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EX-MODELI
GdR CNRS 2134

# Direct parametrisation of invariant manifold with shell finite element: nonlinear dynamics of thin structures using reduced order modeling

17-18 octobre 2024

Journées annuelles du GdR EX-MODELI | Lyon

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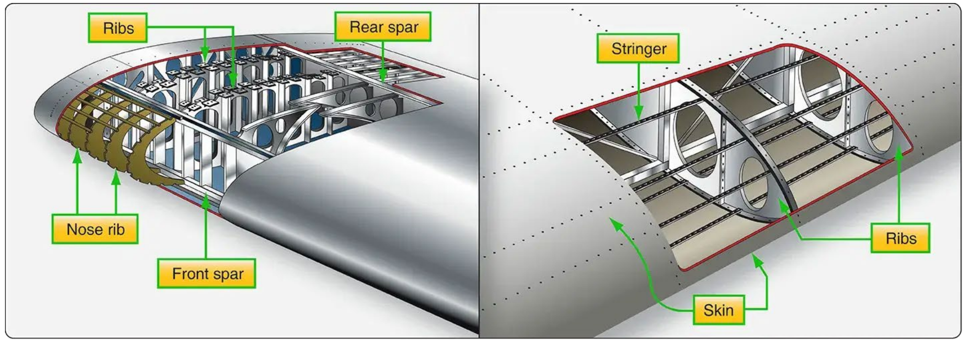
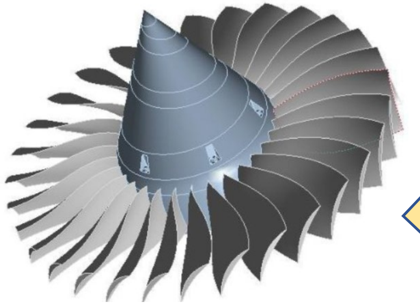
# Abundant application scenarios of thin structures

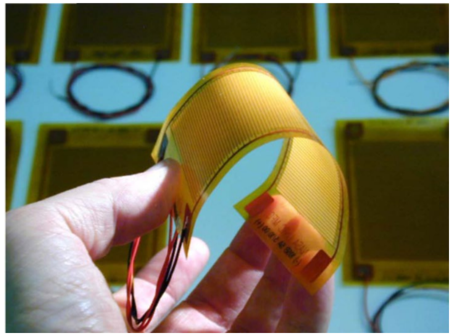


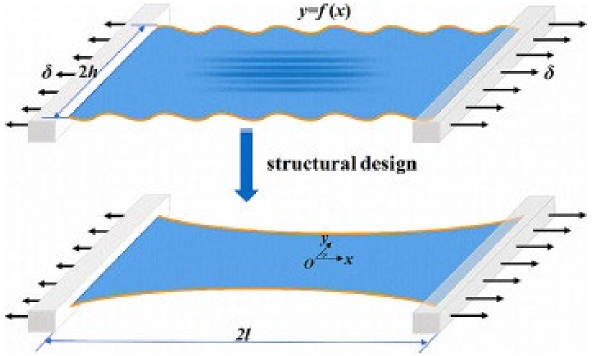
## Engineering modeling



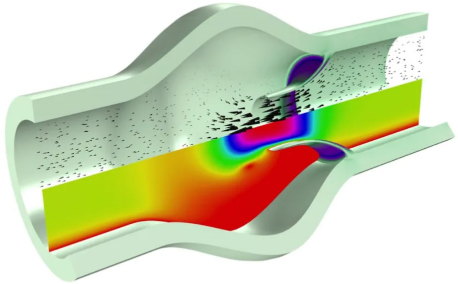



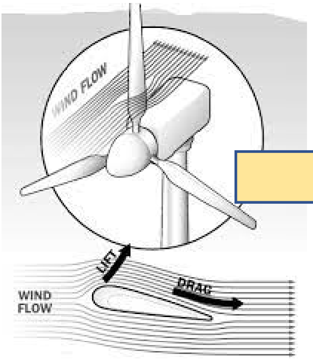
## Thin films system






## Fluid-structure interaction






**Research highlights**

1. How to establish an appropriate finite element method for modeling thin structures ?
2. An appropriate nonlinear dynamics reduced-order method to lower the cost of numerical simulations

# Where do we stand?



**Normal form method**

- Touzé C. A **normal form approach** for nonlinear normal modes
- Vizzaccaro A, et al. **Direct computation** of nonlinear mapping via **normal form** for reduced-order models of finite element nonlinear structures

**DPIM (Direct parametrisation of invariant manifold )**

- Opreni A, et al. High-order **direct parametrisation of invariant manifolds** for model order reduction of finite element structures: application to generic forcing terms and parametrically excited systems
- Vizzaccaro A, et al. Direct parametrisation of invariant manifolds for generic **non-autonomous** systems including superharmonic resonances

**Shell finite element modeling**

- **Fewer degrees of freedom** compared to solid elements
- More convenient definition of the **kinematic description** of thin structures in the transverse direction
- Introduce **assumed natural strain** to prevent Poisson locking

**Nonlinear dynamics solution methods**

- Increment harmonic balance method (For **full order models**)
- Collocation method (For **reduced order models**)

**These theories lay the foundation for our development of a framework for solving thin structures !**

# Curved shell structure modeling: governing equations

## Positional relationship

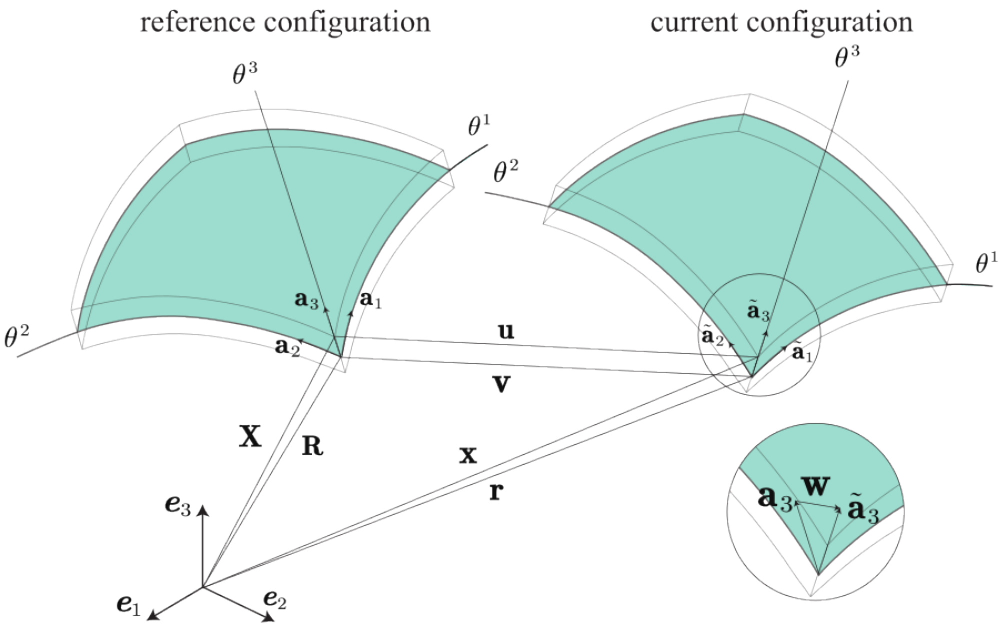
$$\mathbf{X}(\theta^\alpha,\theta^3)=\mathbf{R}(\theta^\alpha)+\theta^3\mathbf{a}_3(\theta^\alpha),\alpha=1,2$$

$$\mathbf{x}(\theta^\alpha,\theta^3)=\mathbf{r}(\theta^\alpha)+\theta^3\tilde{\mathbf{a}}_3(\theta^\alpha),\alpha=1,2$$

## Covariant base tensor

$$\mathbf{G}_\alpha=\mathbf{X}_{,\alpha}=\mathbf{a}_\alpha+\theta^3\mathbf{a}_{3,\alpha} \quad \mathbf{g}_\alpha=\mathbf{x}_{,\alpha}=\tilde{\mathbf{a}}_\alpha+\theta^3\tilde{\mathbf{a}}_{3,\alpha}$$

$$\mathbf{G}_3=\mathbf{X}_3=\mathbf{a}_3 \quad \mathbf{g}_3=\mathbf{x},3=\tilde{\mathbf{a}}_3$$




## Green-Lagrange strain

$$\begin{cases} E_{ij}=\frac{1}{2}(\mathbf{g}_i\cdot\mathbf{g}_j-\mathrm{G}_i\cdot\mathrm{G}_j) \\ E_{ij}=\frac{1}{2}\left(\mathbf{G}_i\cdot\frac{\partial\mathbf{u}}{\partial\theta^j}+\mathbf{G}_j\cdot\frac{\partial\mathbf{u}}{\partial\theta^i}+\frac{\partial\mathbf{u}}{\partial\theta^i}\frac{\partial\mathbf{u}}{\partial\theta^j}\right) \end{cases}$$

## Constitutive relation

$\mathbf{S}=\mathbb{D}:\mathbf{E}$ **Traditional shell elements lead to locking issues**

$$E_{33}^{(0)}+\theta^3E_{33}^{(1)}\simeq-\frac{D^{33ij}}{D^{3333}}(E_{ij}^{(0)}+\theta^3E_{ij}^{(1)})$$

## Enhanced assumed strain

$$\mathbf{E}^{\text{full}}=\mathbf{E}+\tilde{\mathbf{E}} \quad \text{with} \quad \int_\Omega\mathbf{S}:\delta\tilde{\mathbf{E}}\mathrm{d}\Omega=0$$

## Hu-Washizu functional

$$\Pi_{HW}(\mathbf{u},\mathbf{E}^{\text{full}},\mathbf{S})=\int_\Omega\frac{1}{2}\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\mathrm{d}\Omega+\int_\Omega\frac{1}{2}\mathbf{E}^{\text{full}}:\mathbb{D}:\mathbf{E}^{\text{full}}\mathrm{d}\Omega$$
$$-\int_\Omega\mathbf{S}:(\mathbf{E}^{\text{full}}-\mathbf{E})\mathrm{d}\Omega-\mathbf{F}(t)\cdot\mathbf{u}$$

$$\begin{cases} \int_\Omega\rho\ddot{\mathbf{u}}\cdot\delta\mathbf{u}\mathrm{d}\Omega+\int_\Omega\mathbf{S}:\delta\mathbf{E}\mathrm{d}\Omega-\mathbf{F}(t)\cdot\delta\mathbf{u}=0 & \textbf{Traditional shell elements} \\ \int_\Omega\mathbf{S}:\delta\tilde{\mathbf{E}}\mathrm{d}\Omega=0 & \textbf{Additional equations} \end{cases}$$

**Extend traditional variational equations to avoid locking issues**

# Curved shell structure modeling: finite element discretisation



## Shape function interpolation

$$\{\mathbf{u}\}=[\mathbf{N}]\{\mathbf{u}^{(e)}\},\quad \{\delta\mathbf{u}\}=[\mathbf{N}]\{\delta\mathbf{u}^{(e)}\},\quad \{\nabla\mathbf{u}\}=[\boldsymbol{\Xi}]\{\mathbf{u}^{(e)}\}$$

## Green-Lagrange strain

$$\{\mathbf{E}\}=\left([\mathbf{B}_l]+\frac{1}{2}[\mathbf{B}_{nl}(\mathbf{u})]\right)\{\mathbf{u}^{(e)}\}=\left([\mathbf{R}]+\frac{1}{2}[\mathbf{A}(\mathbf{u})]\right)[\boldsymbol{\Xi}]\{\mathbf{u}^{(e)}\}$$

$$\{\delta\mathbf{E}\}=([\mathbf{B}_l]+[\mathbf{B}_{nl}(\mathbf{u})])\{\mathbf{u}^{(e)}\}=([\mathbf{R}]+[\mathbf{A}(\mathbf{u})])[\boldsymbol{\Xi}]\{\delta\mathbf{u}^{(e)}\}$$

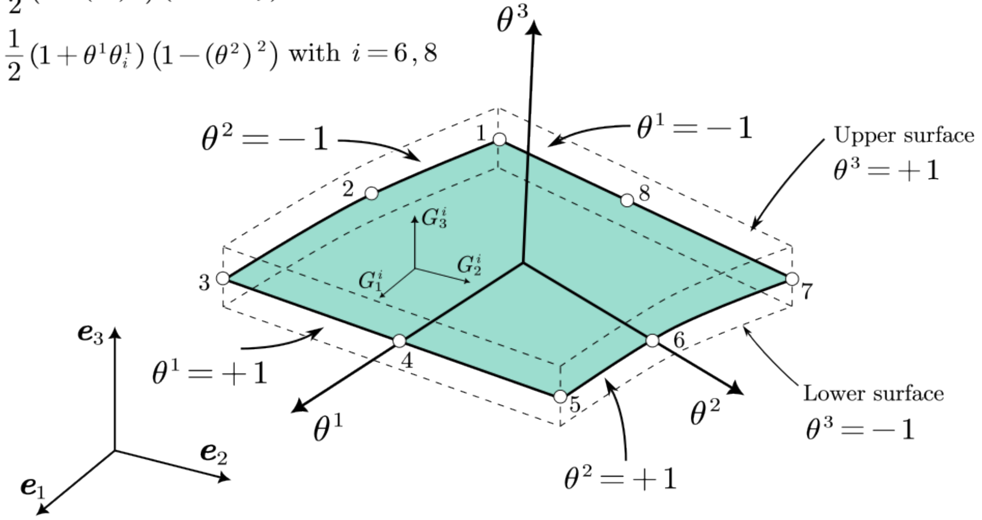
## Interpolation of enhanced assumed strain

$\{\tilde{E}\}=[\mathbf{B}_\alpha]\{\alpha^{(e)}\}$ with $\tilde{E}_{33}=\alpha_1+\alpha_2\theta^1+\alpha_3\theta^2+\alpha_4\theta^1\theta^2$

$$N^i=\frac{1}{4}(1+\theta^1\theta_i^1)(1+\theta^2\theta_i^2)(\theta^1\theta_i^1+\theta^2\theta_i^2-1)\ \text{with}\ i=1,2,3,4$$

$$N^i=\frac{1}{2}(1-(\theta^1)^2)(1+\theta^2\theta_i^2)\ \text{with}\ i=5,7$$

$$N^i=\frac{1}{2}(1+\theta^1\theta_i^1)(1-(\theta^2)^2)\ \text{with}\ i=6,8$$




## Development of the dynamic equation

$$[\mathbf{M}]\{\ddot{\mathbf{U}}\}+[\mathbf{C}]\{\dot{\mathbf{U}}\}+[\mathbf{K}_l]\{\mathbf{U}\}+\{\mathbf{G}(\mathbf{U},\mathbf{U})\}+\{\mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U})\}=\{\mathbf{F}(t)\}$$

## where

$$[\mathbf{M}]=\bigwedge_{k=1}^{N}\int_\Omega\rho[\mathbf{N}]^T[\mathbf{N}]\mathrm{d}\Omega$$

$$[\mathbf{K}_l]=\bigwedge_{k=1}^{N}\left(\int_\Omega[\mathbf{B}_l]^T[\mathbf{D}][\mathbf{B}_l]\mathrm{d}\Omega-\int_\Omega[\mathbf{B}_l]^T[\mathbf{D}][\mathbf{B}_\alpha]\mathrm{d}\Omega\times[\mathbf{k}_{\alpha\alpha}]^{-1}\times\int_\Omega[\mathbf{B}_\alpha]^T[\mathbf{D}][\mathbf{B}_l]\mathrm{d}\Omega\right)$$

$$\{\mathbf{G}(\mathbf{U},\mathbf{U})\}=\bigwedge_{k=1}^{N}\Bigg(\int_\Omega[\mathbf{B}_{nl}(\mathbf{u})]^T[\mathbf{D}][\mathbf{B}_l]\mathrm{d}\Omega-\frac{1}{2}\int_\Omega[\mathbf{B}_l]^T[\mathbf{D}][\mathbf{B}_{nl}(\mathbf{u})]\mathrm{d}\Omega$$
$$-\frac{1}{2}\int_\Omega[\mathbf{B}_l]^T[\mathbf{D}][\mathbf{B}_\alpha]\mathrm{d}\Omega\times[\mathbf{k}_{\alpha\alpha}]^{-1}\times\int_\Omega[\mathbf{B}_\alpha]^T[\mathbf{D}][\mathbf{B}_{nl}(\mathbf{u})]\mathrm{d}\Omega$$
$$-\int_\Omega[\mathbf{B}_{nl}(\mathbf{u})]^T[\mathbf{D}][\mathbf{B}_\alpha]\mathrm{d}\Omega\times[\mathbf{k}_{\alpha\alpha}]^{-1}\times\int_\Omega[\mathbf{B}_\alpha]^T[\mathbf{D}][\mathbf{B}_l]\mathrm{d}\Omega\Bigg)\{\mathbf{u}^{(e)}\}$$

$$\{\mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U})\}=\bigwedge_{k=1}^{N}\Bigg(\frac{1}{2}\int_\Omega[\mathbf{B}_{nl}(\mathbf{u})]^T[\mathbf{D}][\mathbf{B}_{nl}(\mathbf{u})]\mathrm{d}\Omega$$
$$-\frac{1}{2}\int_\Omega[\mathbf{B}_{nl}(\mathbf{u})]^T[\mathbf{D}][\mathbf{B}_\alpha]\mathrm{d}\Omega\times[\mathbf{k}_{\alpha\alpha}]^{-1}\times\int_\Omega[\mathbf{B}_\alpha]^T[\mathbf{D}][\mathbf{B}_{nl}(\mathbf{u})]\Bigg)\{\mathbf{u}^{(e)}\}$$

$$[\mathbf{C}]=p_1[\mathbf{M}]+p_2[\mathbf{K}_l]$$

**Establish standard nonlinear dynamic equations to address DPIM solutions**

# Direct parametrisation of invariant manifold



**Second-order dynamic equation**

$$[\mathbf{M}]\{\ddot{\mathbf{U}}\}+[\mathbf{C}]\{\dot{\mathbf{U}}\}+[\mathbf{K}_l\{\mathbf{U}\}+\{\mathbf{G}(\mathbf{U},\mathbf{U})\}+\{\mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U})\}=\{\mathbf{F}(t)\}$$

**First-order dynamic equation**

$$[\mathbf{B}]\{\dot{\mathbf{y}}\}=[\mathbf{A}]\{\mathbf{y}\}+\{\boldsymbol{Q}(\mathbf{y},\mathbf{y})\}+\{\boldsymbol{\Upsilon}\}\tilde{z} \quad \text{and} \quad \dot{\tilde{z}}=\tilde{\lambda}\tilde{z}$$

**Nonlinear mapping**

$$\{\mathbf{y}\}=\mathbf{W}(\mathbf{z})$$

**ROM equations**

$$\{\dot{\mathbf{z}}\}=\mathbf{f}(\mathbf{z})$$

$$\mathbf{W}(\mathbf{z})=\sum_{p=1}^{o}\langle\mathbf{W}(\mathbf{z})\rangle_p$$

$$\mathbf{f}(\mathbf{z})=\sum_{p=1}^{o}\langle\mathbf{f}(\mathbf{z})\rangle_p$$

$$[\mathbf{B}]\langle\frac{\partial\mathbf{W}(\mathbf{z})}{\partial\mathbf{z}}\mathbf{f}(\mathbf{z})\rangle_p=[\mathbf{A}]\langle\mathbf{W}(\mathbf{z})\rangle_p+\langle\boldsymbol{Q}(\mathbf{W}(\mathbf{z}),\mathbf{W}(\mathbf{z}))\rangle_p+\langle\boldsymbol{\Upsilon}\tilde{z}\rangle_p$$

$$[\mathbf{B}]\langle\frac{\partial\mathbf{W}(\mathbf{z})}{\partial\mathbf{z}}\mathbf{f}(\mathbf{z})\rangle_1=[\mathbf{A}]\langle\mathbf{W}(\mathbf{z})\rangle_1+\langle\boldsymbol{\Upsilon}\tilde{z}\rangle_1 \qquad p=1$$

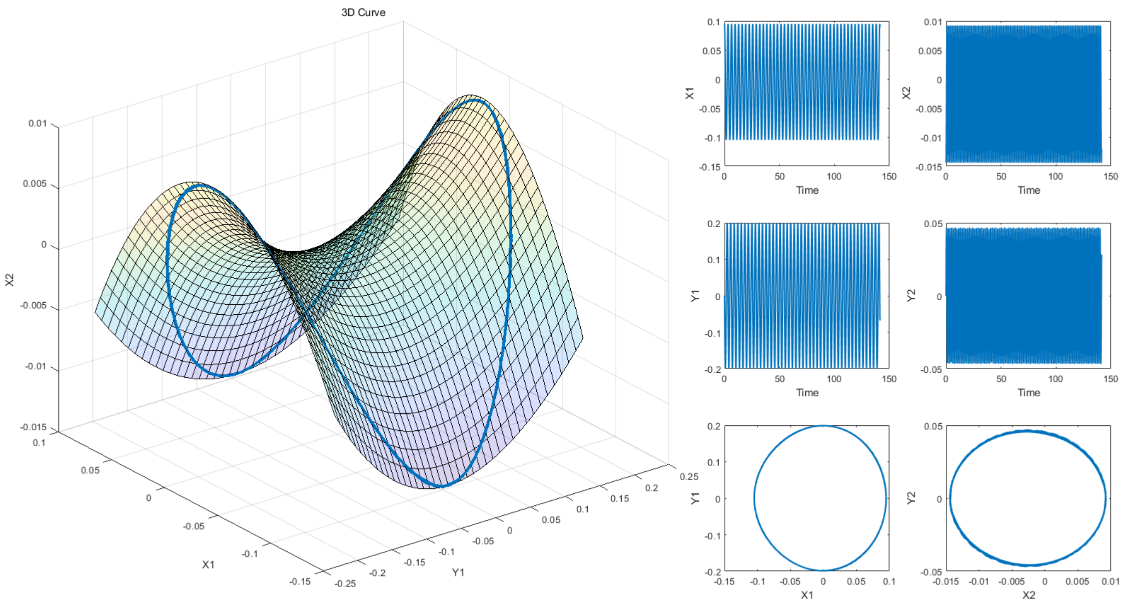
$$[\mathbf{B}]\langle\frac{\partial\mathbf{W}(\mathbf{z})}{\partial\mathbf{z}}\mathbf{f}(\mathbf{z})\rangle_p=[\mathbf{A}]\langle\mathbf{W}(\mathbf{z})\rangle_p+\langle\boldsymbol{Q}(\mathbf{W}(\mathbf{z}),\mathbf{W}(\mathbf{z}))\rangle_p \qquad p>1$$

**Order-1 homological equation**

$$\begin{bmatrix}\tilde{\lambda}\mathbf{B}-\mathbf{A} & \mathbf{B}\boldsymbol{\Phi}_{\mathcal{R}} & 0\\ \boldsymbol{\Phi}_{\mathcal{R}}^{\dagger}\mathbf{B} & 0 & 0\\ 0 & 0 & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{W}^{(1,d+1)}\\ \mathbf{f}_{\mathcal{R}}^{(1,d+1)}\\ \mathbf{f}_{\not{\mathcal{R}}}^{(1,d+1)}\end{bmatrix}=\begin{bmatrix}\boldsymbol{\Upsilon}\\ 0\\ 0\end{bmatrix}$$

**Order-p homological equation**

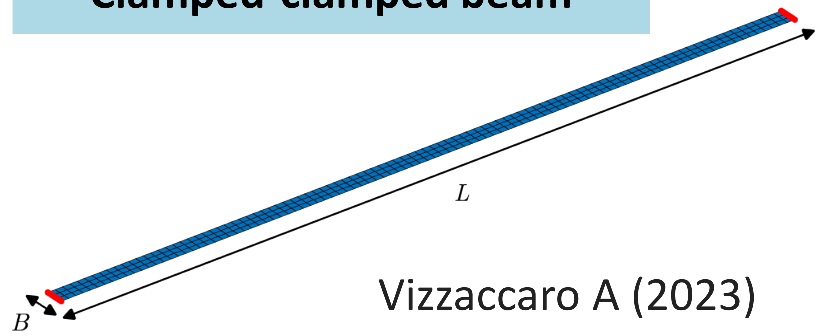
$$\begin{bmatrix}\sigma^{(p,k)}\mathbf{B}-\mathbf{A} & \mathbf{B}\boldsymbol{\Phi}_{\mathcal{R}} & 0\\ \boldsymbol{\Phi}_{\mathcal{R}}^{\dagger}\mathbf{B} & 0 & 0\\ 0 & 0 & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{W}^{(p,k)}\\ \mathbf{f}_{\mathcal{R}}^{(p,k)}\\ \mathbf{f}_{\not{\mathcal{R}}}^{(p,k)}\end{bmatrix}=\begin{bmatrix}\mathbf{R}^{(p,k)}\\ 0\\ 0\end{bmatrix}$$



# Numerical example: modal analysis



**Clamped-clamped beam**




Vizzaccaro A (2023)

**Geometric parameters**

$L = 1000\ \mu\text{m}$

$B = 24\ \mu\text{m}$

$H = 10\ \mu\text{m}$

**Material parameters**

$E = 160\ \text{Gpa}$

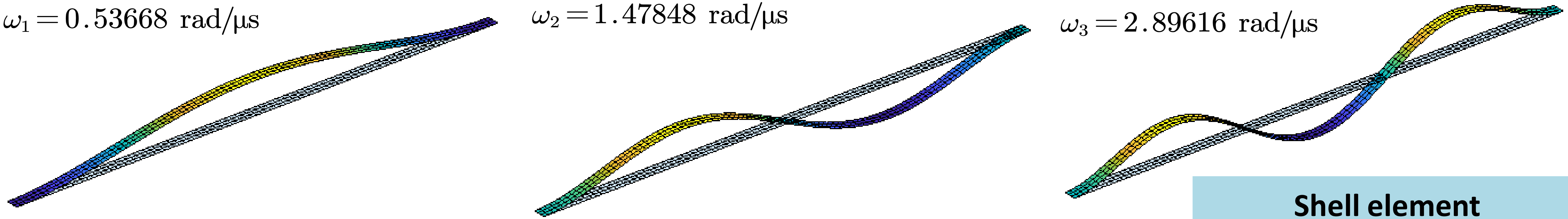
$\upsilon = 0.22$

$\rho = 2320\ \text{kg/m}^3$

$\omega_1 = 0.53668\ \text{rad/}\mu\text{s}$

$\omega_2 = 1.47848\ \text{rad/}\mu\text{s}$

$\omega_3 = 2.89616\ \text{rad/}\mu\text{s}$

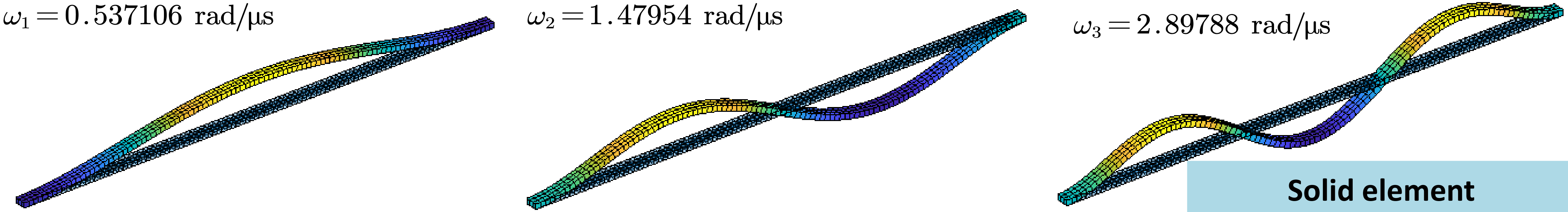


**Shell element**

$\omega_1 = 0.537106\ \text{rad/}\mu\text{s}$

$\omega_2 = 1.47954\ \text{rad/}\mu\text{s}$

$\omega_3 = 2.89788\ \text{rad/}\mu\text{s}$

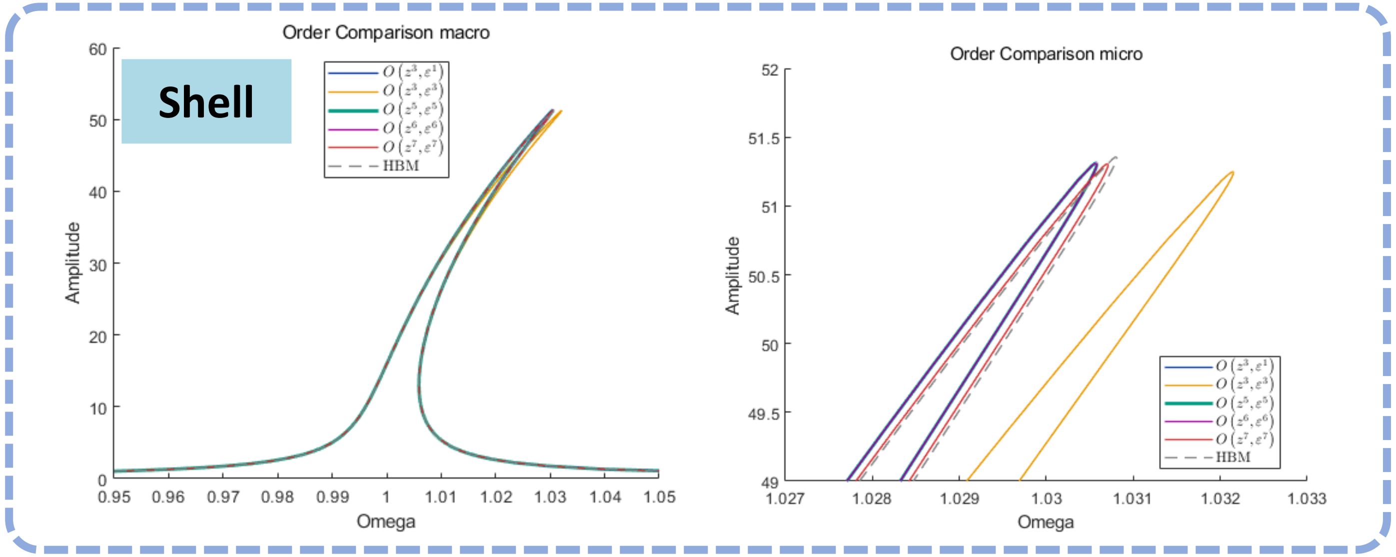


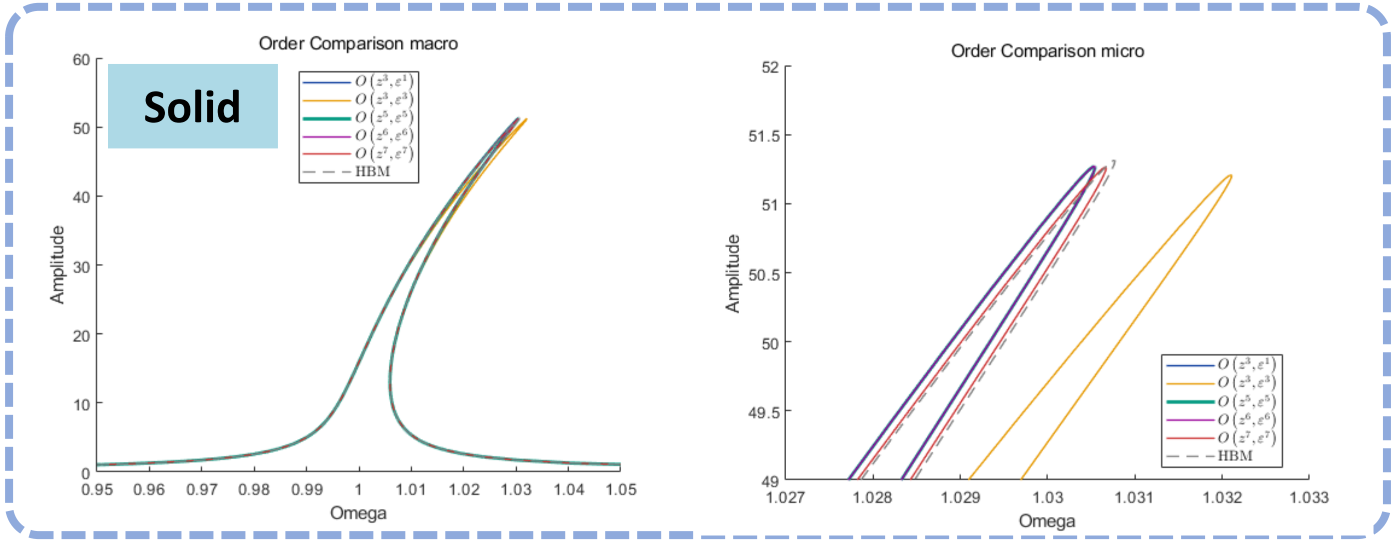
**Solid element**

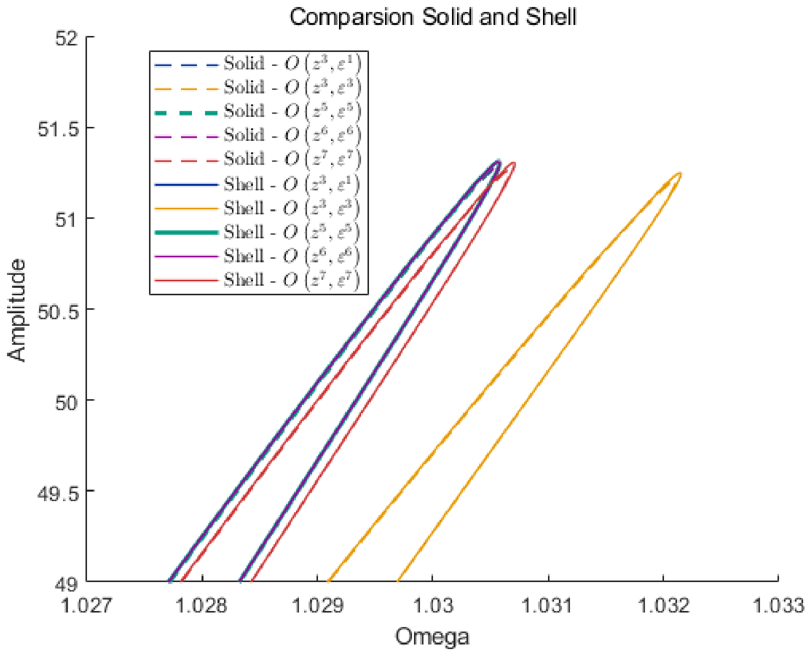
# Numerical example: primary mode

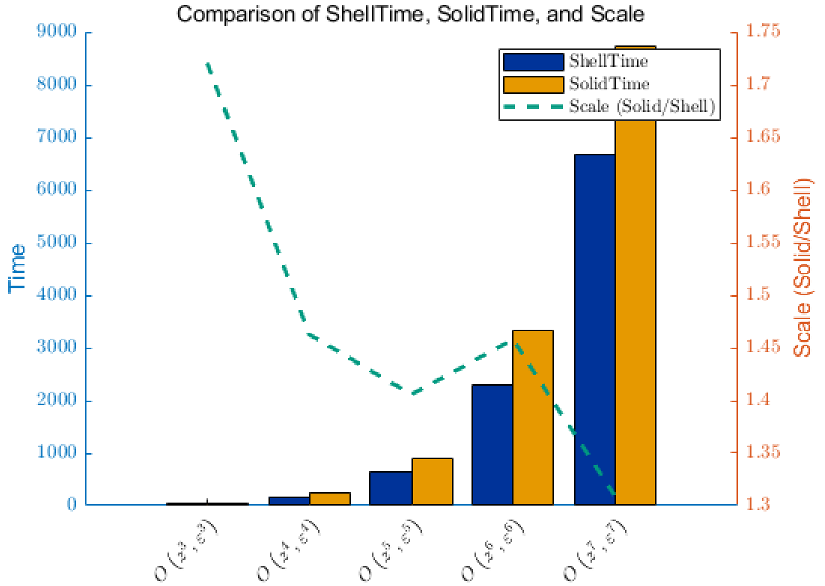


$$[\mathbf{M}]\{\ddot{\mathbf{U}}\}+[\mathbf{C}]\{\dot{\mathbf{U}}\}+[\mathbf{K}_l\{\mathbf{U}\}+\{\mathbf{G}(\mathbf{U},\mathbf{U})\}+\{\mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U})\}=0.03[\mathbf{M}]\{\mathbf{\Phi}_1\}\cos\Omega t$$



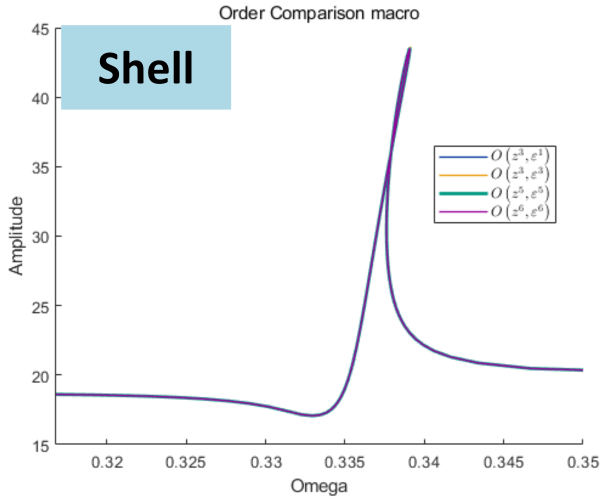



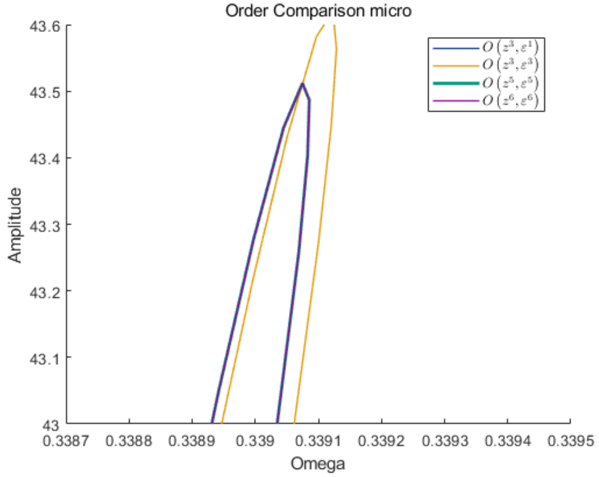



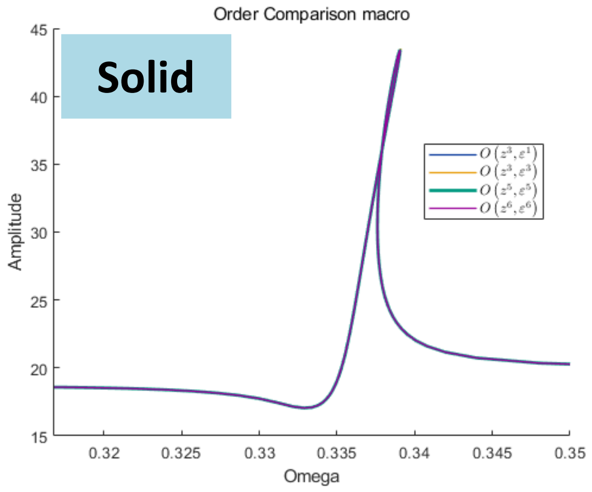



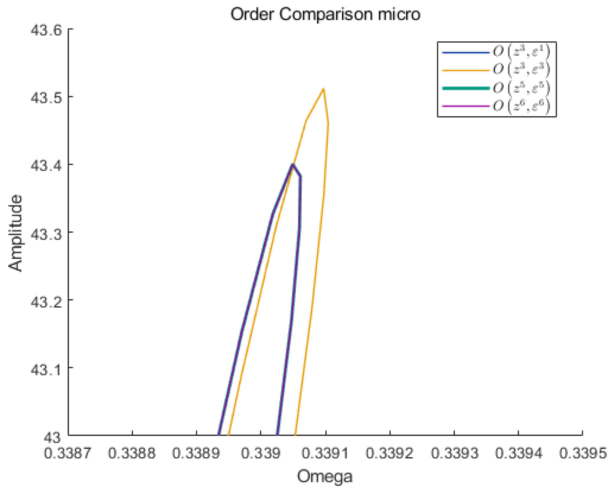
# Numerical example: 1:3 superharmonic resonance

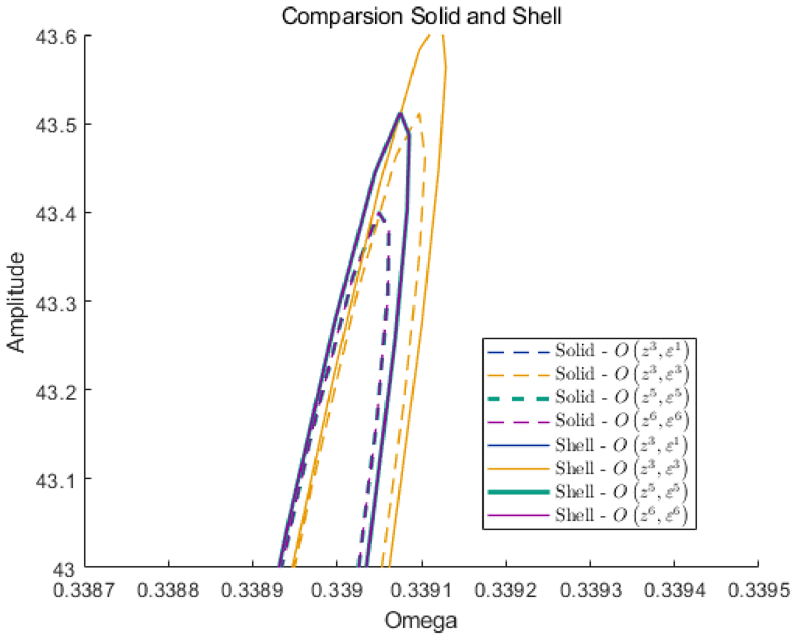
$$[\mathbf{M}]\{\ddot{\mathbf{U}}\}+[\mathbf{C}]\{\dot{\mathbf{U}}\}+[\mathbf{K}_l\{\mathbf{U}\}+\{\mathbf{G}(\mathbf{U},\mathbf{U})\}+\{\mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U})\}=5\,[\mathbf{M}]\,\{\boldsymbol{\Phi}_1\}\cos\Omega t$$

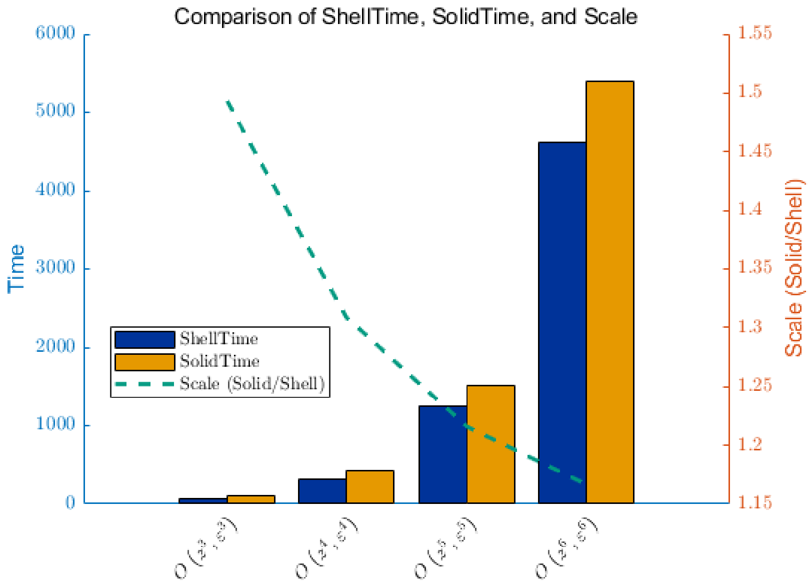


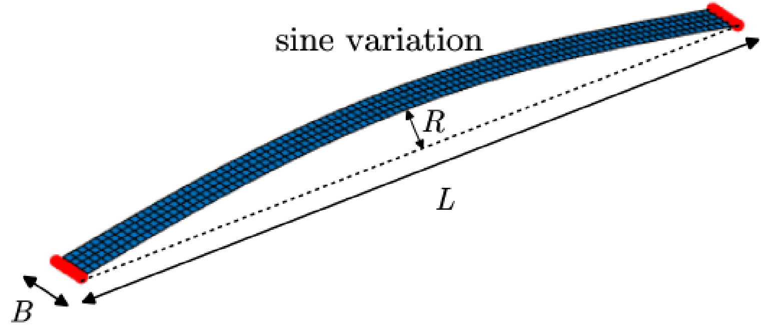



# Numerical example: 1:2 superharmonic resonance

**Clamped-clamped curve beam**




**Geometric parameters**

$L = 640\ \mu\text{m}$

$B = 32\ \mu\text{m}$

$R = 3.84\ \mu\text{m}$

$H = 6.4\ \mu\text{m}$

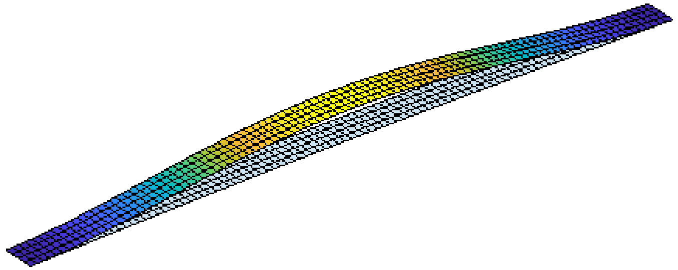
**Material parameters**

$E = 160\ \text{Gpa}$

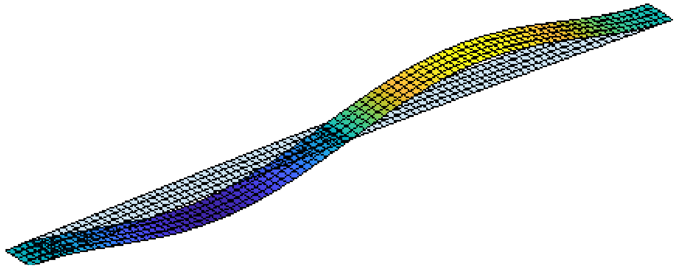
$\upsilon = 0.22$

$\rho = 2320\ \text{kg/m}^3$

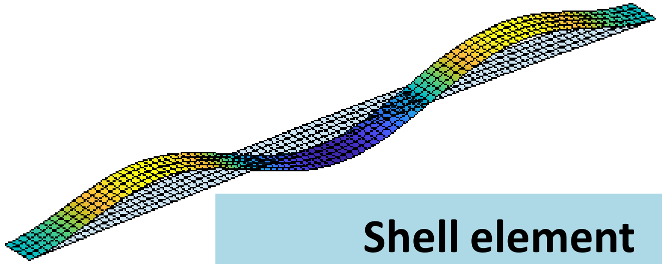
$\omega_1 = 0.996082\ \text{rad}/\mu\text{s}$



$\omega_2 = 2.31348\ \text{rad}/\mu\text{s}$

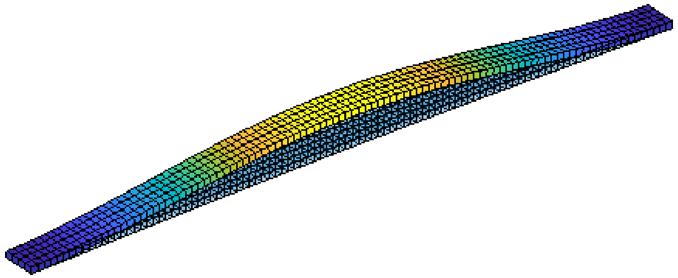


$\omega_3 = 4.53315\ \text{rad}/\mu\text{s}$

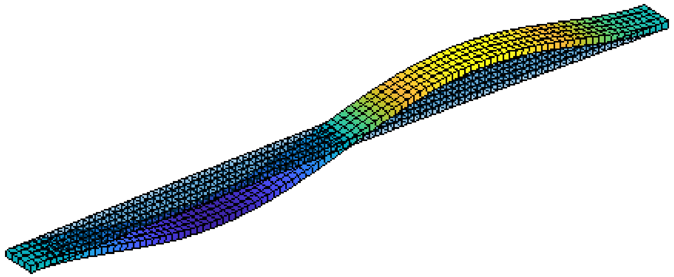


**Shell element**

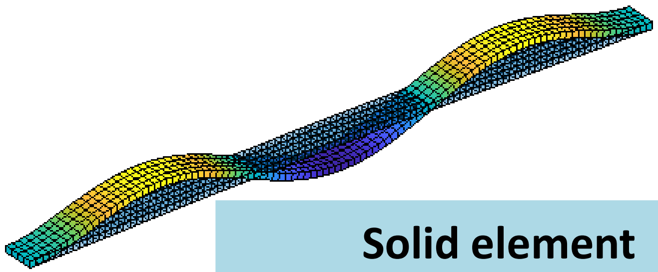
$\omega_1 = 0.996442\ \text{rad}/\mu\text{s}$



$\omega_2 = 2.31507\ \text{rad}/\mu\text{s}$



$\omega_3 = 4.53592\ \text{rad}/\mu\text{s}$

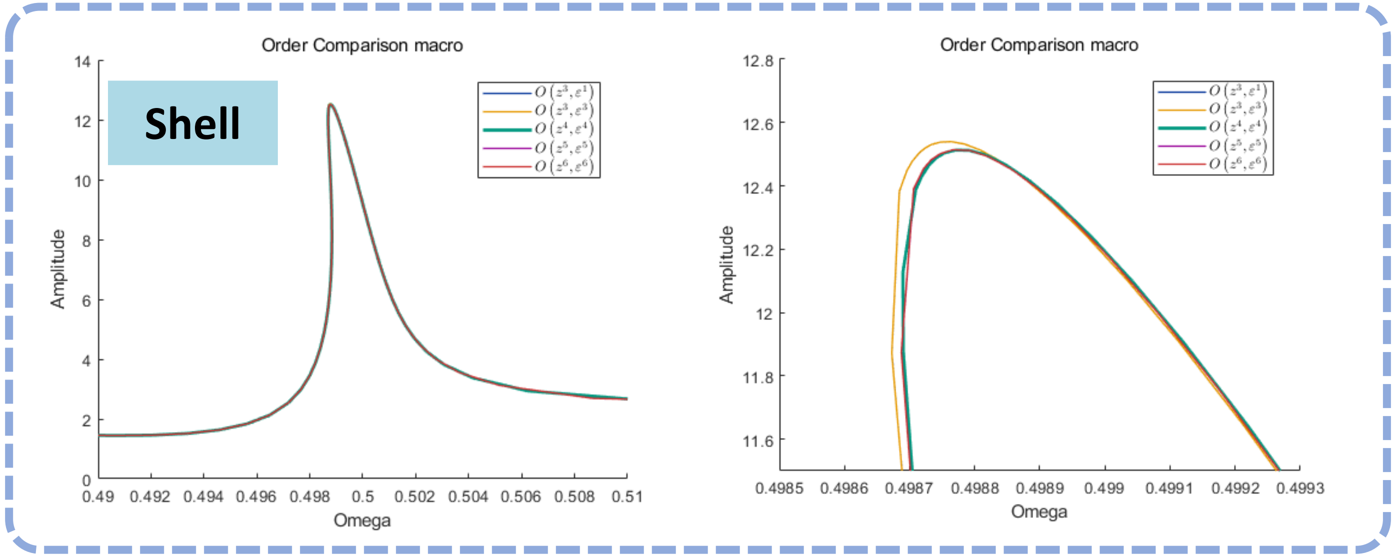


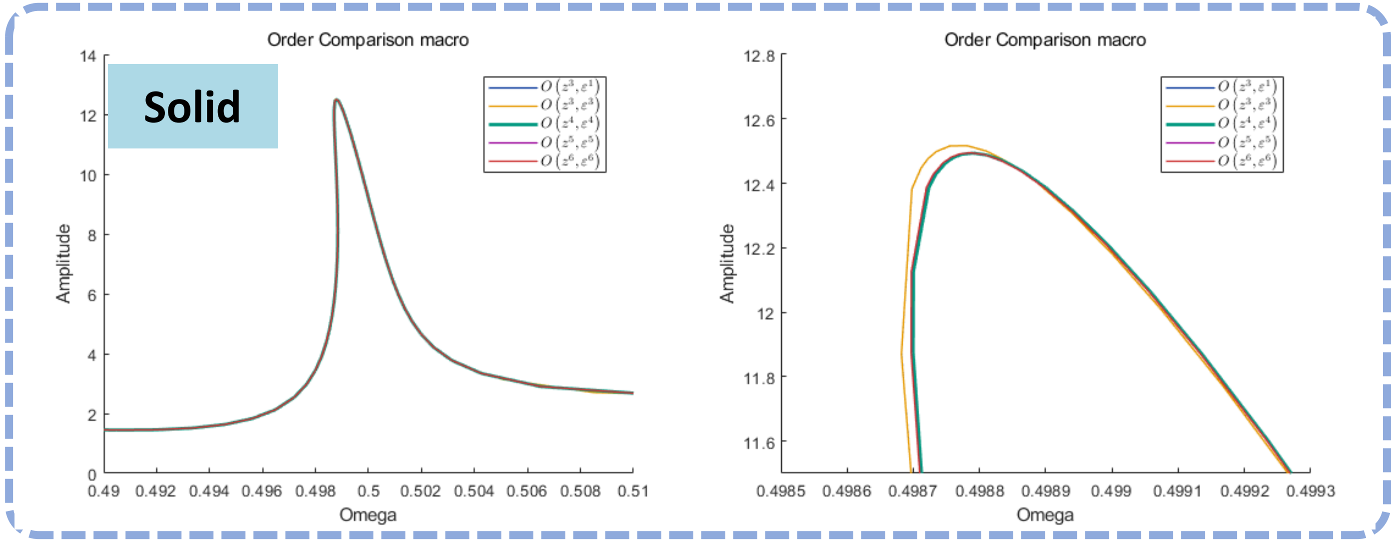
**Solid element**

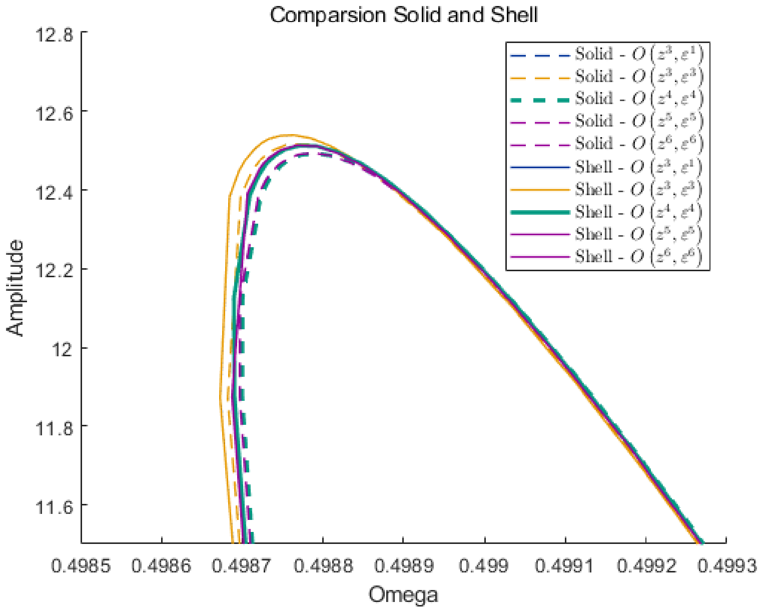
# Numerical example: 1:2 superharmonic resonance

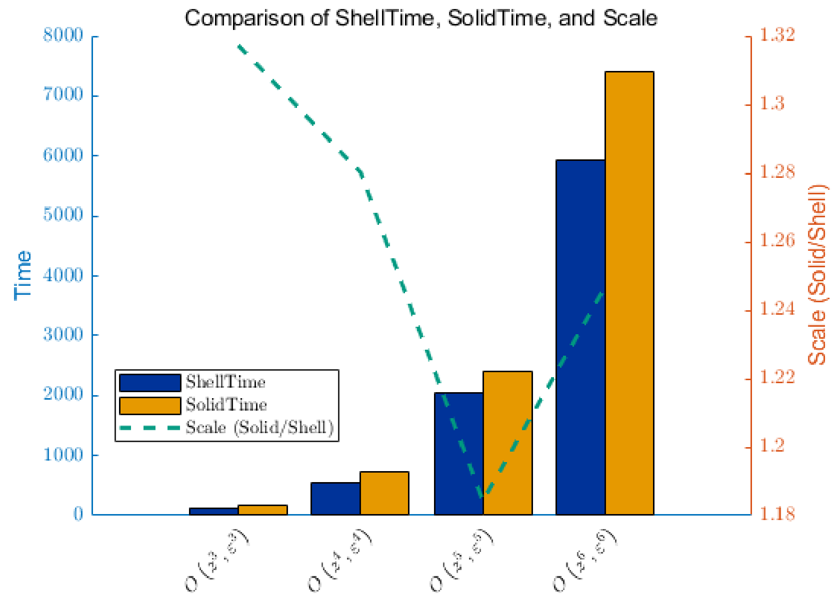


$$[\mathbf{M}]\{\ddot{\mathbf{U}}\}+[\mathbf{C}]\{\dot{\mathbf{U}}\}+[\mathbf{K}_l\{\mathbf{U}\}+\{\mathbf{G}(\mathbf{U},\mathbf{U})\}+\{\mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U})\}=1.5[\mathbf{M}]\{\mathbf{\Phi}_1\}\cos\Omega t$$



# Conclusion & Future works



**Highlights**

1. Established a **shell element model** based on DPIM for application in reduced-order methods for nonlinear dynamics of thin structures

2. By comparing with solid elements and full-order model computation methods, we verified the **effectiveness of the shell element** developed in this work in reduced-order methods

3. Compared to solid elements, the **computational cost is lower**

4. The **kinematic assumptions** of thin structures can be conveniently applied in the thickness direction

**Future works**

1. Based on the validation results of this work, we will proceed to study more **practical engineering problems** concerning complex nonlinear geometrically thin structures, which are not convenient to analyze using solid elements

2. On this basis, we will also consider more complex working conditions, including **fluid-structure interaction** problems and **material nonlinearity** issues

# Thanks for your listening

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**Supervisor** A/Prof. Yu CONG