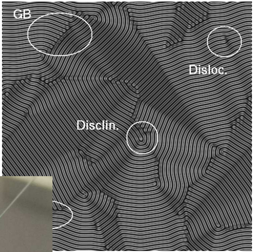
$$\mathbf{L} = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}, \quad v = \begin{pmatrix} A \\ \overline{A} \end{pmatrix}, \quad \boxed{\mathbf{N}(v) = \begin{pmatrix} AQ(|A|^2) \\ \overline{AQ}(|A|^2) \end{pmatrix}}$$

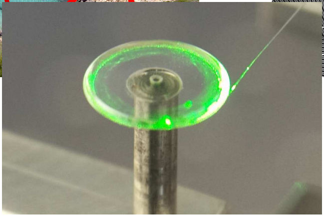
$$Q : \mathbb{C} \to \mathbb{C}, \quad Q(0) = 0$$

Further properties

- *parameter-dependent version*
- *symmetries of the original system are inherited by the reduced system, e.g.,*
    - *equivariance*
    - *reversibility*

Nonlinear waves and patterns





Approach

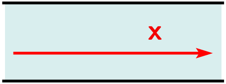
- **spatial dynamics**

Idea of spatial dynamics

Klaus Kirchgässner
(1931 – 2011)

Spatial dynamics

- **nonperiodic bounded solutions of PDEs in infinite strips**



[Kirchgässner, 1982]

  - **x timelike coordinate**

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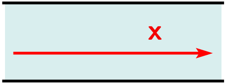


[Kirchgässner, 1982]

  - **x timelike coordinate**

- **Dynamical system**

$$\frac{d}{dx}U = F(U,\mu), \quad U(x) \in \mathcal{X}$$

  - $U(x)$ *belongs to a Hilbert (Banach) space* $\mathcal{X}$ *of functions depending upon the "space" variables;*
  - $\mu \in \mathbb{R}^m$ *parameters.*

Example

$$\boxed{\mathbf{u_{xx}} + \mathbf{u_{yy}} = \mathbf{f(u)}, \quad (\mathbf{x}, \mathbf{y}) \in \mathbb{R} \times (\mathbf{0}, \mathbf{1})}$$

- Set $\boxed{\mathbf{v} = \mathbf{u_x}}$
- Dynamical system:

$$\boxed{\frac{\mathbf{d}}{\mathbf{dx}} \mathbf{U} = \mathbf{F}(\mathbf{U}), \quad \mathbf{U} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad F(\mathbf{U}) = \begin{pmatrix} v \\ -u_{yy} + f(u) \end{pmatrix}}$$

- Phase space: $\boldsymbol{U(x)} \in \mathcal{X}$, $\boxed{\mathcal{X} = L^2(0,1) \times L^2(0,1), \ldots}$

Spatial Dynamics Approach

1. **Dynamical system**

$$\boxed{\frac{d}{dx}U = F(U, \mu), \quad U(x) \in \mathcal{X}}$$

- look for bounded solutions

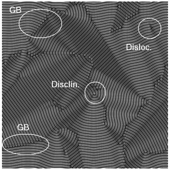
Spatial Dynamics Approach

❶ **Dynamical system**

$$\boxed{\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{U} = \mathbf{F}(\mathbf{U}, \mu), \quad \boldsymbol{U}(\boldsymbol{x}) \in \mathcal{X}}$$

- look for bounded solutions

- *difficult*:



1. **Dynamical system**

$$\boxed{\frac{d}{dx}\mathbf{U} = \mathbf{F}(\mathbf{U}, \mu), \quad U(x) \in \mathcal{X}}$$

Reduction

❶ **Dynamical system**

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❷ **Center manifold reduction:** *obtain a reduced system of ODEs . . .*

$$\boxed{\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{v} = \mathbf{g}(\mathbf{v}, \mu), \quad v(x) \in \mathbb{R}^d}$$

(pass from infinite to finite dimensions)

Reduction

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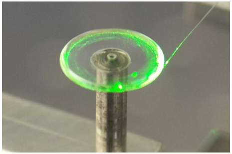

Reduced System

1. Dynamical system
2. **Center manifold reduction:** reduced system of ODEs

$$\frac{d}{dx} v = g(v, \mu), \quad v(x) \in \mathbb{R}^d$$

3. **Bounded orbits of the reduced system of ODEs**
   - *e.g., use* **normal forms**
   - *study a truncated system, then show persistence of the truncated dynamics*

Reduced System

❶ Dynamical system

❷ **Center manifold reduction:** reduced system of ODEs

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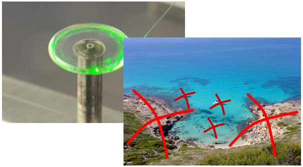
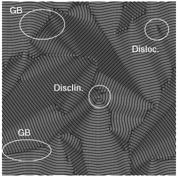
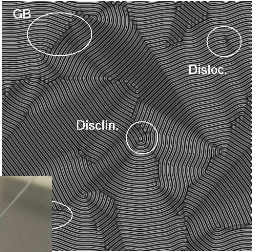
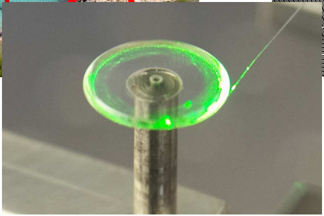
- *e.g., use* **normal forms**
- *study a truncated system, then show persistence of the truncated dynamics*
- *difficult*:





Nonlinear waves and patterns





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Computation of the reduced system

- **Taylor expansion of**

$$\boxed{\mathcal{R}_0(\mathbf{U_0}) = \mathbf{P}_c\mathcal{R}(\mathbf{U_0} + \mathbf{\Phi}(\mathbf{U_0}))}$$

1. $\mathcal{R}$ known
2. $\mathbf{U_0}(x)$ belongs to the center space $X_c = \bigoplus_{i\kappa\in\sigma(\mathcal{A}_*)} E_{i\kappa}$
   - $\mathbf{U_0}(x)$ is a finite linear combination of basis vectors
   - *basis of X_c:* consists of generalized eigenvectors which form a basis for the generalized eigenspaces $E_{i\kappa}$
3. $\mathbf{P}_c$ is the *spectral projector onto* tel que
   - $P_c^2 = P_c$ (projector)
   - $P_c\mathcal{A}_* = \mathcal{A}_*P_c$ (commutes with $\mathcal{A}_*$)
4. $\mathbf{\Phi}$: it is possible to compute its Taylor expansion (often, not necessary)