Front propagation in granular chains

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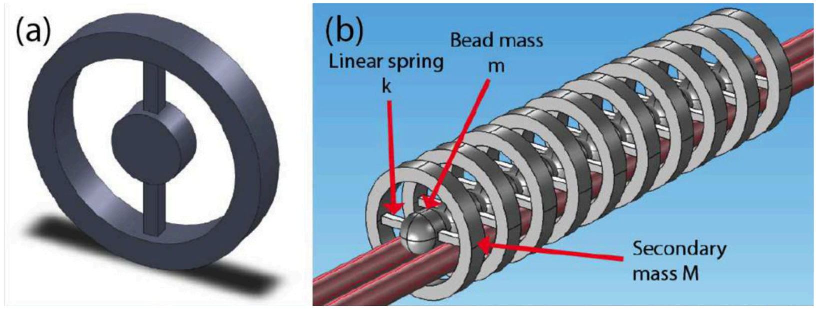
← granular chain

granular chain with ring resonators

↓



Gantzounis et al, J. Appl. Phys. 114 (2013)

- Applications of granular chains and more general nonlinear mechanical metamaterials :

absorption of vibrations or shocks,

shock redirection,

energy focusing / trapping,

acoustic diode,...

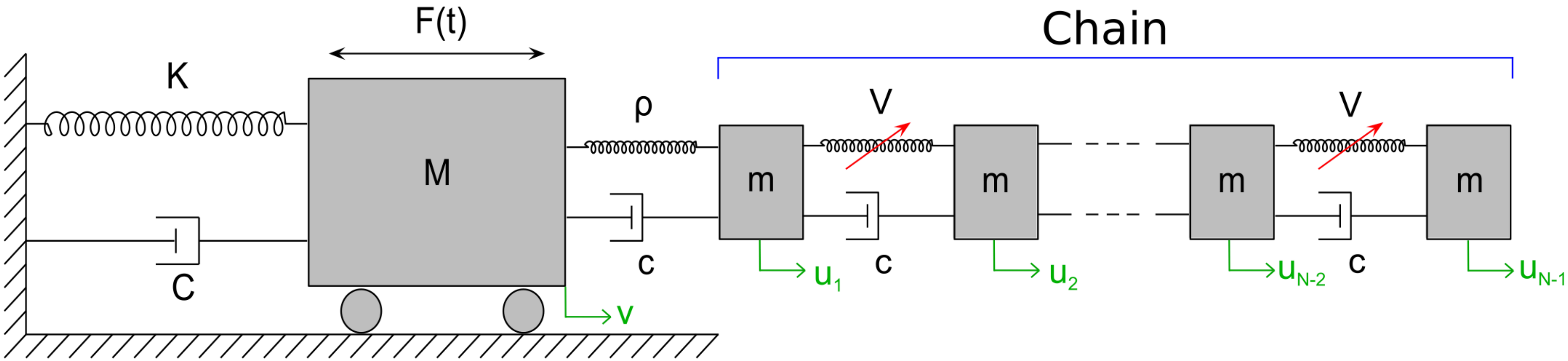
- Analytical study challenging :

nonlinear waves in spatially discrete media,

strong or nonsmooth nonlinearities can make multiscale analysis more difficult

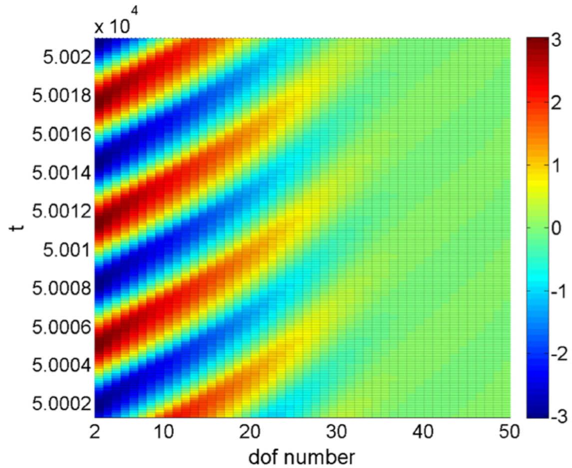
Example 1 : vibratory control of a linear system by addition of a chain of nonlinear oscillators

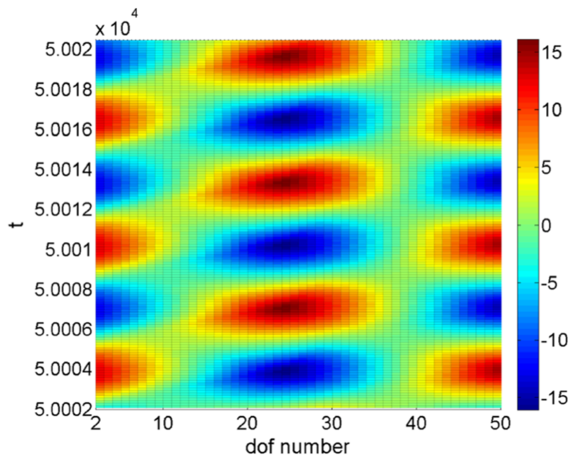
S. Charlemagne, C.-H. Lamarque, A. Ture Savadkoohi, Acta Mech 228, 2017



Amplitude of oscillations :

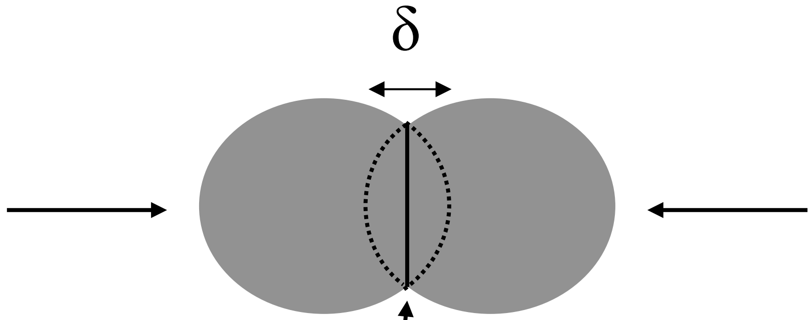
Different nonlinear oscillation patterns depending on parameters and initial conditions :





Example 2 : impacts and solitary waves in granular chains

Chain of spherical beads :



repulsive Hertz contact force $F \approx \delta^{3/2}$

Model for a nondissipative granular chain :

(Fermi-Pasta-Ulam lattice with Hertzian interactions)

$$\ddot{x}_n = (x_{n-1} - x_n)_+^{3/2} - (x_n - x_{n+1})_+^{3/2}$$

$$(a)_+ = \max(a,0)$$

$x_n =$ displacement of bead n

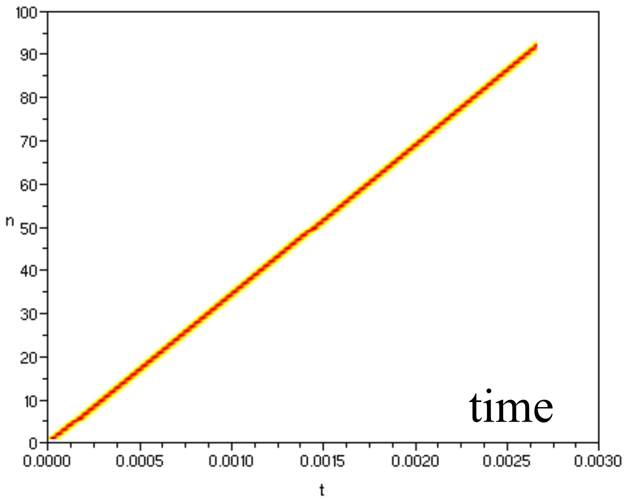
Example 2 : impacts and solitary waves in granular chains

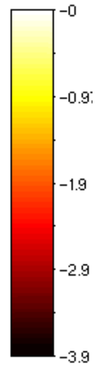
Numerical computation of contact forces after an impact :

homogeneous chain

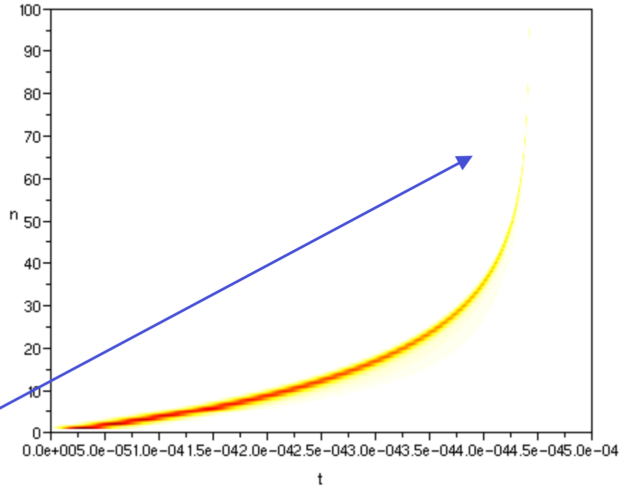
Particle

number





exponentially tapered chain



Compression pulse (Nesterenko's solitary wave) ➔

pulse attenuation during propagation (shock absorber)

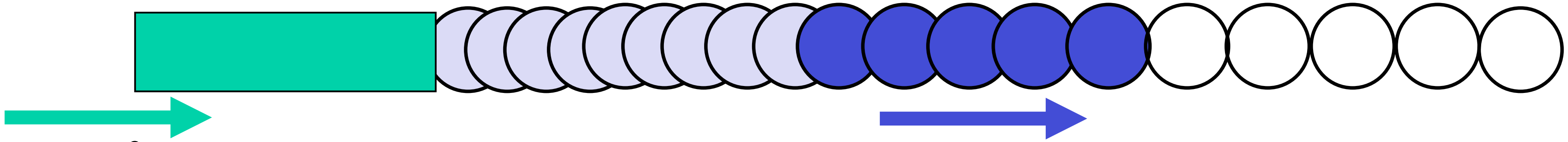
impact➔

Melo et al, Phys Rev E 73, 041305 (2006)

Our focus in this talk : traveling compression fronts

upstream : $x_n(t) \sim 0$

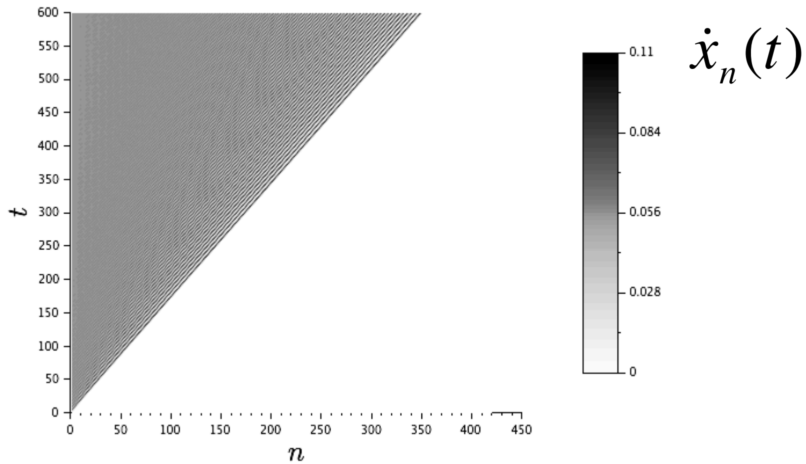
downstream : $x_n(t) - x_{n+1}(t) \sim \delta > 0$



$v = c\delta$

wave velocity c

Numerical time-integration of Hertzian chain (δ=0.1) →



-Experiments: Nesterenko and Lazaridi `87, Nesterenko `94, Molinari and Daraio `09

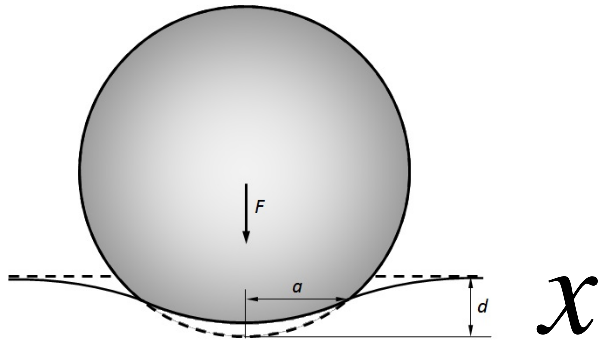
-Formal continuum limits:

Herbold and Nesterenko `07, Liang et al `19 (precompression), G.J. `21.

The front profile heavily depends on dissipation

Dissipative contact force model of Kuwabara and Kono (KK)

$$F = -k\,(x^{\alpha} + \gamma\,\alpha\,x^{\alpha-1}\frac{dx}{dt})$$



$k > 0, \quad \alpha > 1 \ \ (\alpha = 3/2$ for Hertz contact)

$\gamma > 0$ dissipation constant (material parameter)

❖ Derived from continuum mechanics (viscoelastic solids) for α=3/2:

Kuwabara, Kono `87, Brilliantov et al `15, Goldobin et al `15

❖ For α<2, not Lipschitz-continuous at x=0 (onset of contact)

Other contact models :

-Simon-Hunt-Crossley (`67, `75)

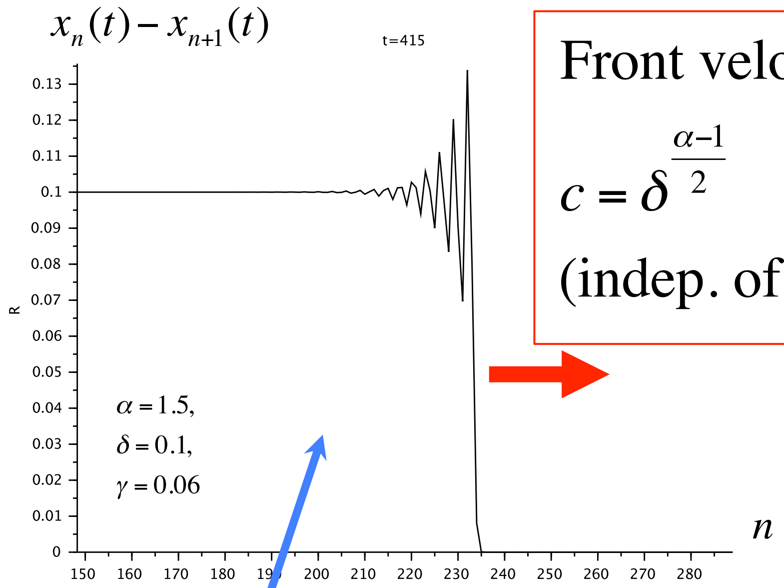
-Hertz with linear dashpot

application to compression fronts : Herbold and Nesterenko `07, Liang et al `19

Dissipative fronts in the KK model

G.J. ‘21

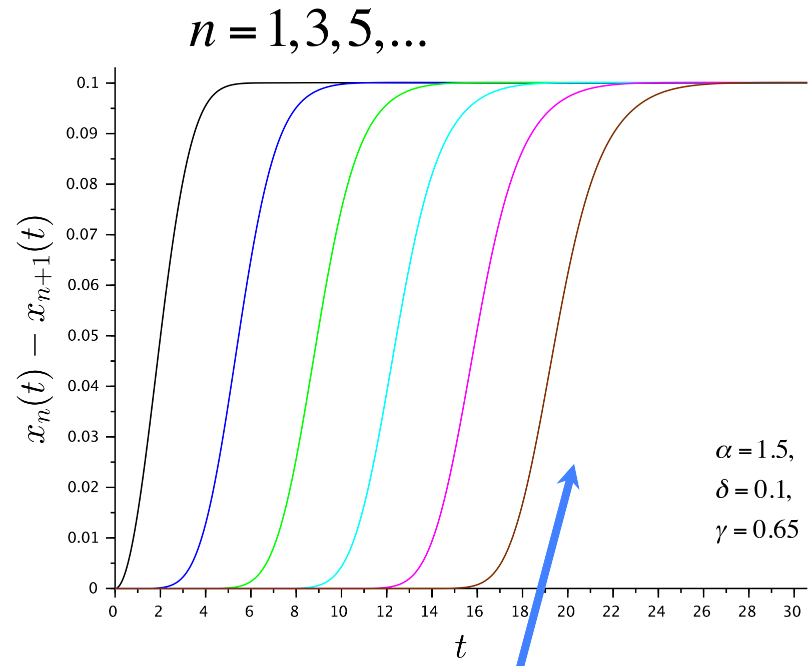
$$\frac{d^2 x_n}{dt^2} = [1+\gamma \frac{d}{dt}]\Big((x_{n-1}-x_n)_+^\alpha - (x_n - x_{n+1})_+^\alpha\Big),\ \ n \geq 1 \qquad \text{(pressure at } n=1 : x_0 - x_1 = \delta > 0\text{)}$$



Front velocity :

$$c = \delta^{\frac{\alpha-1}{2}}$$

(indep. of γ)



Fronts underdamped for $\gamma < \gamma_c = \delta^{(1-\alpha)/2}\Gamma_c(\alpha) = v^{(1-\alpha)/(1+\alpha)}\Gamma_c(\alpha)$, overdamped for $\gamma \geq \gamma_c$.
Critical rescaled damping $\Gamma_c(\alpha)$ computed numerically.

$\rightarrow$ For a given granular chain with KK contact law, underdamped regime occurs when impact velocity v is small enough (opposite situation for Hertz springs with linear dashpot)

Analytical study of dissipative fronts :

♦ Granular chain with KK contact law:

$$\ddot{x} = -\delta^- \left[\left(-\delta^+ x\right)_+^\alpha + \gamma \frac{d}{dt} \left(-\delta^+ x\right)_+^\alpha \right] \qquad \alpha > 1, \;\; \gamma > 0$$

$$x(t) = (x_n(t))_{n \in \mathbb{Z}}, \quad (\delta^+ x)(n) = x(n+1) - x(n), \quad (\delta^- x)(n) = x(n) - x(n-1)$$

♦ First order integro-differential equation for traveling waves :

Ansatz $x_n(t) = X(\xi), \quad \xi = n - c\,t$ $\Rightarrow$ advance-delay differential equation :

$$c^2 X'' = -\delta^- \left[\left(-\delta^+ X\right)_+^\alpha - \gamma c \frac{d}{d\xi} \left(-\delta^+ X\right)_+^\alpha \right]$$

Using $(\delta^- f)(\xi) = \dfrac{d}{d\xi} \displaystyle\int_0^1 f(\xi - \varphi)\, d\varphi$ and integrating (λ = constant):

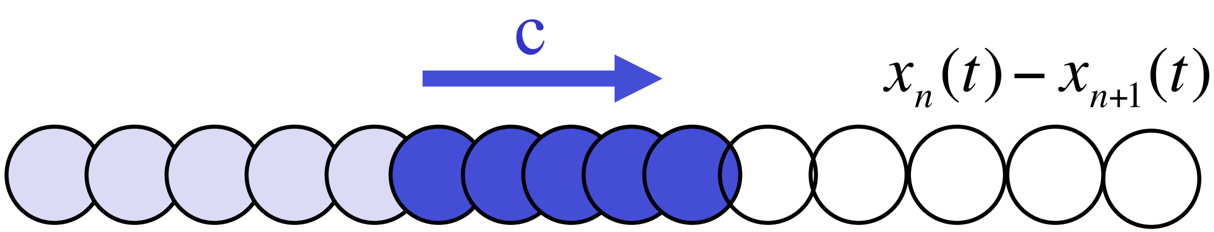
$$c^2 X' = -\int_0^1 \left[\left(-\delta^+ X\right)_+^\alpha \right](\xi - \varphi)\, d\varphi + \gamma c\, \delta^- \left(-\delta^+ X\right)_+^\alpha + \lambda$$

♦ Application: front velocity

At $-\infty$:

$x_n \sim -n\delta + c\delta t$

$x_n(t) - x_{n+1}(t) \to \delta$



At $+\infty$:

$x_n(t) - x_{n+1}(t) \to a$

$$c^2 X' + \int_0^1 \left[\left(-\delta^+ X\right)_+^\alpha\right](\xi - \varphi)\,d\varphi - \gamma c\,\delta^-\left(-\delta^+ X\right)_+^\alpha = \text{constant}$$

Assuming $X \sim -a\xi + b$ when $\xi \to +\infty$ and $X \sim -\delta\xi + \chi$ when $\xi \to -\infty$ with $a, \delta \geq 0 \to$ jump condition (independent of damping constant):

$$c^2(\delta - a) = \delta^\alpha - a^\alpha$$

For a chain at rest at $+\infty$, one has $a = 0$ and

$$c = \delta^{\frac{\alpha-1}{2}}$$

Fronts broaden for α close to 1 → continuum limit approximation using α-1 as a small parameter (asymptotics previously introduced for solitary waves by Chatterjee '99)

$$x_n(t) - x_{n+1}(t) \approx \delta\, y_{\mathrm{h}}^{1/\alpha}(s) \text{ with } s = 2\sqrt{3(1-\frac{1}{\alpha})}(n - \delta^{(\alpha-1)/2} t)$$

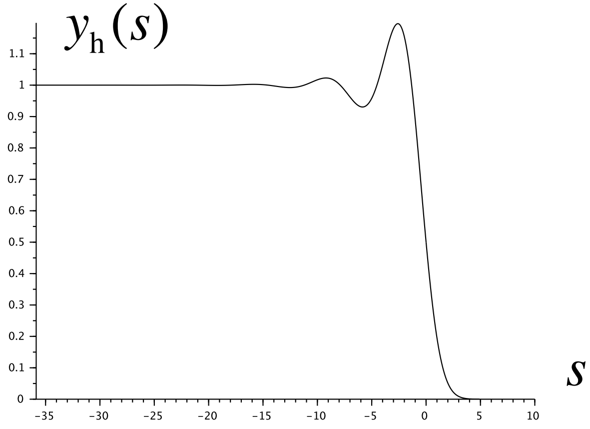
y_{h} solves the nonlinear boundary value problem:

$$\frac{d^2 y_{\mathrm{h}}}{ds^2} - 2\gamma\sqrt{\frac{3\alpha\delta^{\alpha-1}}{\alpha-1}}\frac{dy_{\mathrm{h}}}{ds} + y_{\mathrm{h}}\ln(y_{\mathrm{h}}) = 0,$$

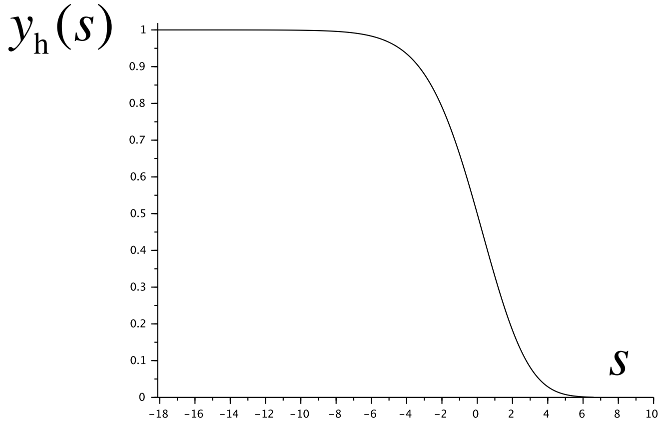
$$\lim_{s\to+\infty} y_{\mathrm{h}}(s) = 0,\ \lim_{s\to-\infty} y_{\mathrm{h}}(s) = 1$$

y_{h} underdamped for

$$\gamma < \gamma_{\mathrm{c}} = \sqrt{\frac{\alpha-1}{3\alpha\delta^{\alpha-1}}}$$



underdamped case



overdamped case

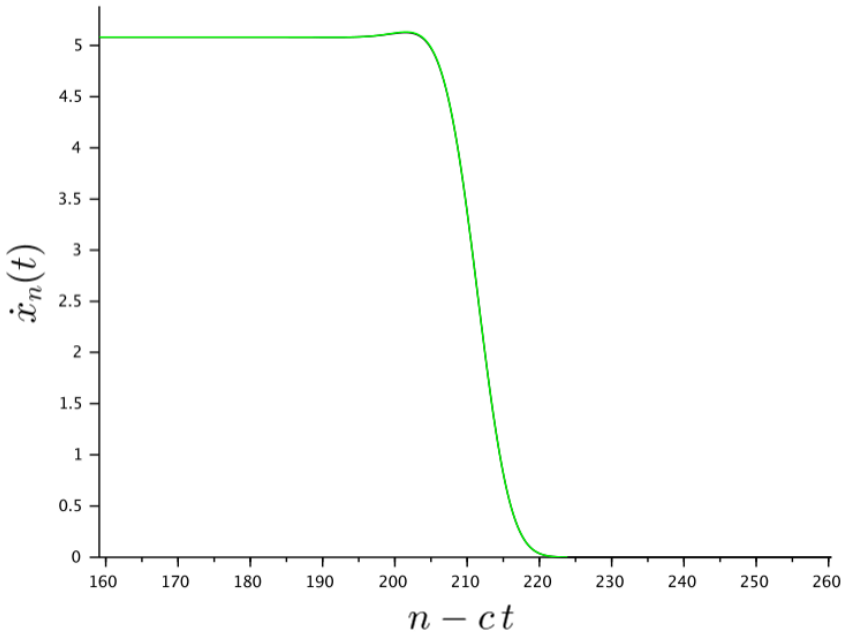
Continuum approximation versus numerical front profiles :

Parameters: $\alpha = 1.02, \quad \delta = 5$

Green curve: approximate front solution (continuum approximation) for $\gamma = 0.06$ $\rightarrow$

Black curve (almost superposed): front solution computed iteratively

Numerical $\gamma_c \approx 0.078$, cont. approx. $\gamma_c \approx 0.08$

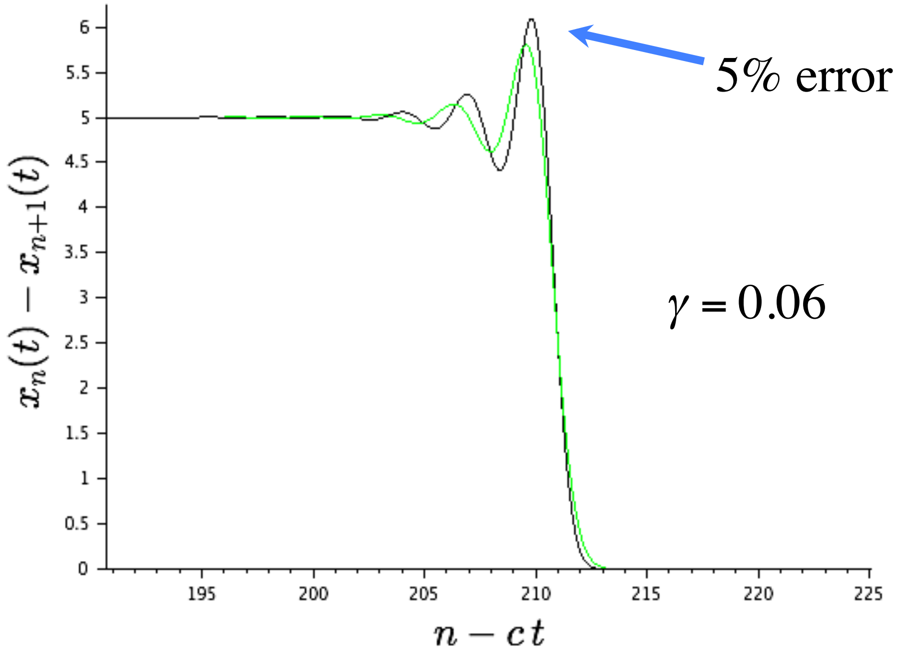


Parameters: $\alpha = 1.5, \quad \delta = 5$

the continuum approximation remains meaningful $\rightarrow$

Numerical $\gamma_c \approx 0.17$

Continuum approx. $\gamma_c \approx 0.22$



Conclusion and remarks

- Impacts on granular chains generate nonlinear waves: solitary waves, fronts due to strongly nonlinear Hertzian interactions

- Front profiles broaden for nonlinearity exponent $\alpha>1$ close to unity
  $\rightarrow$ continuum limit approximation

- Numerical validation of this continuum approximation :
  -convergence towards theoretical front profiles for $\alpha\rightarrow 1$
  -qualitatively correct results for $\alpha=3/2$ (Hertz)

- The continuum limit $\alpha\rightarrow 1$ can also be applied to solitary waves and breathers
  Chatterjee `99, G.J and Pelinovsky `14, G.J. and Starosvetsky `14, G.J. `18, Gzal et al `23

Thank you for your attention !

Reference :

G.J., Traveling fronts in dissipative granular chains and nonlinear lattices, Nonlinearity 34 (2021), 1758

A strongly nonlinear continuum limit:

Traveling waves $x_n(t) = X(n - c\,t)$ satisfy the advance-delay diff. equation

$$c^2 X'' = -\delta^- \left[\left(-\delta^+ X\right)_+^\alpha - \gamma c \frac{d}{d\xi} \left(-\delta^+ X\right)_+^\alpha \right]$$

where $(\delta^+ X)(\xi) = X(\xi + 1) - X(\xi), \quad (\delta^- X)(\xi) = X(\xi) - X(\xi - 1)$

Setting $\mathrm{c} = \delta^{(\alpha-1)/2},\ u = -\delta^+ X$ and assuming $u > 0$ (no gap between beads):

$$\delta^{\alpha-1} u'' = \Delta_d \left[u^\alpha - \gamma \delta^{(\alpha-1)/2} \left(u^\alpha\right)' \right], \quad (\Delta_d u)(\xi) = u(\xi + 1) - 2u(\xi) + u(\xi - 1)$$

We look for long waves $u(\xi) = \delta y^{1/\alpha}(s)$ with $s = 2\sqrt{3}\,\varepsilon\,\xi,\ \varepsilon \approx 0^+$:

$$\left(y^{1/\alpha}\right)'' = \left[\frac{d^2}{ds^2} + \varepsilon^2 \frac{d^4}{ds^4} + O(\varepsilon^4) \right] \left(y - 2\sqrt{3}\,\varepsilon\gamma\delta^{(\alpha-1)/2} \left(y\right)' \right)$$

$$\left(y^{1/\alpha}\right)'' = \left[\frac{d^2}{ds^2} + \varepsilon^2 \frac{d^4}{ds^4} + O(\varepsilon^4)\right]\left(y - 2\sqrt{3}\,\varepsilon\gamma\delta^{(\alpha-1)/2}\left(y\right)'\right)$$

We assume small dissipation $\gamma = O(\varepsilon)$,

set $2\sqrt{3}\,\varepsilon\gamma\delta^{(\alpha-1)/2} = \mu\varepsilon^2 > 0$, neglect $O(\varepsilon^4)$, integrate:

$$y'' - \mu y' + \frac{1 - y^{\frac{1}{\alpha}-1}}{\varepsilon^2} y = 0$$

Assuming $\alpha \approx 1$ and setting $\varepsilon^2 = 1 - \dfrac{1}{\alpha} = \mathrm{O}(\alpha - 1)$ leads to a log nonlinearity:

$$y\frac{1 - y^{\frac{1}{\alpha}-1}}{\varepsilon^2} = y\frac{1 - \exp(-\varepsilon^2 \ln(y))}{\varepsilon^2} = y\ln(y) + O(\varepsilon^2)$$

(Chatterjee \`99, G.J. and Pelinovsky \`14, G.J. and Starosvetsky \`14, G.J. \`18, \`20)

When $\alpha \to 1$ we obtain the limit problem:

$$\frac{d^2 y}{ds^2} - \mu\frac{dy}{ds} + y\ln(y) = 0$$

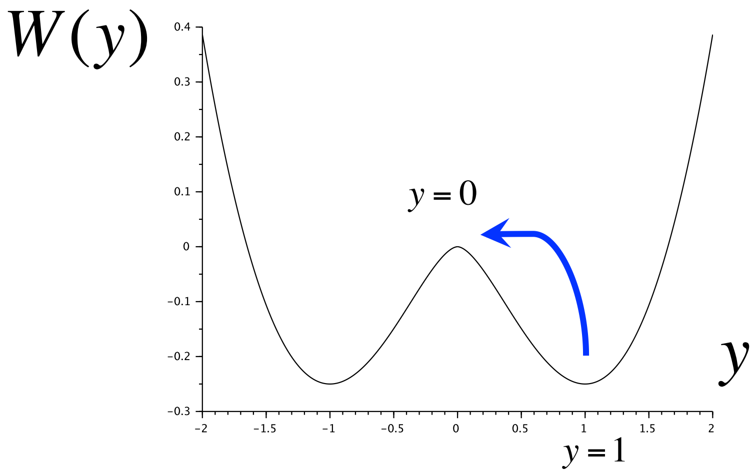
Dissipative fronts in the continuum limit :

Setting $\varepsilon = \sqrt{1 - \frac{1}{\alpha}}$ and $\alpha \to 1$, we have obtained:

$$\frac{d^2 y}{ds^2} - \mu \frac{dy}{ds} + y \ln(y) = 0$$

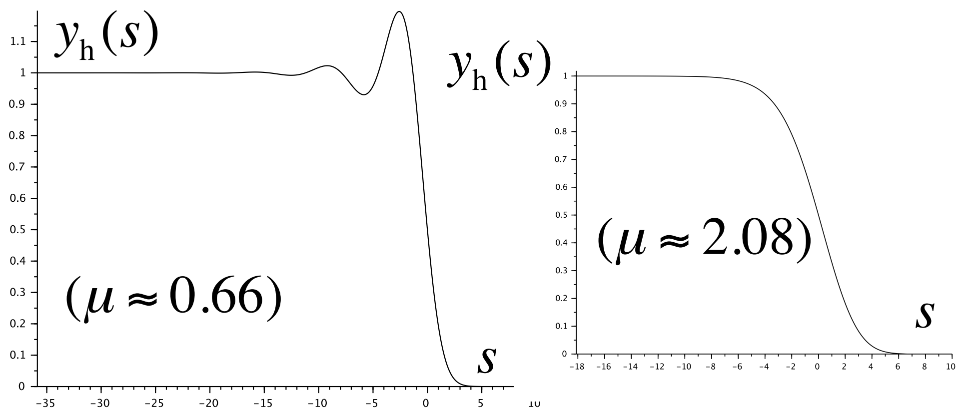
Positive solution ~ motion of a particle in potential $W(y) = \frac{y^2}{2}\left(\ln|y| - \frac{1}{2}\right)$

with a negative damping $(\mu > 0)$:



$\Rightarrow$ Heteroclinic solution :

$y_{\mathrm{h}}(s) \to 1$ for $s \to -\infty$, $y_{\mathrm{h}}(s) \to 0$ for $s \to +\infty$



$$x_n(t) - x_{n+1}(t) \approx \delta\, y_{\mathrm{h}}^{1/\alpha}\left(2\sqrt{3(1-\frac{1}{\alpha})}(n - \delta^{(\alpha-1)/2} t)\right)$$

is overdamped for $\mu \geq 2$, i.e. $\gamma \geq \sqrt{\frac{\alpha - 1}{3\alpha\delta^{\alpha-1}}}$