First Return Time near a Grazing Linear Mode

for N-degree-of-freedom vibro-impact systems

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GDR EX-MODELI
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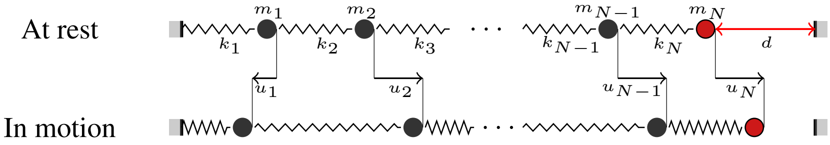
1. Mass-spring system
2. Poincaré section = contact hyperplane $\mathcal{H}$
3. Only one impact for all time
4. First return time
   - n loops, $n \to +\infty$
   - Square-root singularity
5. Comparison with Nordmark's works
6. Conclusion....

A discrete model : N degree-of-freedom system

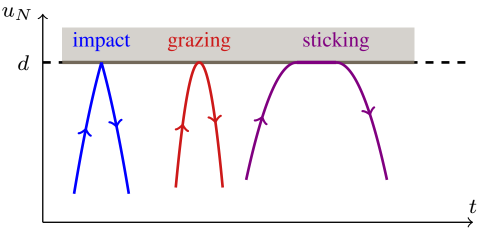


$$
\begin{cases}
\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{r}, & \text{(1a)} \\
\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 & \text{(1b)} \\
u_N(t) \leq d,\ d > 0, \quad R(t) \leq 0, \quad (u_N(t) - d)\, R(t) = 0, \forall t \geq 0 & \text{(1c)} \\
\dot{\mathbf{u}}^\top \mathbf{M}\dot{\mathbf{u}} + \mathbf{u}^\top \mathbf{K}\mathbf{u} = \mathbf{E}(\mathbf{u}, \dot{\mathbf{u}}) = \mathbf{E}(\mathbf{u}_0, \dot{\mathbf{u}}_0), & \text{(1d)}
\end{cases}
$$

where $\mathbf{u}(t) = [u_1(t), \ldots, u_N(t)]^\top$, $\mathbf{r}(t) = [0, \ldots, 0, R(t)]^\top$.
When $u_N(t) = d > 0$, the reflection law is $\dot{u}_N^+(t) = -e\dot{u}_N^-(t)$ with $e = 1$.

Three types of contact with $e = 1$

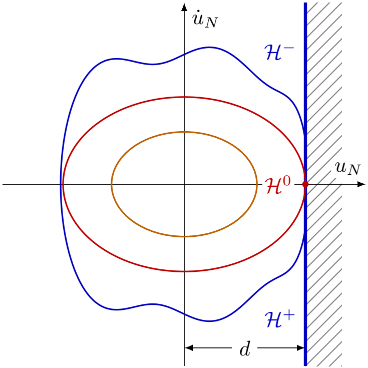
1. impulsive impact
2. grazing contact
3. closing contact



Contact hyperplane $\mathcal{H} = \{[\mathbf{u}, \dot{\mathbf{u}}]^\top \in \mathbb{R}^{2N},\ u_N = d\}$

$\mathcal{H} = \mathcal{H}^- \cup \mathcal{H}^+ \cup \mathcal{H}^0$ where

$$\mathcal{H}^- = \{[\mathbf{u}, \dot{\mathbf{u}}^-]^\top \in \mathbb{R}^{2N},\ u_N = d \text{ and } \dot{u}_N^- > 0\}$$
$$\mathcal{H}^+ = \{[\mathbf{u}, \dot{\mathbf{u}}^+]^\top \in \mathbb{R}^{2N},\ u_N = d \text{ and } \dot{u}_N^+ < 0\}$$
$$\mathcal{H}^0 = \{[\mathbf{u}, \dot{\mathbf{u}}]^\top \in \mathbb{R}^{2N},\ u_N = d \text{ and } \dot{u}_N = 0\}$$



Return to $\mathcal{H} = \{[\mathbf{u}, \dot{\mathbf{u}}],\ u_N = d\} \subset \mathbb{R}^{2N}\}$?

Theorem (0, 1 or ∞
(H. LeThi, S. Junca, M. Legrand) PhD 2017, DCDS 2022, 44 p.)

Assume no internal resonance then the number of contact times is:

0 - linear dynamics: $u_N(t) < d \quad \forall t \in \mathbb{R}$

1 - linear dynamics: $t = t_1$

∞ - nonlinear dynamics: $-\infty \leftarrow t_{n-1} < t_n < t_{n+1} \rightarrow +\infty,\ n \in \mathbb{Z}$

$$\mathcal{H}^0 = \{[\mathbf{u}, \dot{\mathbf{u}}],\ u_N = d, \dot{u}_N = 0\} = \mathcal{H}^0_\infty \cup \mathcal{H}^0_1$$

where $\mathcal{H}^0_1 = \{[\mathbf{u}, \dot{\mathbf{u}}] \in \mathcal{H}^0,\ \textit{only one contact}\}$

Poincaré section $\mathcal{H}_\mathcal{P}$ where the First Return Time is well defined:

$$\mathcal{H}_\mathcal{P} = \mathcal{H}^- \cup \mathcal{H}^0_\infty \subset \{[\mathbf{u}, \dot{\mathbf{u}}],\ u_N = d, \dot{u}_N \geq 0\}$$

Only one impact for all time

Theorem (Caracterization of $\mathcal{H}_1^0$
(H. Le Thi, S. Junca, M. Legrand) 2023)

$\mathcal{H}_1^0$ is the convex hull of the GLMs on $\mathcal{H}$, without GLMs.

- GLM= Grazing linear mode,
  identified with the contact point on the the contact hyperplane $\mathcal{H}$.
-
  - $\mathcal{H}_1^0 \cap \mathcal{H}_{\mathcal{P}} = \emptyset$
  - $\mathcal{H}_1^0 \neq \emptyset$
  - $N - 1$ dimensional bounded set $\mathcal{H}_1^0 \subset \mathcal{H} \sim \mathbb{R}^{2N-1}$
- " Instability "near $\mathcal{H}_1^0$, orbits coming back with large time.

Dynamics and FRT (First Return Time)

- Dynamics governed by the FRM (First Return Map)

$$\mathbf{FRM}(W) = \mathbf{R}(FRT(W))\,\mathbf{S}\,W$$

  - $\mathbf{S} = \mathbf{diag}(1, \ldots, 1, -1)$, impact law
  - $W = (\mathbf{u}, \dot{\mathbf{u}})$, the initial state defined on $\mathcal{H}$.
  - $\mathbf{R}(t) = e^{t\,A}$, the exponential matrix associated to the free flow.

- Square-root dynamics (Nordmark, 2001)
  $\approx$ square-root singularity of FRT near grazing orbits

n loops, $n \to +\infty$

1. Perturbation $\varepsilon \ll 1$ of GLM_j with fundamental period T_j, $j \leq N$

$$FRT = n\ T_j + \mathcal{O}(\sqrt{\varepsilon}), \qquad n \in \mathbb{N}^*$$

   $\to$ $FRT \neq 2024.1017\ T_j$ for ε small enought
   $\to$ FRT is discontinuous near GLM_j

2. $\mathcal{V}_0^\varepsilon = \emptyset$, $n = 0$ means no micro-contact
3. $\mathcal{V}_1^\varepsilon \neq \emptyset$, $n = 1$, existence   Nordmark 2001, PhD Huong Le Thi 2017
4. $\mathcal{V}_2^\varepsilon \neq \emptyset$??? $n = 2$. Idem $n = 3, 4, \dots$
5. $n \gg 1$, infinitely many $\mathcal{V}_n^\varepsilon \neq \emptyset$, existence
6. $\mathcal{H}_\mathcal{P} \cap \mathcal{V}^\varepsilon = \bigcup_{0<n<+\infty} \mathcal{V}_n^\varepsilon$, $\dfrac{FRT}{T_j} \approx n$, $n \in \mathbb{N}$
7. $\mathcal{H}_1^0 \cap \mathrm{Ball}(GLM_j, \varepsilon) = \mathcal{V}_{+\infty}^\varepsilon \neq \emptyset$

Square-root singularity and Poincaré map

1. Poincaré section $\mathcal{H}_{\mathcal{P}}$ outside $\mathcal{H}$ and tranverse to the free flow
2. discontinuity mapping

Shaw & Holmes, "square-root singularity ", *J. Sound Vib.* 1983

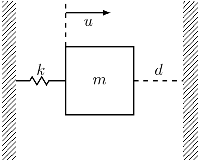
Nordmark , "bifurcation for grazing impact", *J. Sound Vib.* 1991

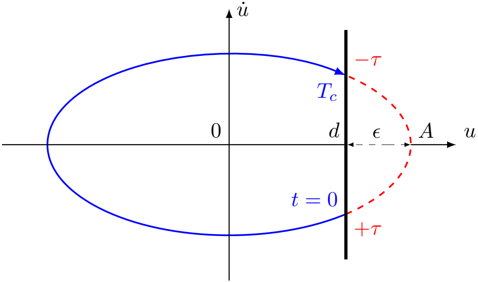
A. Nordmark, Existence of periodic orbits in grazing bifurcations of impacting mechanical oscillators, *Nonlinearity* 2001

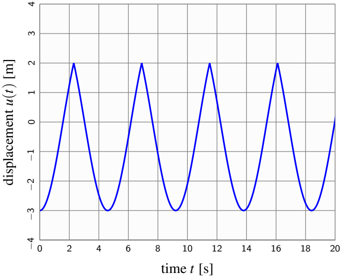
M. Di Bernardo, C. Budd, A. Champneys, P. Kowalczyk, Piecewise-smooth dynamical systems: theory and applications, *Applied Mathematical Sciences. Springer* 2008

1. Poincaré section in $\mathcal{H}$ and not tranverse
2. Nonlinear Normal Modes

- Legrand, Heng, J., 1IPP (1 impact per period), 2017
- Thorin, Delezoide, Legrand, kIPP (k impacts per period), 2017
- Le Thi, J., Legrand, 1SPP, 2 dof (1 sticking per period), 2018
- Thorin, Delezoide, Legrand, 1SPP, N dof, 2017







$$T(\varepsilon) = T_{linear} - c\sqrt{\varepsilon} + \dots$$

$\sqrt{\text{singularity}}$ of the First Return Time (FRT)

Theorem (Near one grazing contact (Le Thi, Junca, Legrand) 2017)

Consider $W := (\mathbf{u}, \dot{\mathbf{u}})$*, the initial data* $W_0 \in \mathcal{H}^-$*, the FRT* T_0*, and* $(\mathbf{u}(T_0, W_0), \dot{\mathbf{u}}(T_0, W_0)) \in \mathcal{H}^0$ *such that*

$$\cap_{j=1}^N T_j \mathbb{N} = \emptyset, \quad \partial_{W_k} u_N(T_0, W_0) \neq 0, \quad \partial_{tt} u_N(T_0, W_0) \neq 0$$

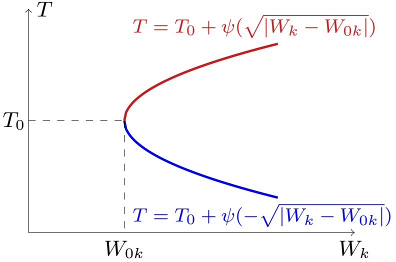
Then, there exists a smooth function ψ *defined on the set* $\mathcal{B}_k = \{W \in V_{W_0}, \, s_k(W_k - \alpha(\underline{W})) \geq 0\}$ *such that the FRT is given by*

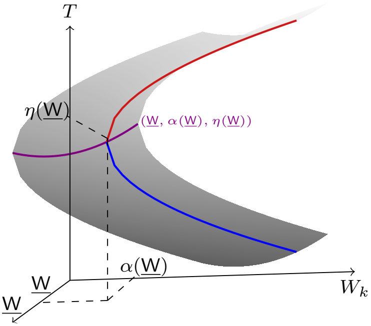
$$T(W) = \eta(\underline{W}) + \psi(\sigma \sqrt{s_k(W_k - \alpha(\underline{W}))}, \underline{W}) \tag{2}$$

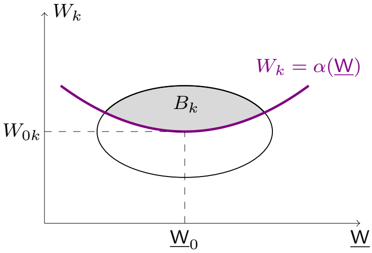
In particular, $\psi(0, \underline{W}_0) = 0$*,* $\partial_1 \psi(0, \underline{W}_0) = |\gamma_k|^{-1/2} \neq 0$*.*

- $\eta, \alpha \in C^\infty(V_{\underline{W}_0}, \mathbb{R})$ such that $\eta(\underline{W}_0) = T_0$ and $\alpha(\underline{W}_0) = W_{0k}$
- $\underline{W} \in \mathbb{R}^{2N-1}$: the reduced vector obtained from W by removing W_k
- $\gamma_k = -\partial_{tt}\Phi(T_0, W_0)/(2\, \partial_{W_k}\Phi(T_0, W_0))$; $s_k = \text{sign}(\gamma_k)$ and $\sigma = \text{sign}(\ddot{u}_N^-(T_0, W_0))$

Illustration of the theorem







Comparisons with Nordmark's seminal works

- Nordmark
  1. Transverse to the orbits
  2. The Discontinuity mapping needs 2 Times
  3. Only one turn to come back to the Poincaré section
  4. $\sqrt{\text{singularity}}$ affects $\approx \frac{1}{2}$ neighborhood of the grazing orbit
  5. Grazing bifurcation: $\exists$ nonlinear periodic orbits
- $\mathcal{H}_{\mathcal{P}}$ on the contact hyperplane
  1. Not transverse but "natural"
  2. Only 1 time FRT is needed to define FRM
  3. 1 or many loops to come back to $\mathcal{H}_{\mathcal{P}}$
  4. $\sqrt{\text{singularity}}$ affects
     - $\approx \frac{1}{4}$ GLM's neighborhood (Huong's phd 2017)
     - ∞ many leaves, 2023
  5. $\exists$ NLM issued from GLMs *if no internal resonance*

Conclusion & Perspectives

1. Solutions with only one contact
2. $FRT \approx 1\ T_j, 2\ T_j, 3\ T_j, \ldots$ near GLM_j
3. number of loops n
4. $\sqrt{\text{singularity}}$ revisited

Few open questions:

- For all $n \in \mathbb{N}^*$ there are orbits with n loops ?
- Is $W \mapsto \mathbf{u}(FRT(W))$ continuous at GLM ?
- Instability of the Linear Grazing Orbits ?
- Internal resonance ?

Thank you for

your attention

Claude-Henri