Optimization of bifurcation structures

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Journées annuelles du GDR EX-MODELI

- Modern mechanical systems
  - Ever-increasing demand for more efficient systems
  - Lighter, more slender structures
  - Smaller functional clearances
- Nonlinear vibrations
  - Multiple solutions
  - Bifurcations
  - Amplitude-jumps, quasi-periodic & chaotic solutions, etc.
- Bifurcations are not accounted for during the design stage
  - Discovered during testing/operation
  - At best, detected using a posteriori stability/bifurcation analysis

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Objectives

- In recent years, development of bifurcation tracking techniques for parametric analyses
- Optimization of bifurcations
  - Alternative to bifurcation tracking analyses capable of handling a large number of design parameters
  - Enforce bifurcations to occur at targeted locations

- Formulation of the optimization problem
- Computational nonlinear analysis
- Results

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Optimization problem

We consider dynamical systems under the following form:

$$\mathbf{R}(\mathbf{q}, \mu) = \mathbf{0}$$

Solution curve:

- Continuum of solutions under variation of μ
- Bifurcation points $\rightarrow$ qualitative and quantitative changes in the dynamics at values $\mu_\star$
- Usually detected by monitoring scalar test functions g whose zeros indicate a bifurcation

Let $\mathcal{T}$ and $\mathcal{P}$ denote the sets of target and predicted bifurcations, respectively

$$\underset{\mathbf{x}}{\text{minimize}} \quad \underbrace{|\mathcal{T} - \mathcal{P}|\Psi(\mathbf{x})}_{\text{Bifurcation measure}} + \underbrace{\frac{1}{|\mathcal{T}|}\sum_{\tau \in \mathcal{T}} \prod_{\pi(\mathbf{x}) \in \mathcal{P}} \left| \frac{\pi(\mathbf{x}) - \tau}{\tau} \right|^{1/|\mathcal{P}|}}_{\text{Error measure}}.$$

$$\text{subject to} \quad b_i^l \leq x_i \leq b_i^u \qquad \forall i \in [\![1, p]\!]$$

- Discontinuous objective function $\rightarrow$ Gradient-free optimizer (from NLopt.jl)

Bifurcation measure

Bifurcation measure: Encourage the presence of bifurcations on the solution curve

$$|\mathcal{T} - \mathcal{P}|\Psi(\mathbf{x})$$

- $|\mathcal{T} - \mathcal{P}| \to$ vanishes when the number of bifurcations on the curve equals the number of targets
- $\Psi(\mathbf{x}) \to$ pushes the optimizer towards states were many bifurcations occur
- $\Psi(\mathbf{x}) \to 0$ when many bifurcations are detected.

$$\Psi(\mathbf{x}) = \frac{\int_{R=0} \frac{|g|}{max|g|} \mathrm{d}s}{\int_{R=0} \mathrm{d}s}$$

Error measure

Error measure: Match bifurcations to targeted locations

$$\frac{1}{|\mathcal{T}|} \sum_{\tau \in \mathcal{T}} \prod_{\pi(\mathbf{x}) \in \mathcal{P}} \left| \frac{\pi(\mathbf{x}) - \tau}{\tau} \right|^{1/|\mathcal{P}|}$$

Formulation with arithmetic and geometric means:

- Errors for all combinations of targets and predictions
- Mitigates the risk of several bifurcations matched to the same target
- Equals zero when all targets are matched

Predictions π and targets τ can be:

- Frequencies
- A measure of states (infinity norm, L^2 norm, etc.)
- Both

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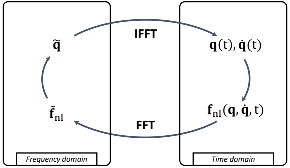
Harmonic Balance Method

HBM-AFT

$$q(t) = \mathfrak{Re}\left(\sum_{k=0}^{\infty} \tilde{\boldsymbol{q}}_k \mathrm{e}^{ik\Omega t}\right) \approx \mathfrak{Re}\left(\sum_{k=0}^{N_h} \tilde{\boldsymbol{q}}_k \mathrm{e}^{ik\Omega t}\right)$$

$$\boldsymbol{R}(\tilde{\boldsymbol{q}}, \Omega) = \boldsymbol{Z}(\Omega)\tilde{\boldsymbol{q}} + \tilde{\boldsymbol{f}}_{nl}(\tilde{\boldsymbol{q}}) - \tilde{\boldsymbol{f}}_{ex} = \boldsymbol{0}$$



Arclength continuation

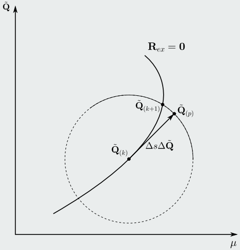
Prediction

$$\begin{bmatrix} \partial_{\tilde{\boldsymbol{q}}} \boldsymbol{R} & \partial_\mu \boldsymbol{R} \\ \Delta\tilde{\boldsymbol{q}}_k^T & \Delta\mu_k \end{bmatrix}_{(k)} \begin{pmatrix} \Delta\tilde{\boldsymbol{q}}_{k+1} \\ \Delta\mu_{k+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}$$

Correction

$$P(\tilde{\boldsymbol{q}}, \mu, s) = (\Delta\tilde{\boldsymbol{q}})^T (\Delta\tilde{\boldsymbol{q}}) + \Delta\mu^2 - \Delta s^2 = 0$$

$$\begin{pmatrix} \tilde{\boldsymbol{q}} \\ \mu \end{pmatrix}_{(k+1)} = \begin{pmatrix} \tilde{\boldsymbol{q}} \\ \mu \end{pmatrix}_{(k)} - \begin{bmatrix} \partial_{\tilde{\boldsymbol{q}}} \boldsymbol{R} & \partial_\mu \boldsymbol{R} \\ \partial_{\tilde{\boldsymbol{q}}} P & \partial_\mu P \end{bmatrix}_{(k)}^{-1} \boldsymbol{R}_{ex}^{(k)}$$



Bifurcation analysis

Local stability - Hill's method

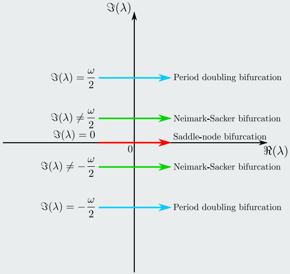
Quadratic eigenvalue problem (QEP)

$$\left(\lambda^2\tilde{\boldsymbol{M}} + \lambda\tilde{\boldsymbol{C}} + \boldsymbol{Z}(\omega) + \partial_{\tilde{\boldsymbol{q}}}\tilde{\boldsymbol{F}}_{nl}\left(\tilde{\boldsymbol{q}}_0\right)\right)\tilde{\boldsymbol{r}} = \boldsymbol{0}$$

$$\tilde{\boldsymbol{M}} = \boldsymbol{I}_{2N_h+1} \otimes \boldsymbol{M}$$
$$\tilde{\boldsymbol{C}} = \boldsymbol{\nabla} \otimes 2\boldsymbol{M} + \boldsymbol{I}_{2N_h+1} \otimes \boldsymbol{C}$$

'Linearization' of the QEP

$$\begin{bmatrix} \tilde{\boldsymbol{C}} & \partial_{\tilde{\boldsymbol{q}}}\boldsymbol{R} \\ -\left[\boldsymbol{I}_{N(2N_h+1)}\right] & \boldsymbol{0} \end{bmatrix} + \lambda \begin{bmatrix} \tilde{\boldsymbol{M}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{N(2N_h+1)} \end{bmatrix} = \boldsymbol{0}$$



Bifurcation detection

Scalar test function g evaluated by solving a bordered linear system

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{d}^{\dagger} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{w} \\ g \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}$$

Where $\boldsymbol{A}$ **depends on the bifurcation of interest**. $\boldsymbol{A} = \partial_{\tilde{\boldsymbol{q}}}\boldsymbol{R}$ for fold bifurcations

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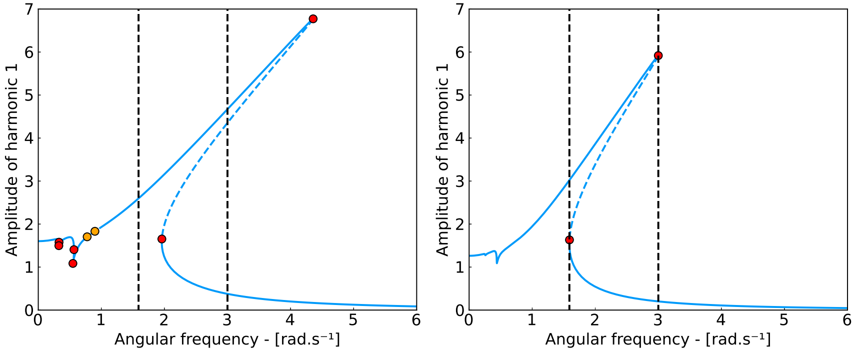
Duffing oscillator - target frequencies

$$m\ddot{x} + c\dot{x} + kx + k_{nl}x^3 = F\cos(\Omega t)$$

- Optimization variables: m, c, k , k_{nl}

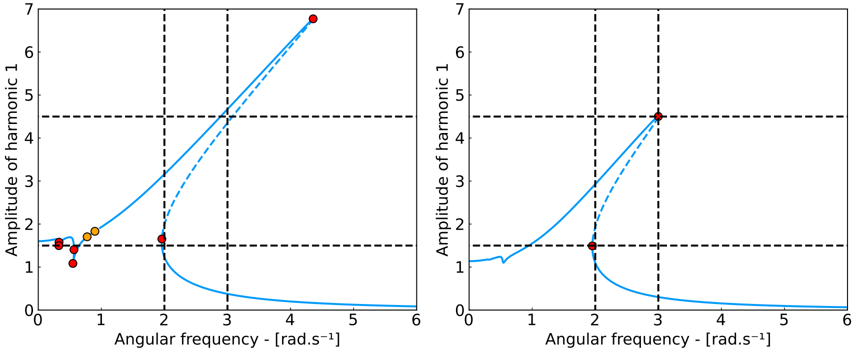


(a) $m = 1,\ c = 0.1,\ k = 1,\ k_{nl} = 0.5$ (b) $m = 1.85,\ c = 0.17,\ k = 1.77,\ k_{nl} = 0.54$

Duffing oscillator - target amplitudes and frequencies

$$m\ddot{x} + c\dot{x} + kx + k_{nl}x^3 = F\cos(\Omega t)$$

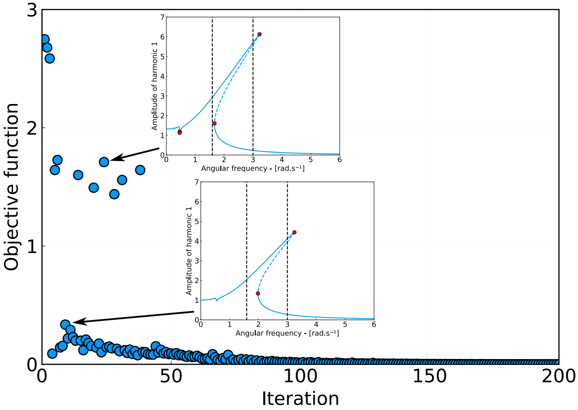
- Optimization variables: m, c, k , k_{nl}



(c) $m = 1,\ c = 0.1,\ k = 1,\ k_{nl} = 0.5$

(d) $m = 1.33,\ c = 0.22,\ k = 2.06,\ k_{nl} = 0.63$

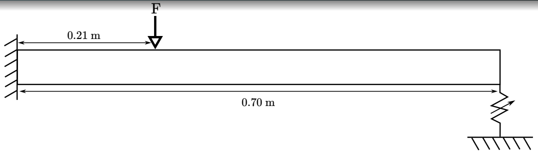
Duffing oscillator - objective function



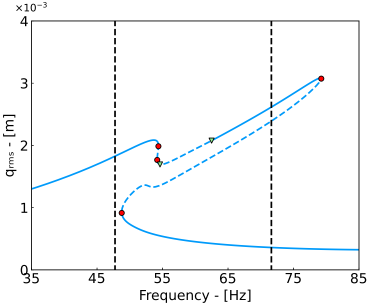
Objective function minimum when:

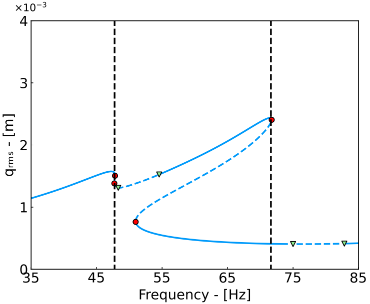
- all targets are matched with at least one bifurcation
- **AND** the number of bifurcations equals the number of targets

Finite element model with ROM



- 2D Euler bernoulli beam elements - Craig-Bampton ROM
- 120 optim. variables (element-wise height/width, length and nonlinear coeff)





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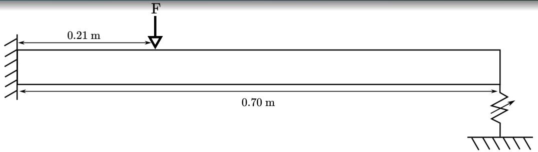

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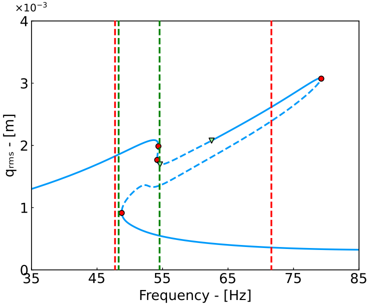

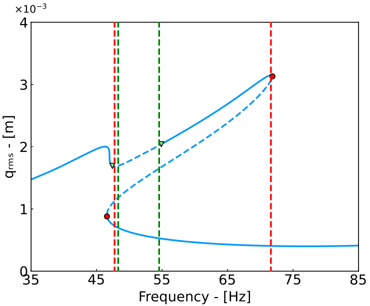

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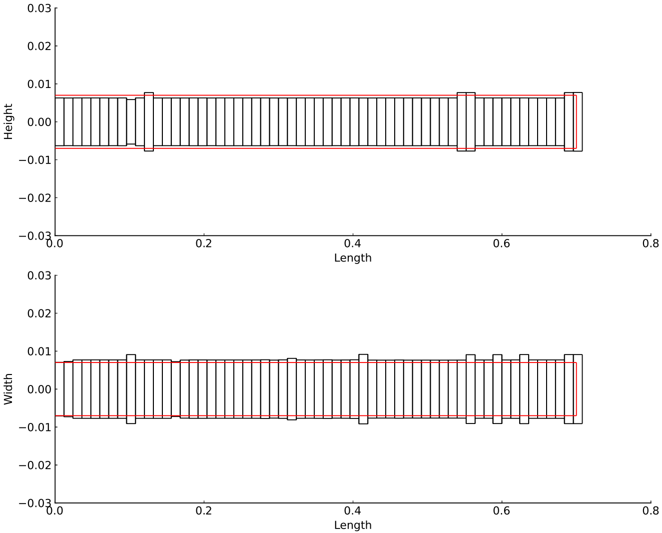

Conclusion

- Optimization framework to enforce the appearance of bifurcation points at targeted locations
  - Capable of handling multiple bifurcations of different types simultaneously
  - Handles target frequencies, amplitudes, both, ...
- Relatively high number of optimization parameters ($\approx$ 1e2)

Perspectives

- Extension to high-dimensional FE models
  - Development of parametric ROMs
  - Development of meta-models
- Investigation of global optimization algorithms

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Thank you for your attention!

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