Computing the dynamics of periodic waveguides with nonlinear boundaries using the Wave Finite Element Method

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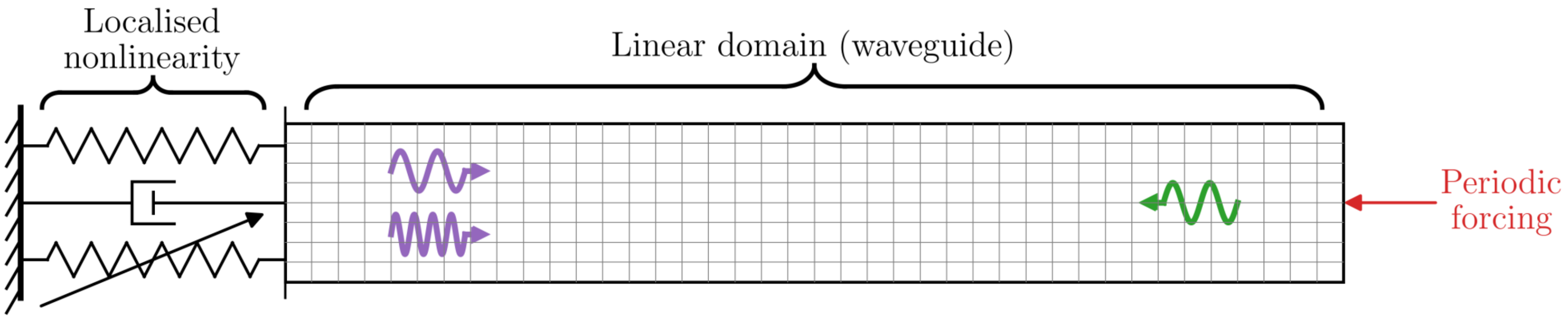
Outline

Wave-based methods – advantages



Interest of wave-based methods

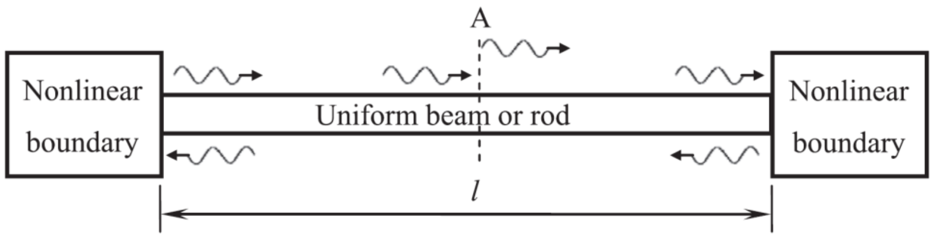
- Linear elements **occupy most of the domain** but have **simple dynamics**
- The **meshing** used in finite element (FEM) approaches grants most of the computational time to linear elements
- Use **wave solutions for the linear elements** to eliminate or reduce their computational cost

FEM	Wave approaches
Many dof on the linear elements	**Wave solutions** for the linear elements
Captures the intricate dynamics of the nonlinear singularity	Captures the intricate dynamics of the nonlinear singularity

Wave-based methods – state of the art

Analytical approaches

- Diffusion coefficients and nonlinear modes of bars and beams with a nonlinear joint [Vakakis and Nayfeh, 1993], [Tang *et al.*, 2018], [Abdi *et al.*, 2022]

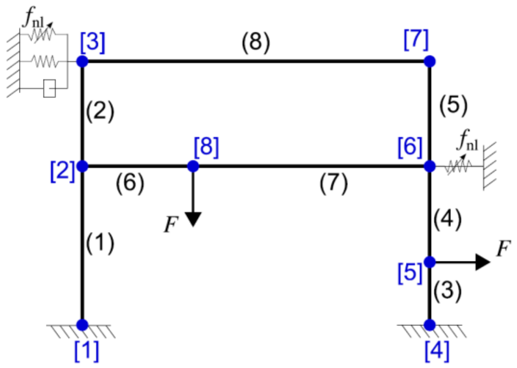


- Nonlinear dynamics of bar assemblies [Balaji *et al.*, 2022]

Limited to
→ Simple waveguide geometries
→ Weak, smooth nonlinearities

Numerical approaches

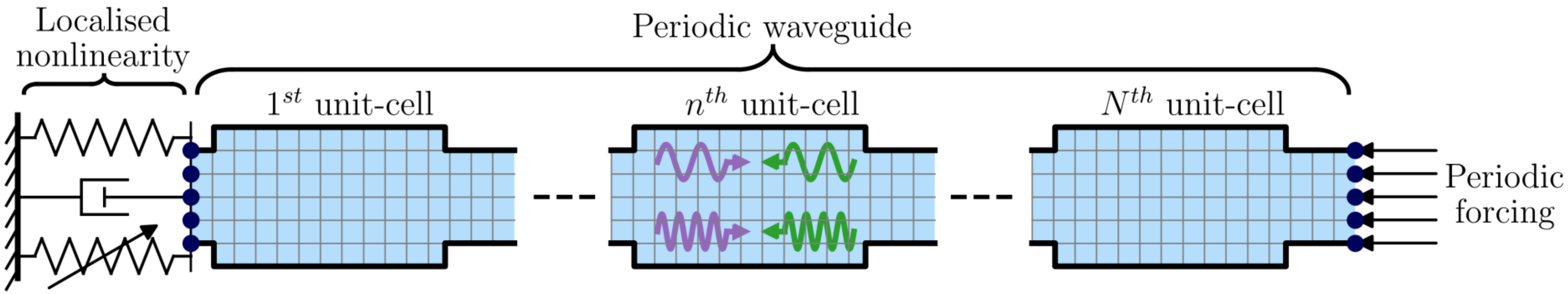
- Nonlinear dynamics of bar and beam assemblies [Chouvion, 2019]



Limited to
→ Simple waveguide geometries, trusses

Need for new **numerical methods** to deal with
→ Complex waveguide geometries
→ Strong nonlinearities

Wave-based methods – The Wave Finite Element Method (WFEM)



Wave Finite Element Method (WFEM)

- Deal with **periodic** waveguides
- Unit-cells (UC) of **arbitrarily complex geometry** discretised with **finite elements**
- **Bloch waves** to represent the waveguide's dynamics

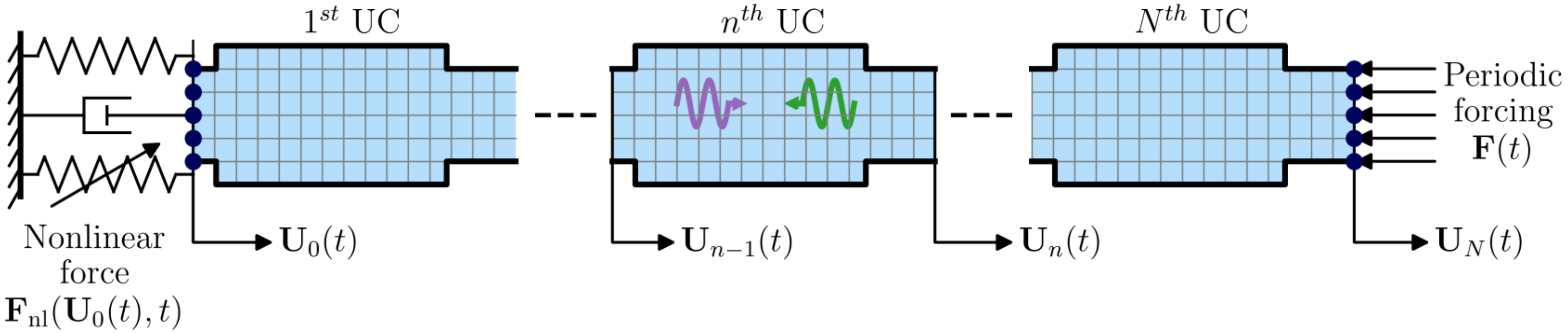
→ Account for **localised nonlinearities**

Method with applications in

- Metamaterial design
- Non-destructive testing
- Vibration control
- Vibro-acoustics

Outline

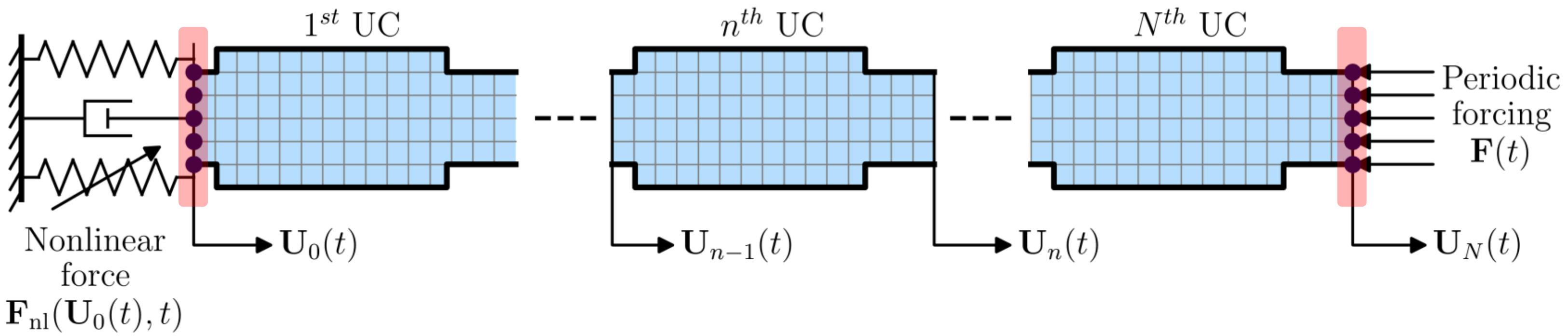
Main steps of the nonlinear WFEM formulation



Main steps

1. Look for **time-periodic solutions**
2. Express the equations governing the motion
   - **Inside** the waveguide
   - At the waveguide **boundaries**
3. Derive a **general solution** of the displacement inside the waveguide using a **Bloch waves expansion**
4. Use the boundary conditions to derive the **Bloch waves' amplitude**
5. **Reconstitute** the displacement field

Equations governing the waveguide dynamics



Look for **periodic solutions**

$$\mathbf{F}(t) = \Re\left[\sum_{h=0}^{H} \mathbf{f}_h e^{jh\Omega t}\right], \qquad \mathbf{U}_n(t) = \Re\left[\sum_{h=0}^{H} \mathbf{u}_{n,h} e^{jh\Omega t}\right]$$

H: number of harmonics retained in the harmonic balance method (HBM)

Distinguish the motion **in the waveguide** and at the **boundaries**

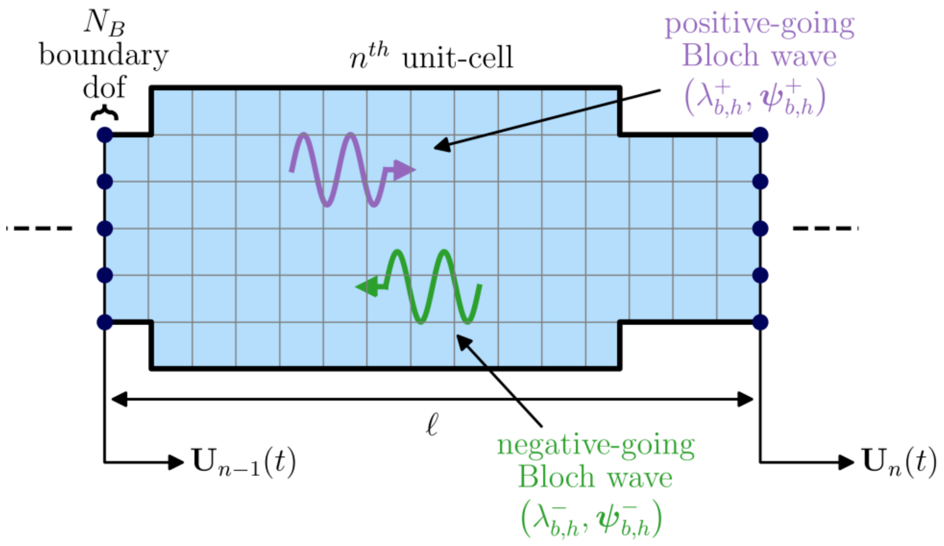
$$\begin{cases} \mathrm{D}_{\mathrm{RL}}^{(h)} \mathbf{u}_{n-1,h} + \left(\mathrm{D}_{\mathrm{LL}}^{(h)} + \mathrm{D}_{\mathrm{RR}}^{(h)}\right) \mathbf{u}_{n,h} + \mathrm{D}_{\mathrm{LR}}^{(h)} \mathbf{u}_{n+1,h} = 0, \quad n = 1, \dots, N-1 & (1) \\ \mathrm{D}_{\mathrm{LL}}^{(h)} \mathbf{u}_{0,h} + \mathrm{D}_{\mathrm{LR}}^{(h)} \mathbf{u}_{1,h} = \langle \mathbf{F}_{\text{nl}}, e^{jh\Omega t} \rangle, & (2) \\ \mathrm{D}_{\mathrm{RL}} \mathbf{u}_{N-1,h} + \mathrm{D}_{\mathrm{RR}} \mathbf{u}_{N,h} = \mathbf{f}_h = \langle \mathbf{F}, e^{jh\Omega t} \rangle & (3) \end{cases}$$

→ Motion **in the waveguide**

(2), (3): **Boundary** conditions

$\mathrm{D}^{(h)}$: condensed dynamic stiffness matrix of a unit-cell for harmonic h

General solution in the waveguide: Bloch waves expansion



Bloch's theorem

Changes in the waveguide's geometry are $\ell -$ **periodic**

→ Look for **Bloch wave** solutions

$$\boldsymbol{u}_{n,h} = \lambda^n \boldsymbol{\psi}$$

λ: propagation constant

k: wavenumber such that $\lambda = e^{-jk\ell} = e^{\Im[k]\ell} e^{-j\Re[k]\ell}$ (Spatial attenuation: $e^{\Im[k]\ell}$; Propagation term: $e^{-j\Re[k]\ell}$)

$\boldsymbol{\psi}$: wave shape ($\ell -$ periodic function)

Bloch waves expansion

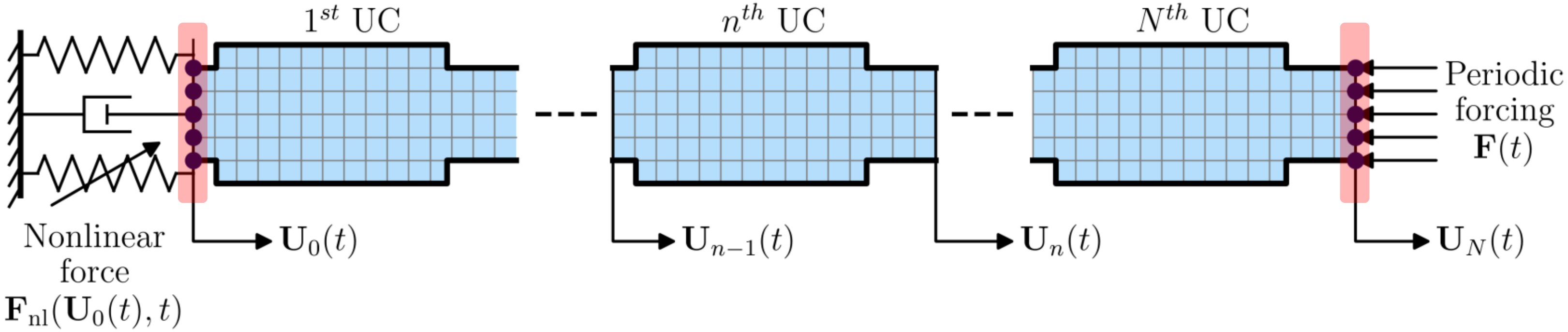
$$\mathbf{U}_n(t) = \mathbf{u}_{n,0} + \Re\left[\sum_{h=1}^{H}\left(\sum_{b=1}^{N_B} q^+_{b,h}\lambda^n_{b,h}\boldsymbol{\psi}^+_{b,h} + q^-_{b,h}\lambda^{N-n}_{b,h}\boldsymbol{\psi}^-_{b,h}\right)e^{jh\Omega t}\right]$$

Static term: $\mathbf{u}_{n,0}$; the bracketed sum is $\boldsymbol{u}_{n,h}$ (Positive-going waves, Negative-going waves)

→ Satisfies the internal waveguide equation ($h \geq 1$)

$q^{\pm}_{b,h}$: wave amplitudes, $\lambda_{b,h} = \lambda^+_{b,h} = 1/\lambda^-_{b,h}$

Satisfy the boundary conditions



General solution

$$\mathbf{U}_n(t) = \mathbf{u}_{n,0} + \Re\left[\sum_{h=1}^{H}\left(\sum_{b=1}^{N_B} q^{+}_{b,h}\lambda^{n}_{b,h}\boldsymbol{\psi}^{+}_{b,h} + q^{-}_{b,h}\lambda^{N-n}_{b,h}\boldsymbol{\psi}^{-}_{b,h}\right)e^{jh\Omega t}\right]$$

Wave amplitudes $q^{\pm}_{b,h}$

Static terms $\mathbf{u}_{n,0}$

Solving the boundary conditions

Equation at boundaries

Dynamic stiffness matrix — External forces

$$\mathbf{Z}\boldsymbol{x} + \mathbf{f}_{nl}(\boldsymbol{x}) = \mathbf{f}$$

Nonlinear forces

Vector of unknowns

$$\boldsymbol{x} = \begin{bmatrix} \mathbf{u}_{0,0} \\ \mathbf{u}_{N,0} \\ \mathbf{q} \end{bmatrix}$$

$\mathbf{u}_{0,0}$, $\mathbf{u}_{N,0}$: Static terms

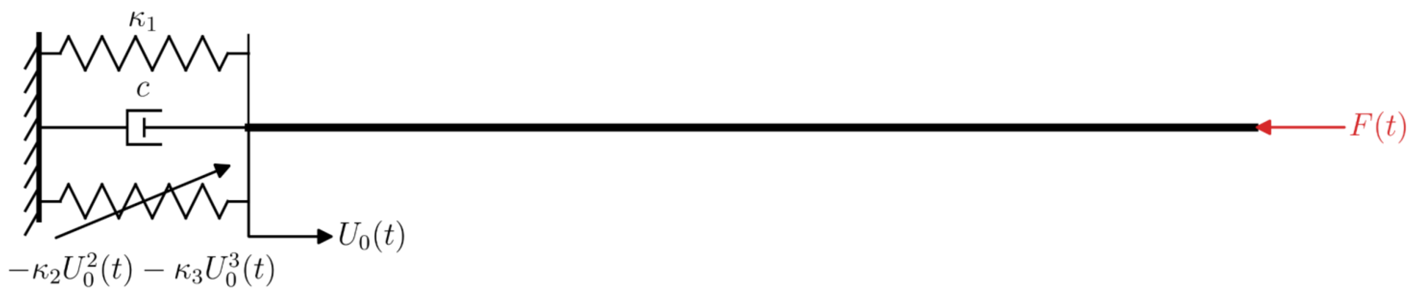
$\mathbf{q}$ → Wave amplitudes

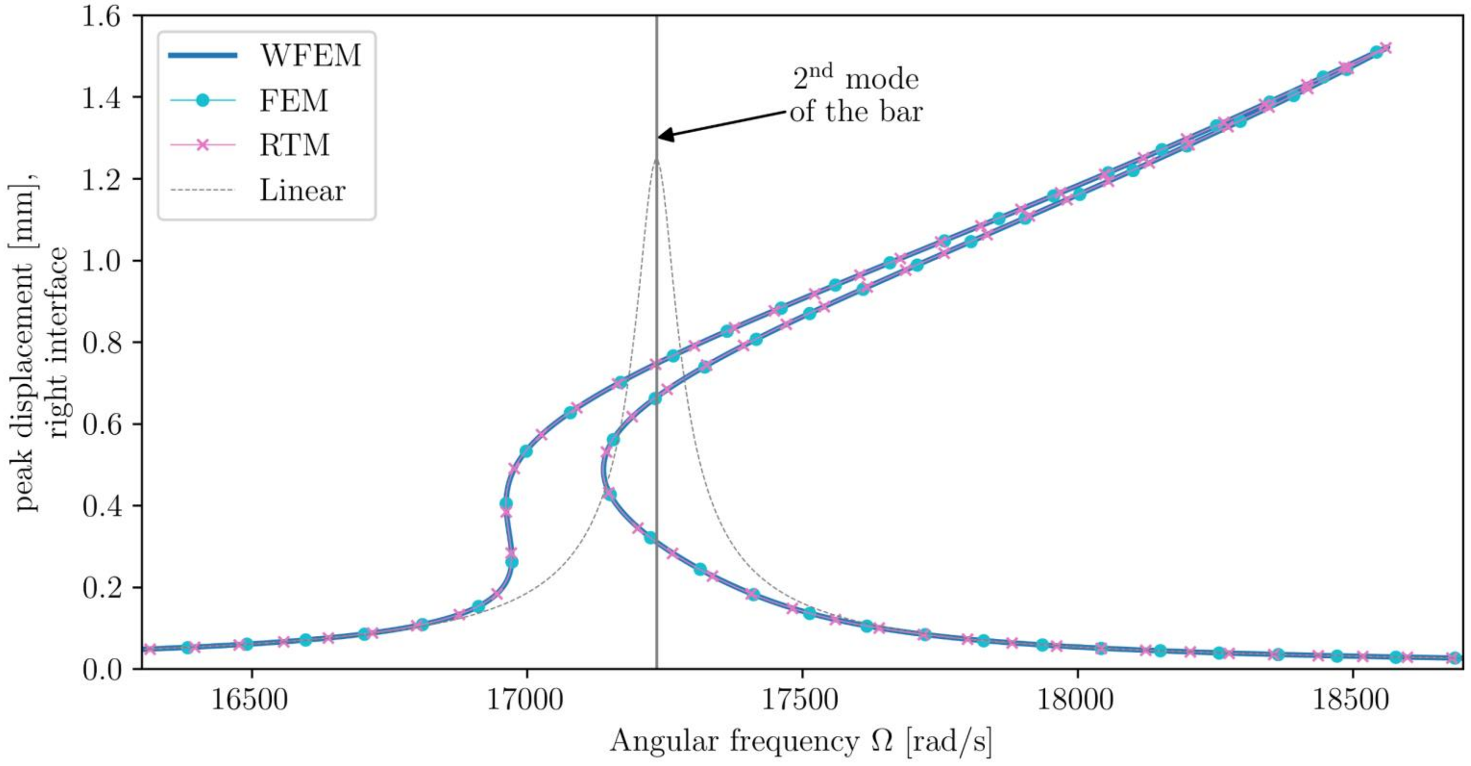
Resolution procedure

- Iterative Newton solver
- Arc-length continuation
- Alternating frequency-time to express nonlinear forces

Outline

Bar with a quadratic and cubic nonlinearity



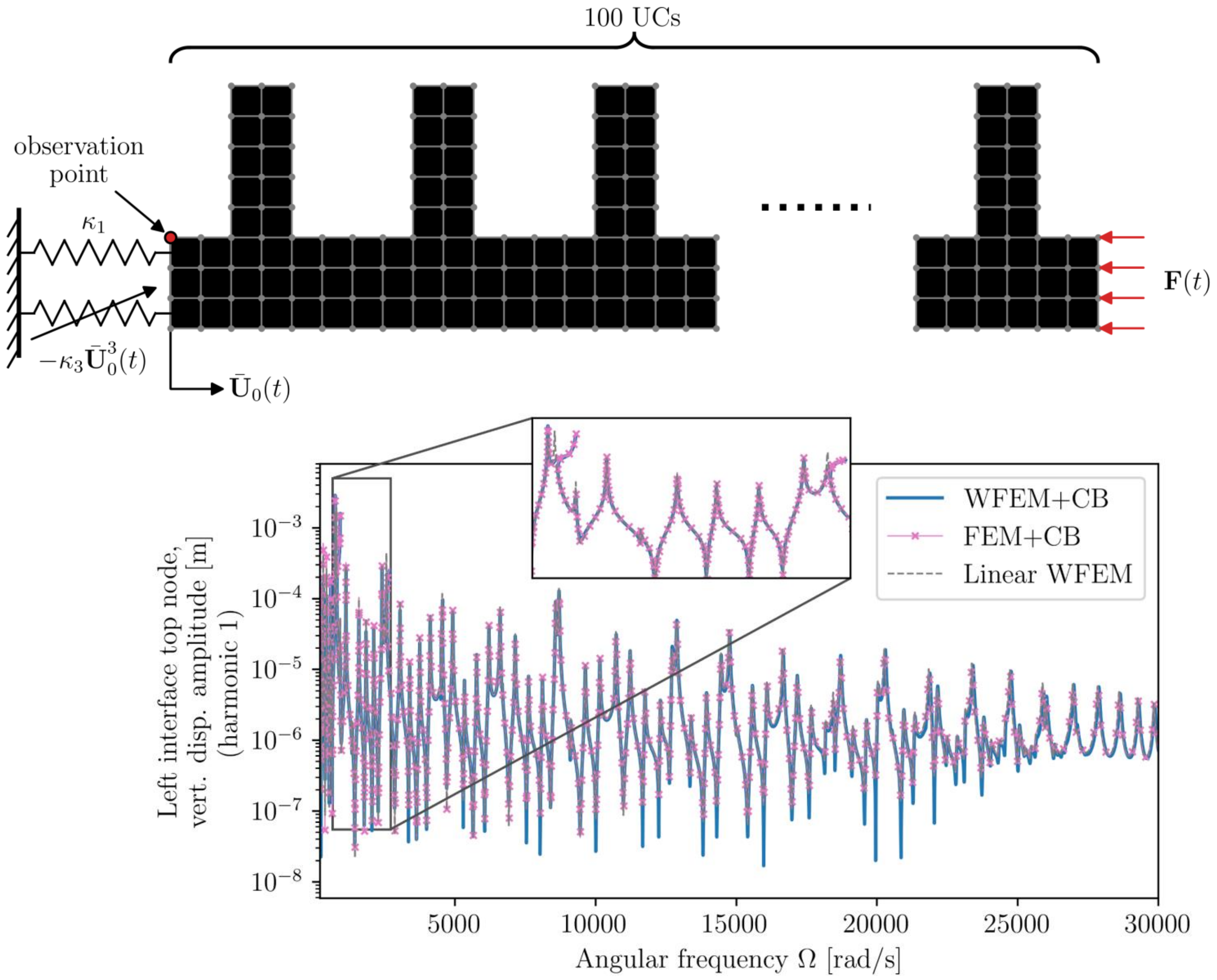


Comparison of the **WFEM** to the **FEM** and the Ray Tracing Method (**RTM**)

Strong nonlinearity

Validation on a simple system

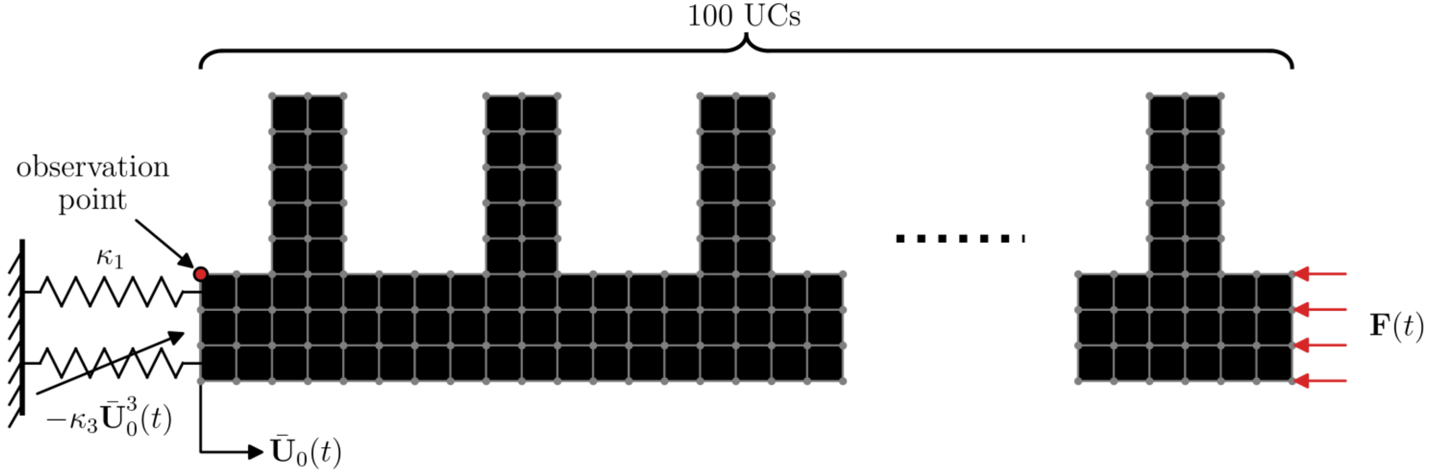
Periodic waveguide discretised by 2D finite elements

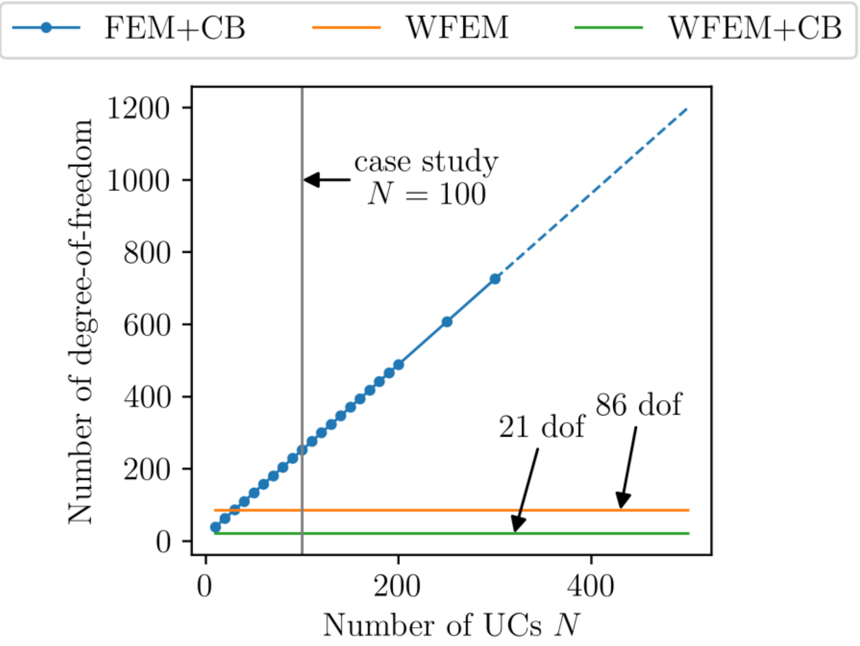


Comparison of the **WFEM** to the **FEM** using Craig-Bampton (CB) procedures

Validation on a periodic finite-element structure

Computational performances



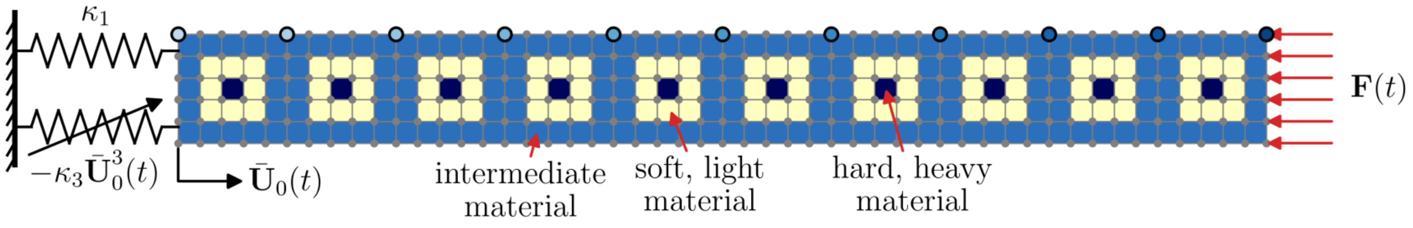


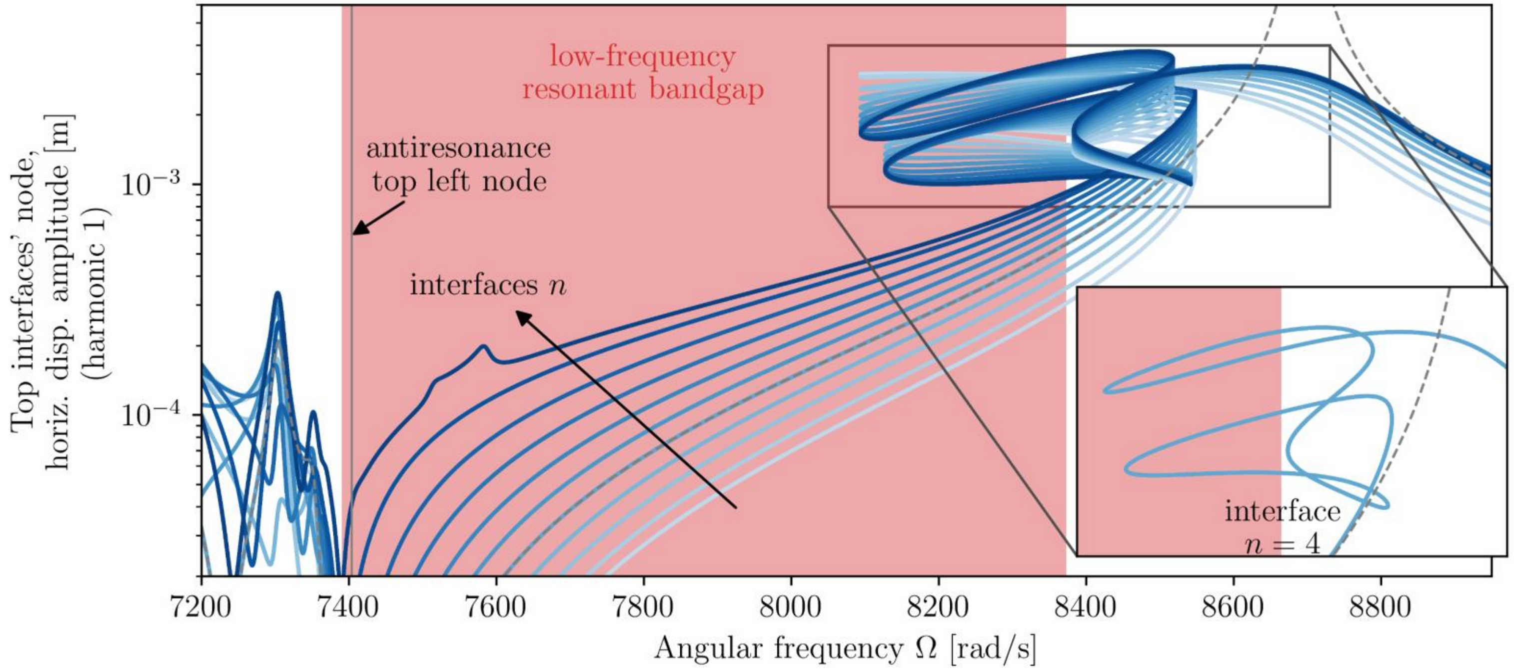
	FEM+CB	WFEM+ CB	Gain (ratio)
System size	252	21	12
Time per point [s]	1.121	0.249	4.5
Total time [s]	9133	1511	6

Superior **computational efficiency** of the WFEM

Larger gain expected on models of larger size

Shifting of a band-edge mode in the bandgap





Bandgap computed with the WFEM

Strongly softening **band-edge mode**

Nonlinear **resonance in the bandgap**

Potential use in **metamaterial design**

Must be avoided for **vibration reduction**

Outline

I. Context and research contributions

II. Nonlinear formulation of the Wave Finite Element Method

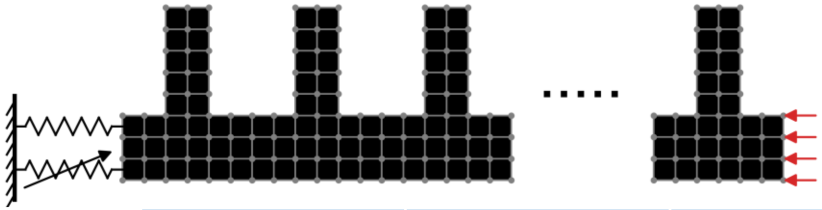
III. Case studies

IV. Conclusion

Conclusion

A **nonlinear WFEM** formulation was presented

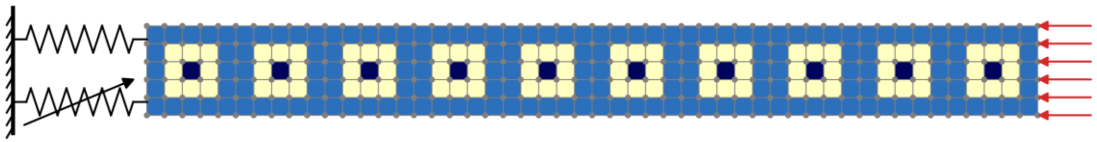
$$\mathbf{U}_n(t) = \underbrace{\mathbf{u}_{n,0}}_{\text{Static term}} + \Re\left[\sum_{h=1}^{H}\left(\underbrace{\sum_{b=1}^{N_B} q_{b,h}^{+}\lambda_{b,h}^{n}\boldsymbol{\psi}_{b,h}^{+} + q_{b,h}^{-}\lambda_{b,h}^{N-n}\boldsymbol{\psi}_{b,h}^{-}}_{\text{Bloch waves expansion}}\right)e^{jh\Omega t}\right]$$



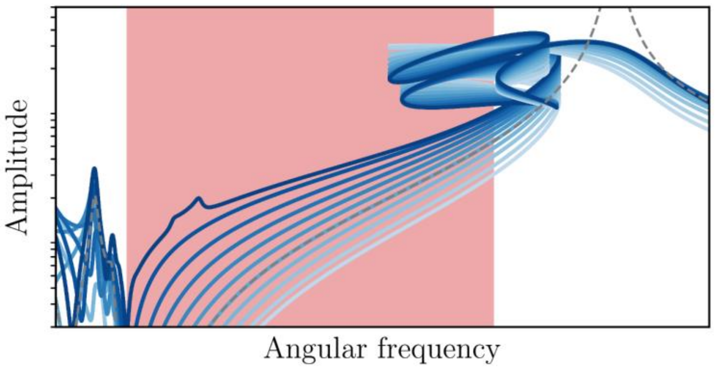
	FEM+CB	WFEM+CB	Gain (ratio)
Total time [s]	9133	1511	6

WFEM **Validated** on a bar and a periodic waveguide discretised with 2D finite elements

The **computational efficiency** was exposed



Observation of **large vibrations in the bandgap** of a locally resonant metamaterial



EX–MODELI
EXPLOITATION & MODÉLISATION des DYNAMIQUES NON LINÉAIRES
GdR CNRS 2134

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Additional slides

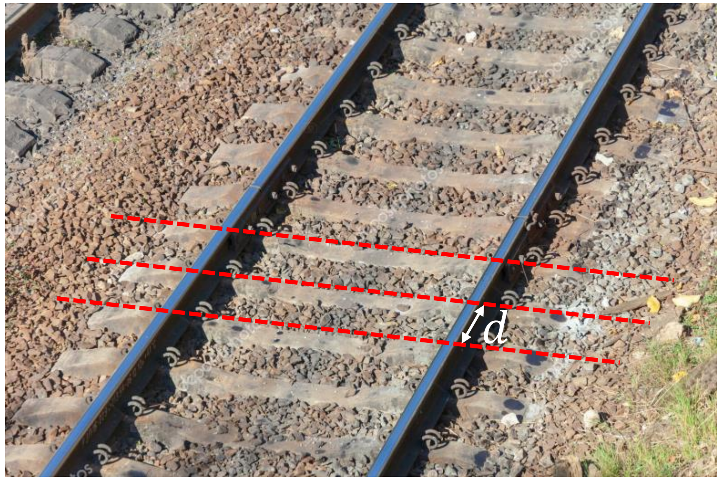
Applications

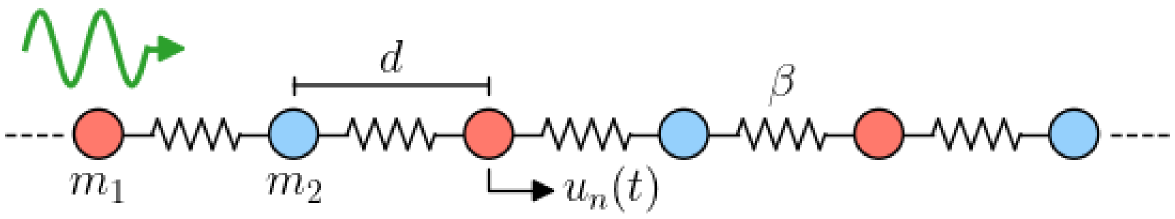
Waveguides in civil engineering

Waveguides are present in many civil engineering structures

Need for **numerical tools** to predict
- The waveguides response to **dynamic excitations**
- The **wave propagation and diffusion** in the waveguides

Periodic waveguides

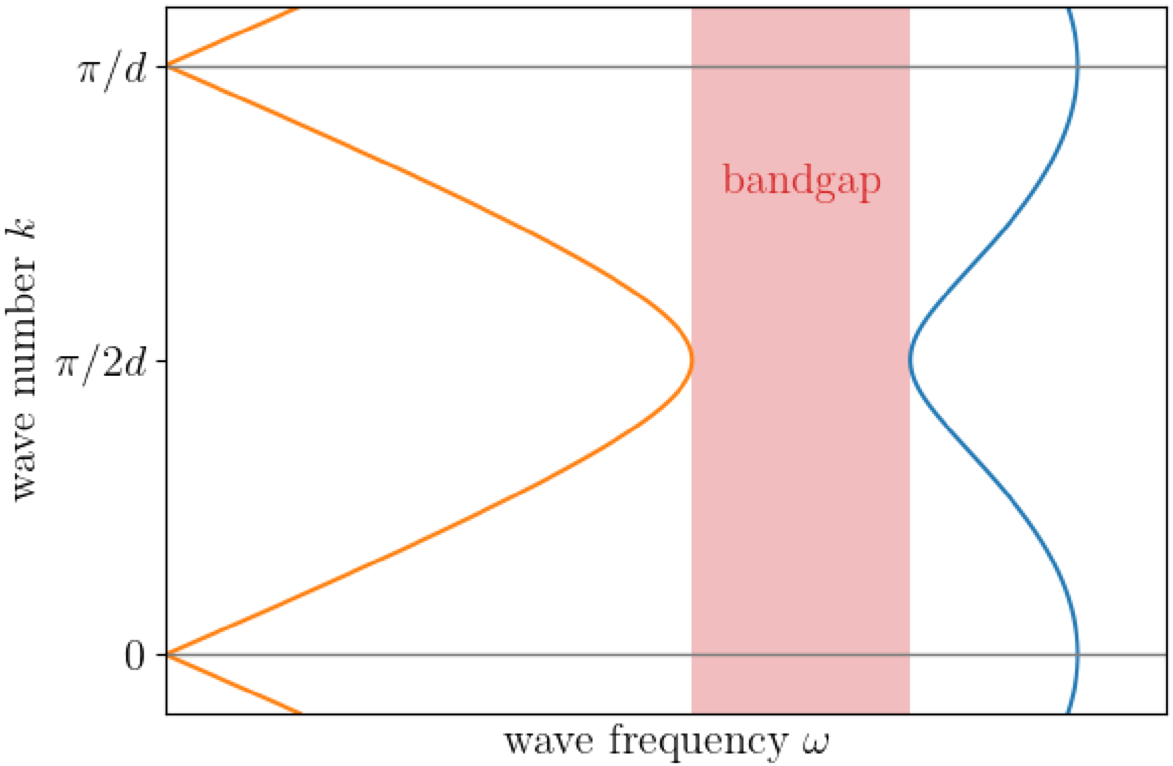




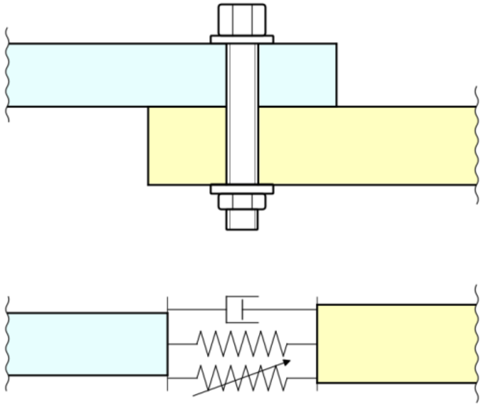
Many waveguides exhibit a **periodic pattern**

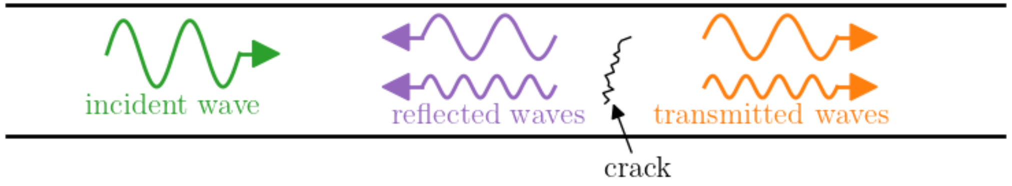
Periodicity causes the presence of **bandgaps**, were waves do not propagate.

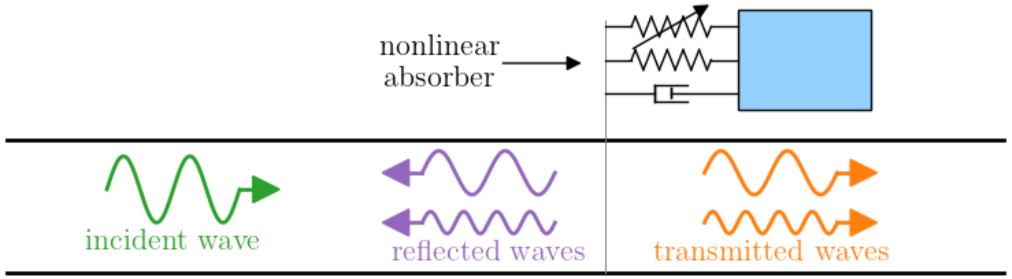
Periodicity can be used to **reduce the size of the model**



Nonlinearities in civil engineering waveguides





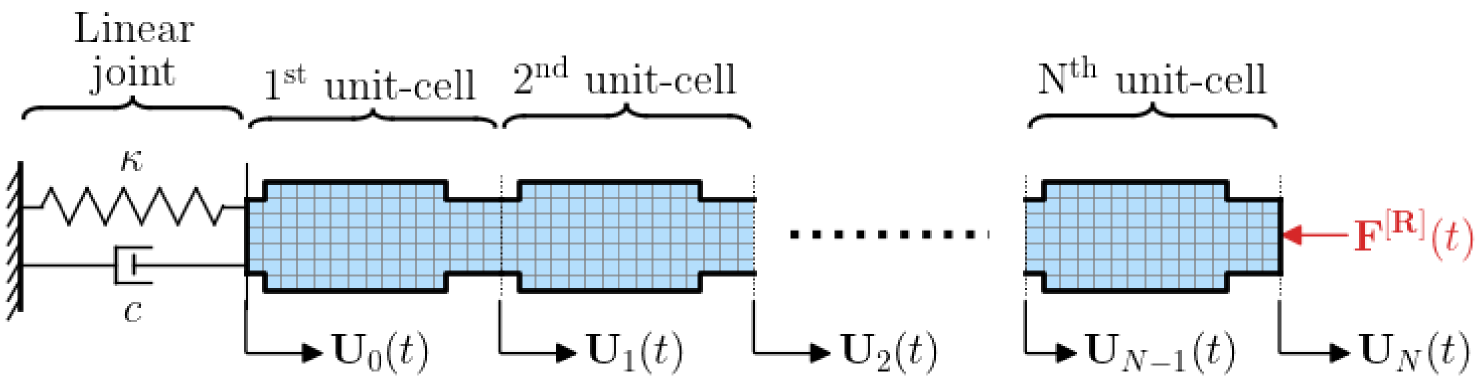


Nonlinearities bring **several difficulties**

- **Amplitude dependency** → The wave amplitudes and phases depend nonlinearly on the source amplitude
- **Harmonics generation** → The diffusion of a single wave generates waves at multiples frequencies
- **Instabilities** → The response depends on the initial conditions, bifurcation points, energy transfers

Theory

Waveguide FEM model



Equation of motion

Structure discretised using the **Finite Element Method** (FEM):

$$\mathcal{M}\ddot{\boldsymbol{\mathcal{U}}}(t) + \mathcal{C}\dot{\boldsymbol{\mathcal{U}}}(t) + \mathcal{K}\boldsymbol{\mathcal{U}}(t) = \boldsymbol{\mathcal{F}}(t)$$

$\boldsymbol{\mathcal{F}}(t)$ has fundamental angular frequency Ω
→ Look for a **periodic response**

Harmonic response
→ $\boldsymbol{\mathcal{U}}(t) = \Re\left[\boldsymbol{u}e^{j\Omega t}\right]$

Notations

N: number of unit-cells (UC)

κ: spring constant

c: viscous damping coefficient

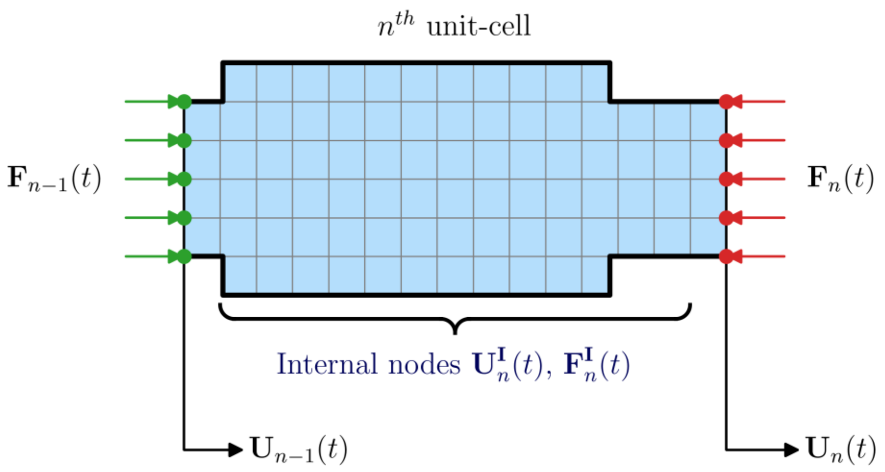
$\mathbf{U}_n(t)$: displacements at interface n

$\mathbf{F}^{[\mathbf{R}]}(t)$: external forces applied on the right side of the waveguide

$\boldsymbol{\mathcal{U}}(t)$: global displacements

$\boldsymbol{\mathcal{F}}(t)$: global forces

UC FEM model



Equation of motion of a unit-cell

$$\mathrm{M}_c \ddot{\mathbf{U}}^{(n)}(t) + \mathrm{C}_c \dot{\mathbf{U}}^{(n)}(t) + \mathrm{K}_c \mathbf{U}^{(n)}(t) = \mathbf{F}^{(n)}(t)$$

$$\mathbf{U}^{(n)}(t) = \begin{bmatrix} \mathbf{U}_{n-1} \\ \mathbf{U}_n^{\mathrm{I}} \\ \mathbf{U}_n \end{bmatrix}, \qquad \mathbf{F}^{(n)}(t) = \begin{bmatrix} \mathbf{F}_{n-1} \\ \mathbf{F}_n^{\mathrm{I}} \\ \mathbf{F}_n \end{bmatrix}$$

Fourier transform

$$\underbrace{\left[-\Omega^2 \mathrm{M}_c + \mathrm{j}\Omega \mathrm{C}_c + \mathrm{K}_c\right]}_{\mathrm{G}(\Omega)} \mathbf{u}^{(n)} = \mathbf{f}^{(n)}$$

$$\mathbf{U}^{(n)}(t) = \Re\left[\mathbf{u}^{(n)} e^{\mathrm{j}\Omega t}\right],$$
$$\mathbf{F}^{(n)}(t) = \Re\left[\mathbf{f}^{(n)} e^{\mathrm{j}\Omega t}\right]$$

Dynamic condensation

$$\mathrm{D}(\Omega) \begin{bmatrix} \mathbf{u}_{n-1} \\ \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{n-1} \\ \mathbf{f}_n \end{bmatrix} \qquad \mathrm{G}(\Omega) \to \mathrm{D}(\Omega)$$

Combine unit-cells

Equation of motion of an infinite waveguide

$$\mathrm{D}_{\mathrm{RL}} \mathbf{u}_{n-1} + (\mathrm{D}_{\mathrm{LL}} + \mathrm{D}_{\mathrm{RR}}) \mathbf{u}_n + \mathrm{D}_{\mathrm{LR}} \mathbf{u}_{n+1} = 0$$

$$\mathrm{D}(\Omega) = \begin{bmatrix} \mathrm{D}_{\mathrm{LL}} & \mathrm{D}_{\mathrm{LR}} \\ \mathrm{D}_{\mathrm{RL}} & \mathrm{D}_{\mathrm{RR}} \end{bmatrix}$$ condensed dynamic stiffness matrix

Computing Bloch waves

EOM of the n^{th} UC

$$\underbrace{\begin{bmatrix} \mathrm{D}_{\mathrm{LL}}^{(h)} & \mathrm{D}_{\mathrm{LR}}^{(h)} \\ \mathrm{D}_{\mathrm{RL}}^{(h)} & \mathrm{D}_{\mathrm{RR}}^{(h)} \end{bmatrix}}_{\mathrm{D}^{(h)}(\Omega)} \begin{bmatrix} \mathbf{u}_{n-1,h} \\ \mathbf{u}_{n,h} \end{bmatrix} = \begin{bmatrix} -\mathbf{f}_{n-1,h} \\ \mathbf{f}_{n,h} \end{bmatrix}$$

EOM inside the waveguide

$$\mathrm{D}_{\mathrm{RL}}^{(h)} \mathbf{u}_{n-1,h} + \left(\mathrm{D}_{\mathrm{LL}}^{(h)} + \mathrm{D}_{\mathrm{RR}}^{(h)}\right) \mathbf{u}_{n,h} + \mathrm{D}_{\mathrm{LR}}^{(h)} \mathbf{u}_{n+1,h} = 0, \qquad n = 1, \dots, N-1$$

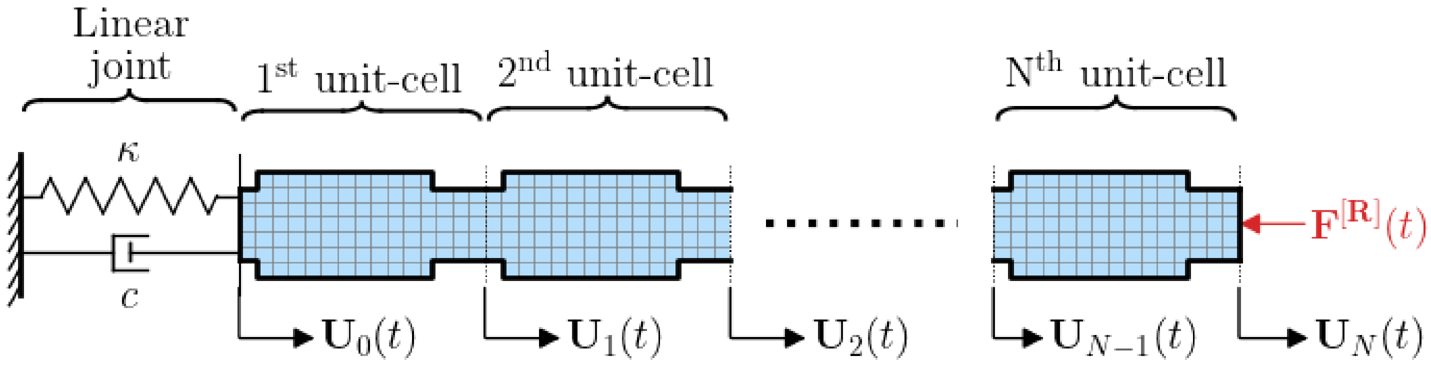
Look for Bloch wave solutions

$$\boldsymbol{u}_{n,h} = \lambda^n \boldsymbol{\psi}$$

Bloch waves equation

$$\left[\lambda^{-1} \mathrm{D}_{\mathrm{RL}}^{(h)} + \left(\mathrm{D}_{\mathrm{LL}}^{(h)} + \mathrm{D}_{\mathrm{RR}}^{(h)}\right) + \lambda \mathrm{D}_{\mathrm{LR}}^{(h)}\right] \boldsymbol{\psi} = 0$$

Forced response – linear



Waveguide equations

$$\begin{cases} \mathrm{D_{RL}}\mathbf{u}_{n-1} + (\mathrm{D_{LL}} + \mathrm{D_{RR}})\mathbf{u}_n + \mathrm{D_{LR}}\mathbf{u}_{n+1} = 0, \quad \forall n \notin \{0, N\} & (1) \\ \mathrm{D_{LL}}\mathbf{u}_0 + \mathrm{D_{LR}}\mathbf{u}_1 = -(\kappa + \mathrm{j}\Omega c)\mathbf{u}_0, & (2) \\ \mathrm{D_{RL}}\mathbf{u}_{N-1} + \mathrm{D_{RR}}\mathbf{u}_N = \mathbf{f}^{[R]} & (3) \end{cases}$$

Bloch wave decomposition

$$\mathbf{u}_n = \sum_{b=1}^{N_b} q_b^+ \lambda_b^n \boldsymbol{\psi}_b^+ + q_b^- \lambda_b^{-n} \boldsymbol{\psi}_b^-$$

Positive-going waves | Negative-going waves

→ (1) Satisfied by definition

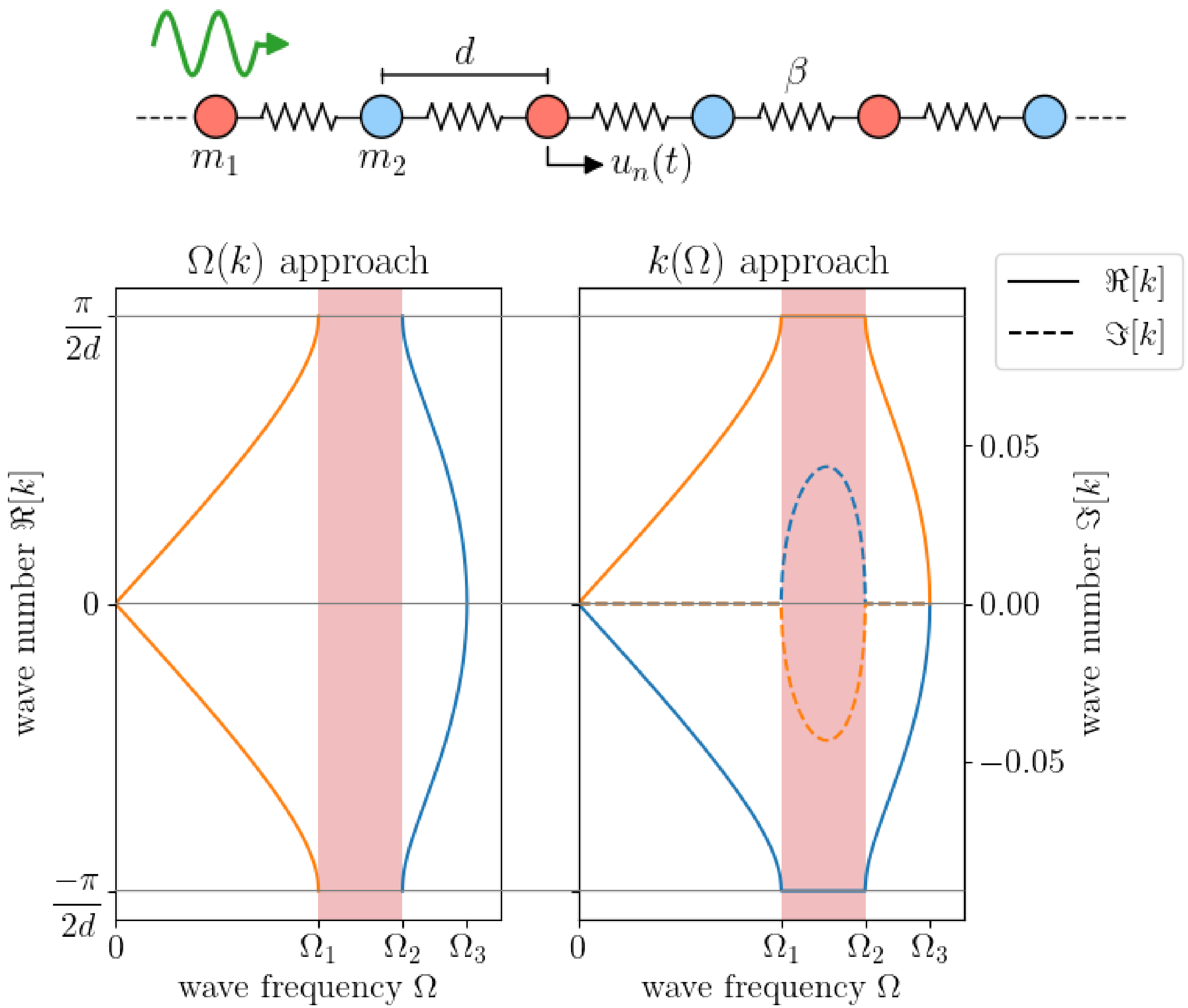
Wave amplitudes

Combine (2) and (3) → $\mathrm{W}\begin{bmatrix}\mathbf{q}^+\\ \mathbf{q}^-\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{f}^{[\mathbf{R}]}\end{bmatrix}$

$$\mathbf{q}^+ = [q_1^+, \dots, q_{N_b}^+]^{\mathrm{T}}$$

Solve and deduce the $\mathbf{u}_n$

Advantage of the $k(\Omega)$ approach over the $\Omega(k)$ approach



Spatial attenuation

$$\lambda = e^{-\mathrm{j}kd} = e^{\Im[k]d} e^{-\mathrm{j}\Re[k]d}$$

Propagation constant — Propagation term

$\Omega(k)$	$k(\Omega)$
Solve linear eigenvalue problem	Solve quadratic eigenvalue problem
Real quantities, no damping	Complex wavenumber, damping considered
No spatial attenuation	Spatial attenuation
No information in the bandgap	Information in the bandgap

WFEM

Numerical methods

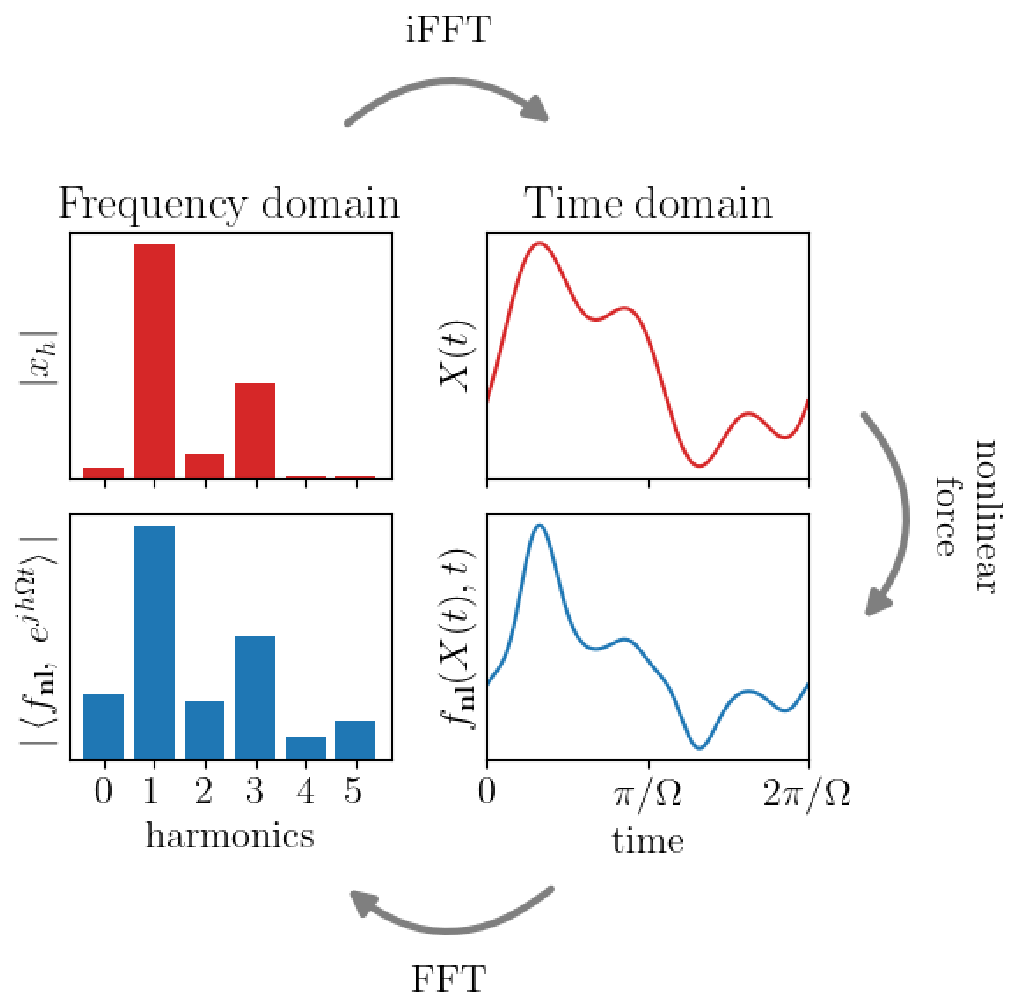
Alternating frequency-time procedure

Alternating frequency-time procedure

System to solve: $\mathrm{Z}\boldsymbol{x} + \mathbf{F}_{\mathrm{nl}}(\boldsymbol{x}) = \mathbf{F}$

1. Express $\boldsymbol{x}$ in the time domain (iFFT):
   $$\boldsymbol{x} \rightarrow \boldsymbol{X}(t)$$
2. Compute the nonlinear forces $\mathbf{f}_{\mathrm{nl}}(t)$:
   $$\mathbf{X}(t) \rightarrow \mathbf{f}_{\mathrm{nl}}(\mathbf{X}(t), t)$$
3. Express $\mathbf{f}_{\mathrm{nl}}(\mathbf{X}(t), t)$ in the frequency domain (FFT):
   $$\mathbf{f}_{\mathrm{nl}}(\mathbf{X}(t), t) \rightarrow \mathbf{F}_{\mathrm{nl}}(\boldsymbol{x})$$

The smoothness of $\mathbf{f}_{\mathrm{nl}}(\mathbf{X}(t), t)$ dictates the number of harmonics to be retained in the HBM



Continuation procedure

System to solve

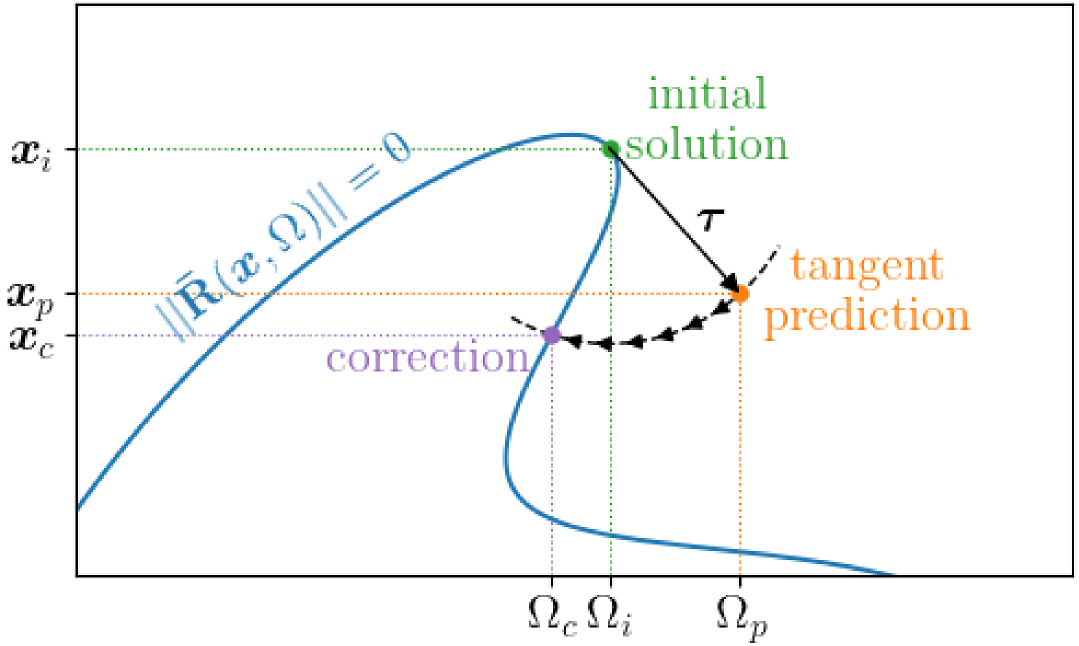
HBM residual: $\mathbf{R}(\boldsymbol{x}) = \mathrm{Z}\boldsymbol{x} + \mathbf{F}_{\mathrm{nl}}(\boldsymbol{x}) - \mathbf{F} \approx \mathbf{0}$

HBM equation

Continuation residual: $\overline{\mathbf{R}}(\boldsymbol{x}, \Omega) \approx \mathbf{0}$

$$\overline{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{x} \\ \Omega \end{bmatrix}$$

- → Additional unknown: Ω
- → Additional equations: Arc-length
- → Unknowns are functions of the arclength parameter s: $\boldsymbol{x}(s)$, $\Omega(s)$

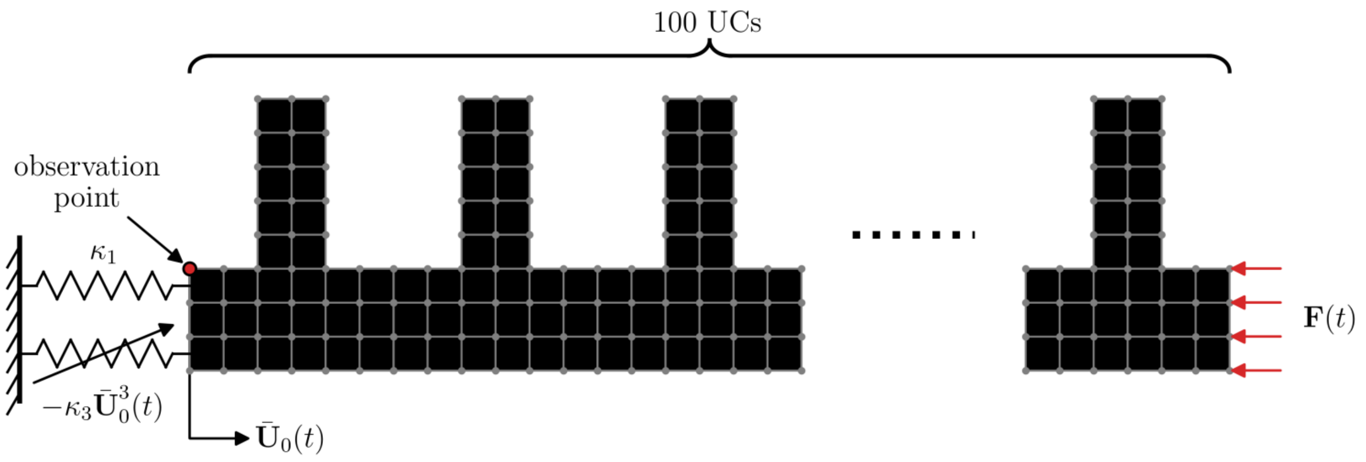


Prediction-correction

1. **Initial** solution $\overline{\boldsymbol{x}}_i$
2. **Predicted** solution $\overline{\boldsymbol{x}}_p = \overline{\boldsymbol{x}}_i + \boldsymbol{\tau}\delta s$
3. **Corrected** solution $\overline{\boldsymbol{x}}_c = \mathrm{solve}(\overline{\mathbf{R}}, \overline{\boldsymbol{x}}_p)$

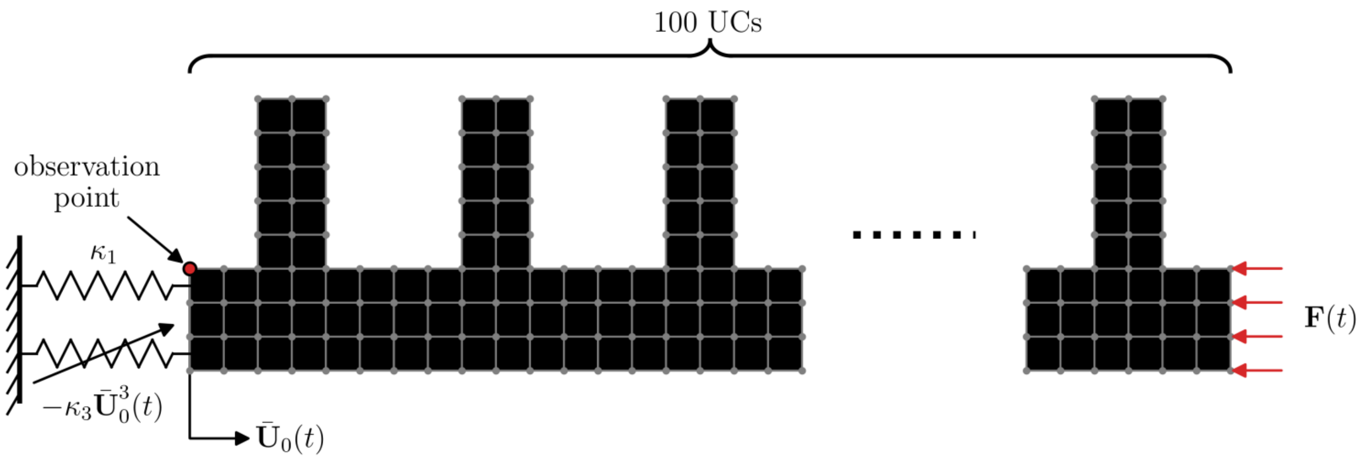
Periodic waveguide

Waveguide parameters



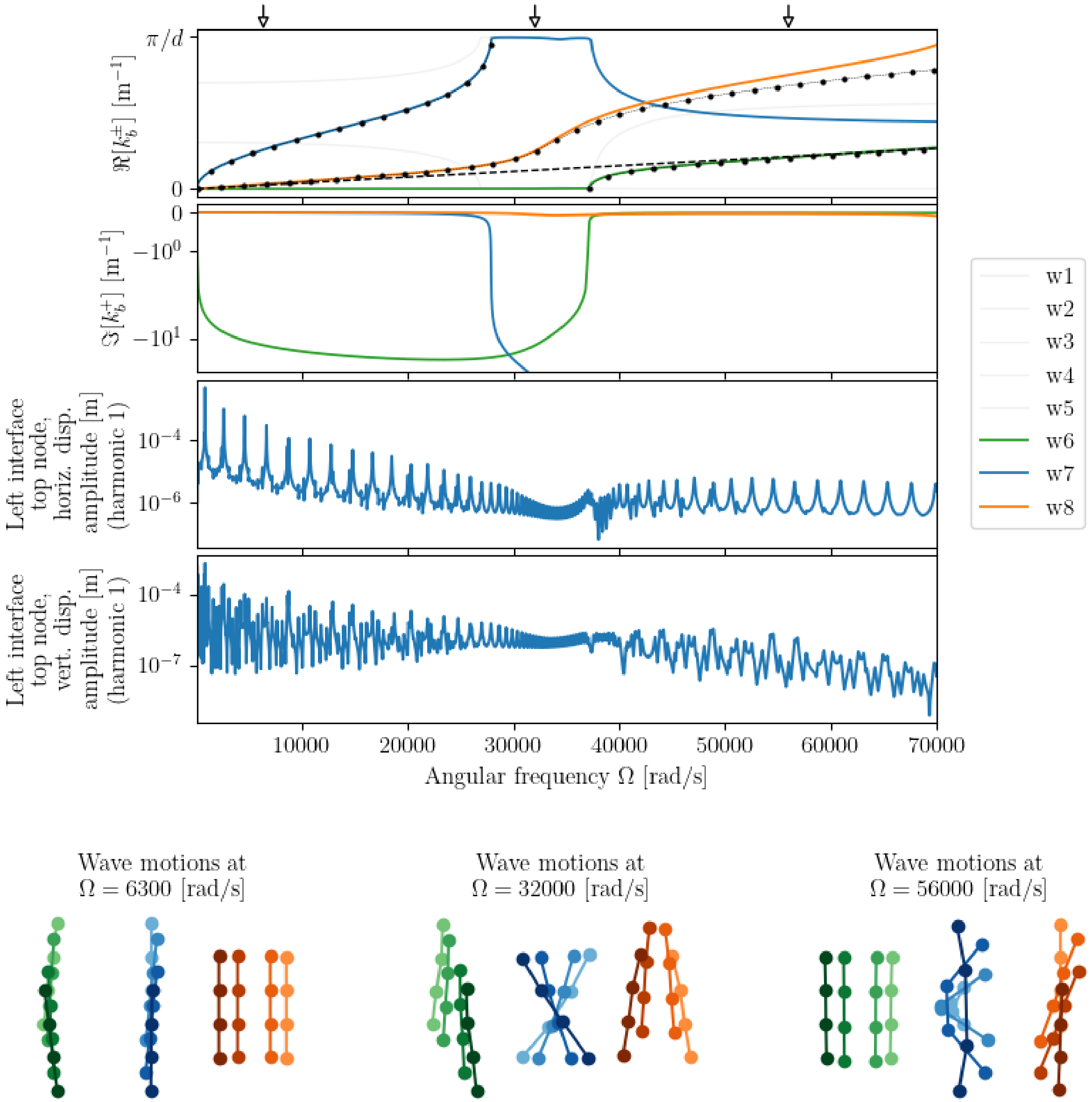
Material parameters	E [GPa]	ν	ρ [kg/m^3]	η [%]				
	200	0.3	7850	0.2				
Waveguide parameters	N	N_B	ℓ_e [m]	ℓ [m]	L [m]	d_e [m]	d [m]	$A_{\max}$ [m^2]
	100	8	0.01	$6\ell_e$	$N\ell$	ℓ_e	$6\ell_e$	$8\ell_e d$
Joint parameters	κ_1 [N/m]	κ_3 [N/m^3]						
	$A_{\max}E/(4L)$	$\kappa_1 \times 2.5 \times 10^5$						

Computational efficiency

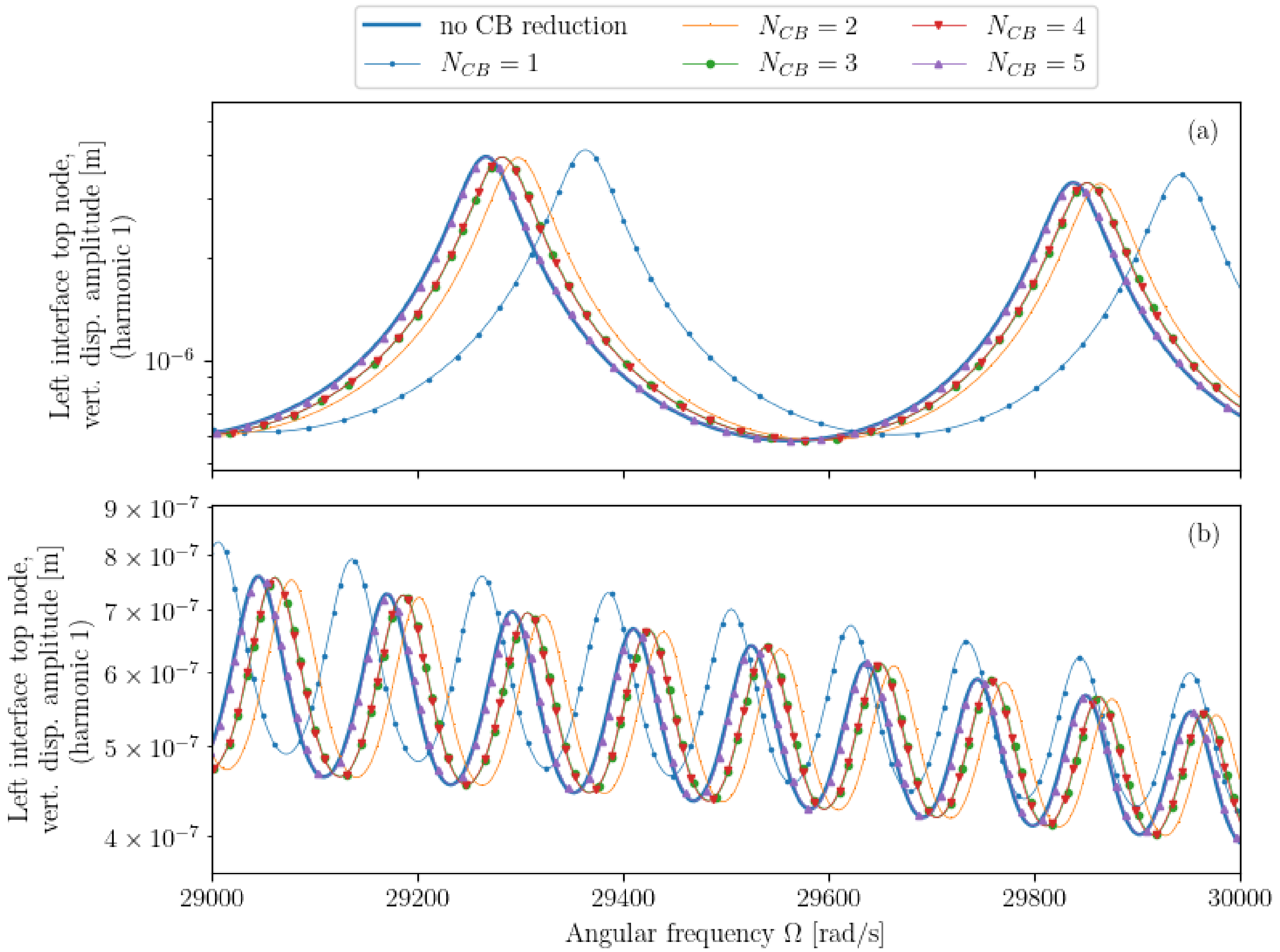


	WFEM+CB	FEM+CB	Gain (ratio)
Pre-processing time [s]	22	135	6.1
System size after pre-processing	21	252	12
Time per continuation point [s]	0.249	1.121	4.5
Post-processing time per continuation point [s]	0.006	0.132	22
Total time [s]	1511	9133	6.0

Waves overview



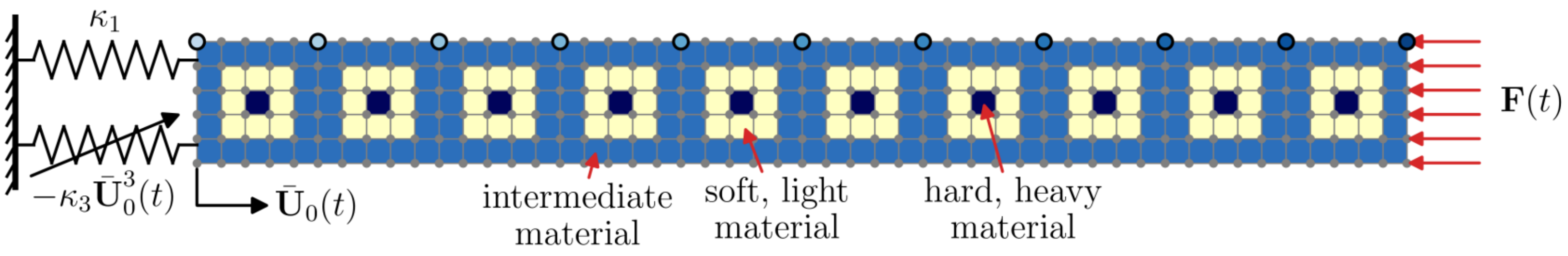
Component modes convergence



CB mode	ω_i [rad/s]
1	74067
2	135205
3	156135
4	240599
5	259797

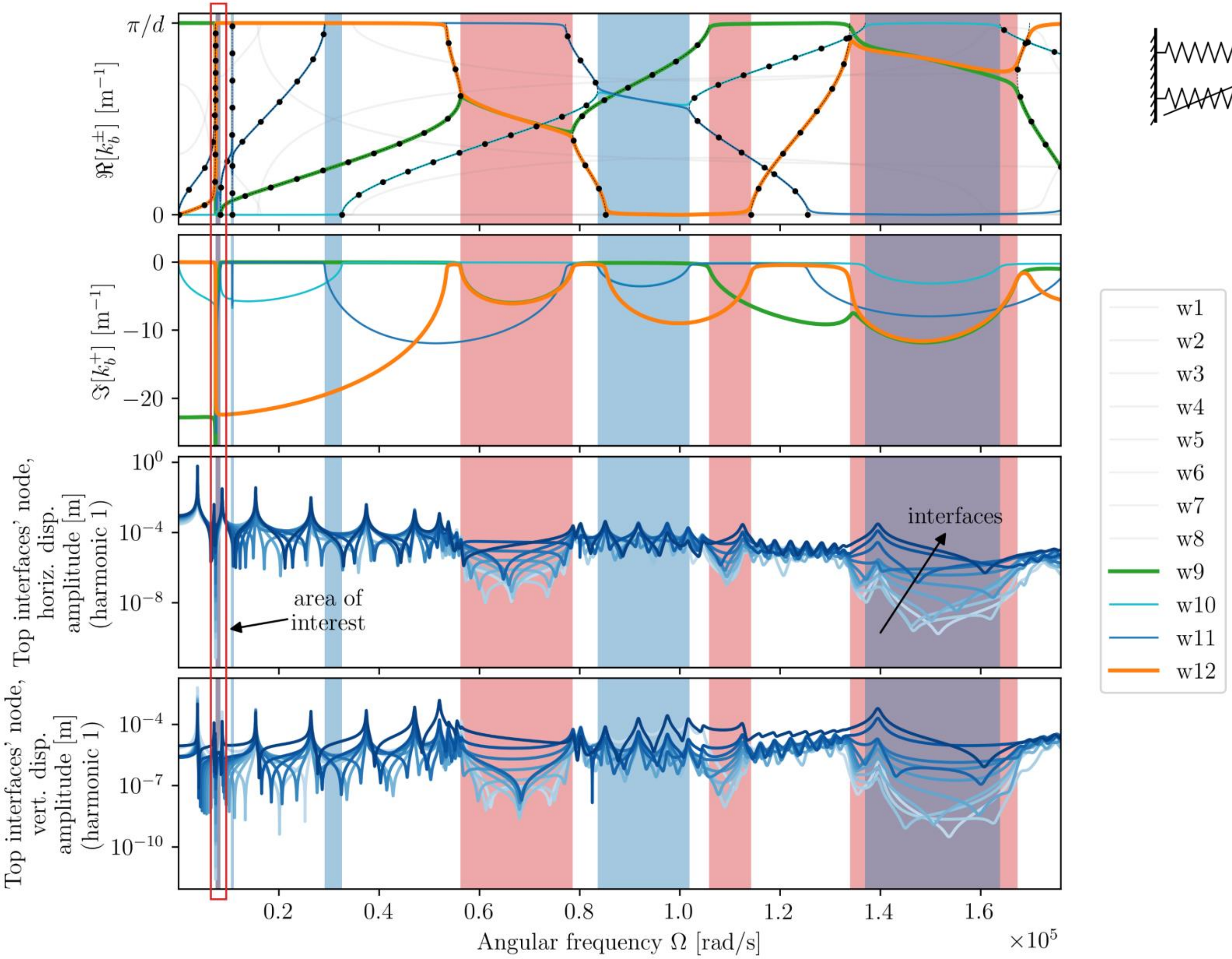
Locally resonant waveguide

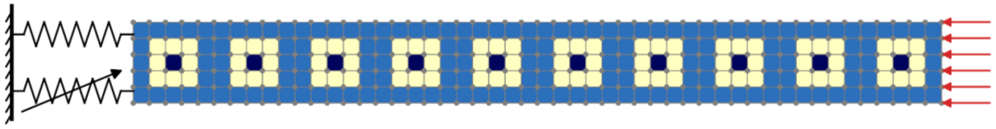
Waveguide parameters



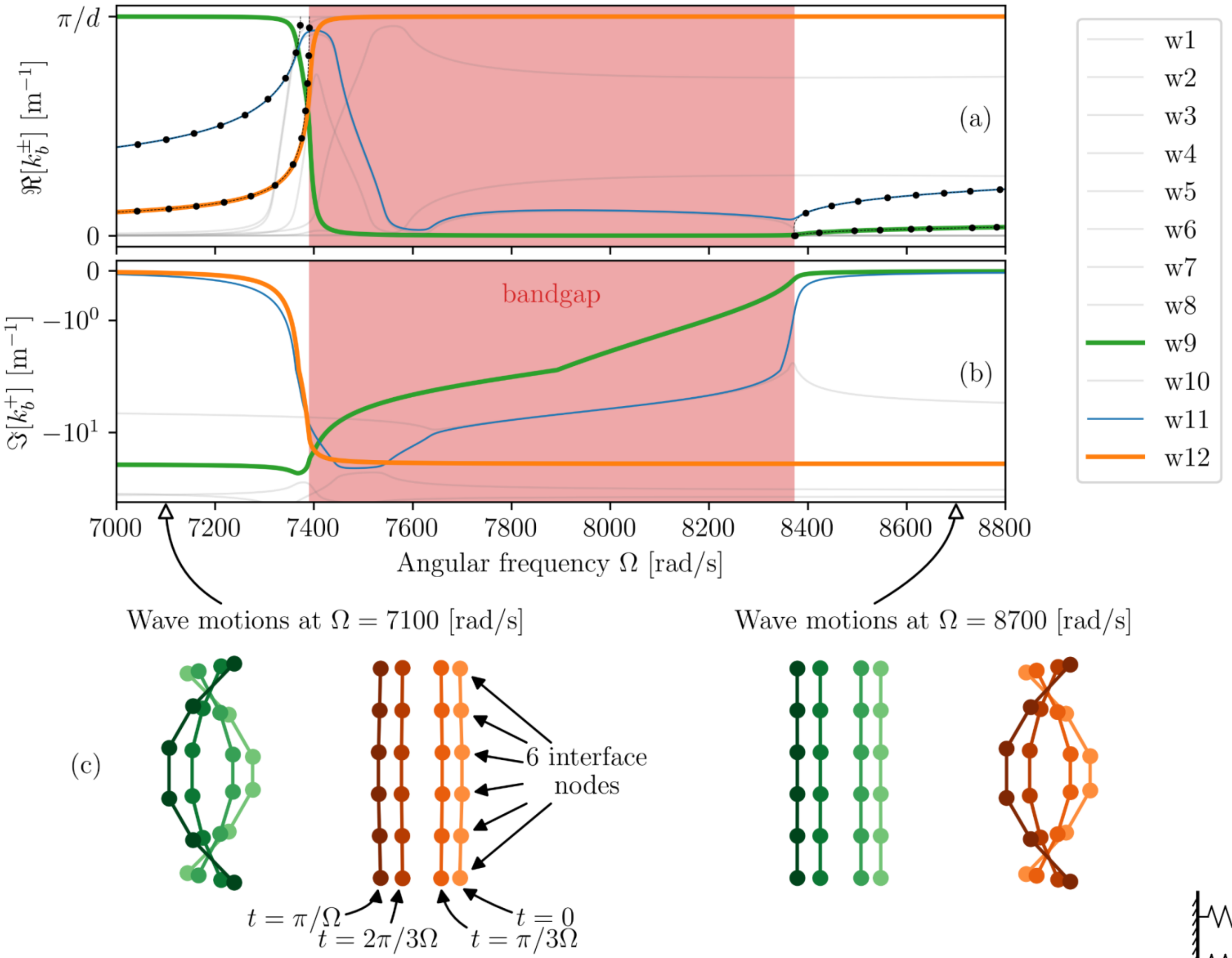
Material parameters	E [GPa]	ν	ρ [kg/m^3]	η [%]	β [s]	
Intermediate material	70	0.3	2700	0.2	4×10^{-8}	
Soft material	0.04	0.5	1300	0.2	4×10^{-8}	
Heavy mass	200	0.3	7850	0.2	4×10^{-8}	
Other parameters	N	N_B	ℓ_e [m]	ℓ [m]	L [m]	d [m]
	10	12	0.02	$5\ell_e$	$N\ell$	0.05
Joint parameters	κ_1 [N/m]	κ_3 [N/m^3]				
	5.83×10^6	-9.72×10^{11}				

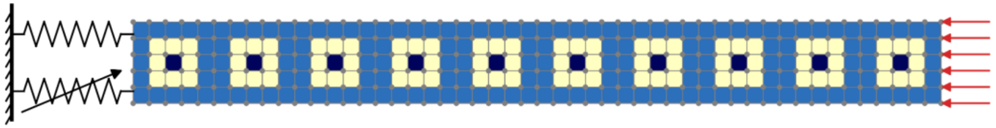
Dynamics overview



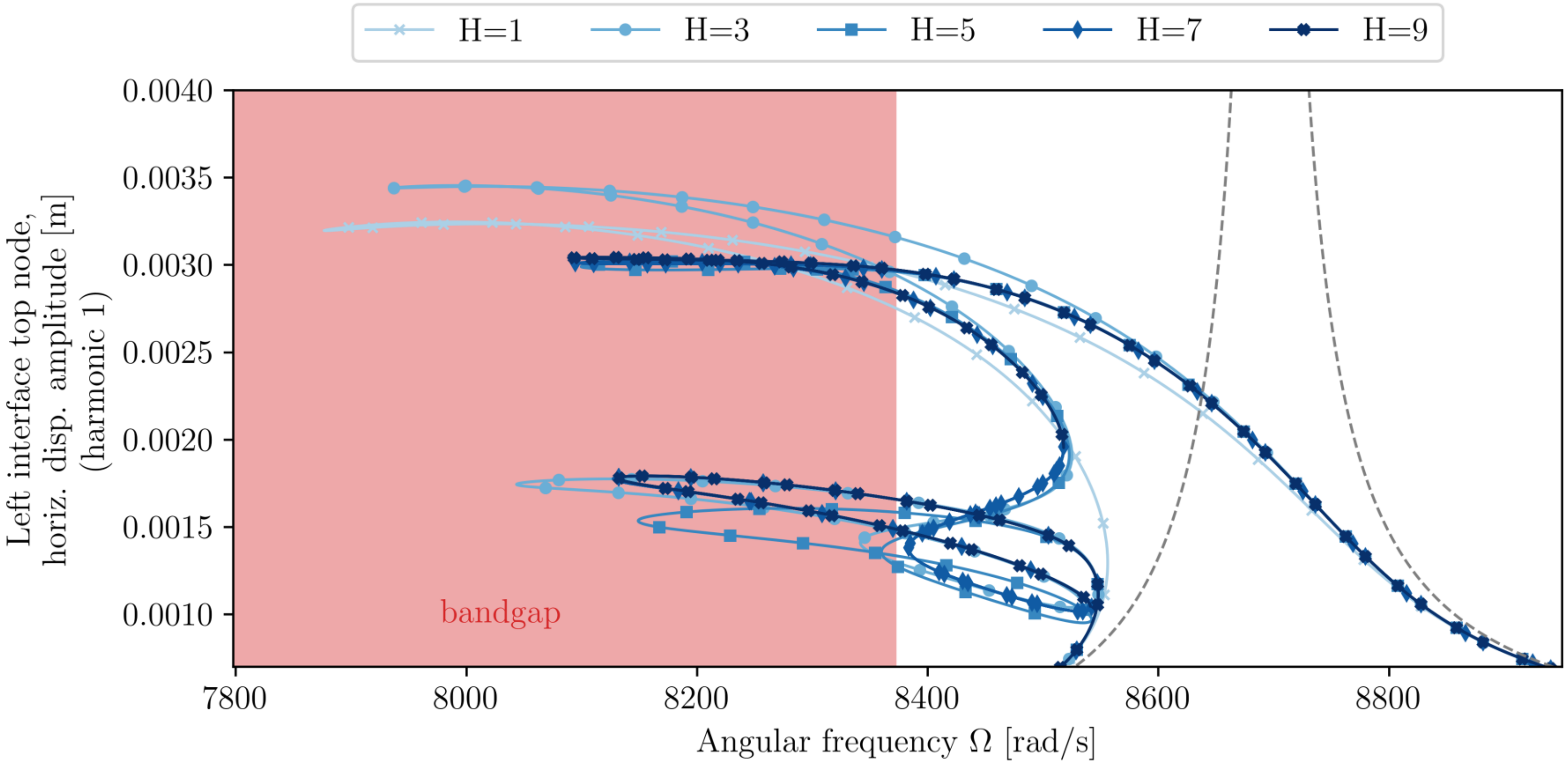


Dynamics around the low frequency bandgap

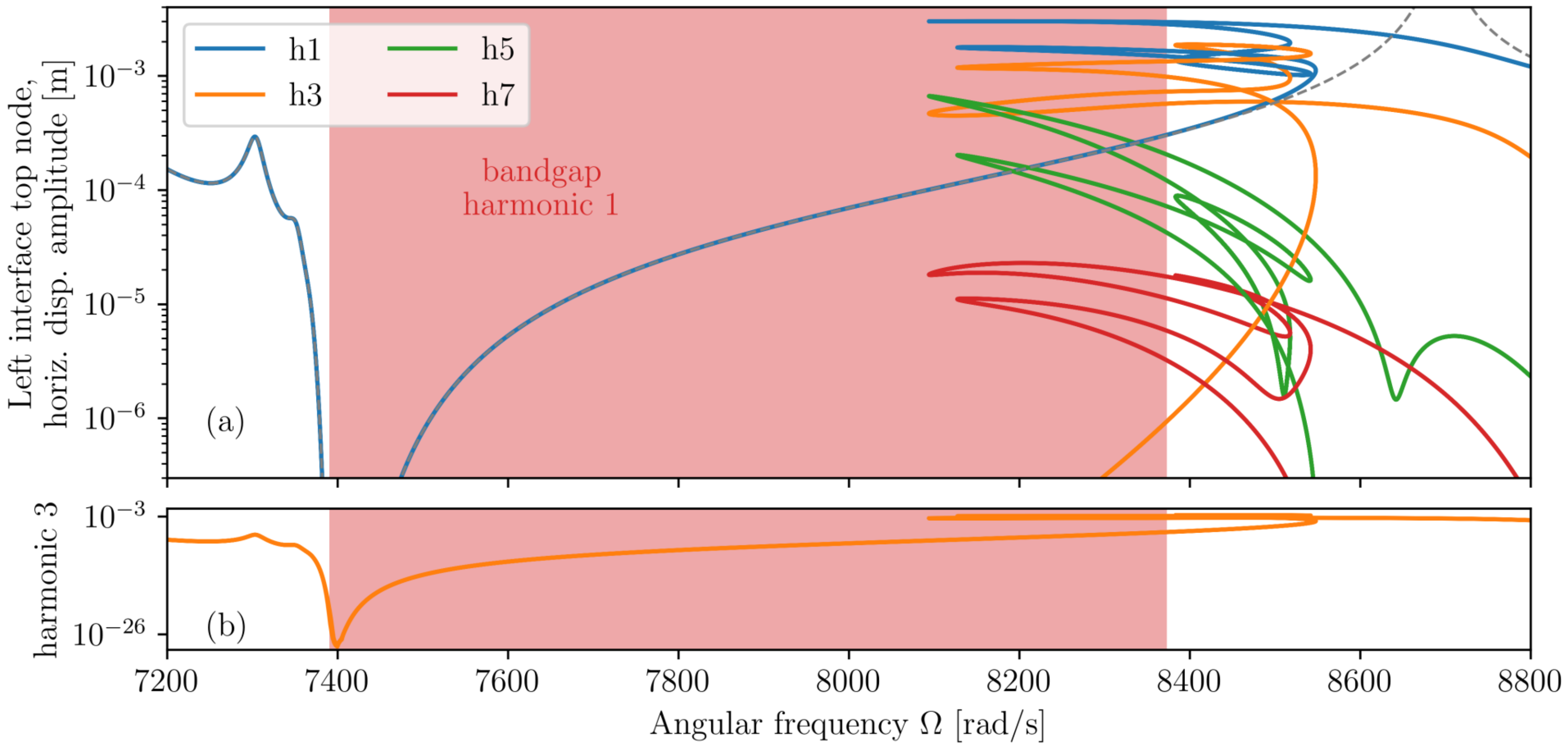




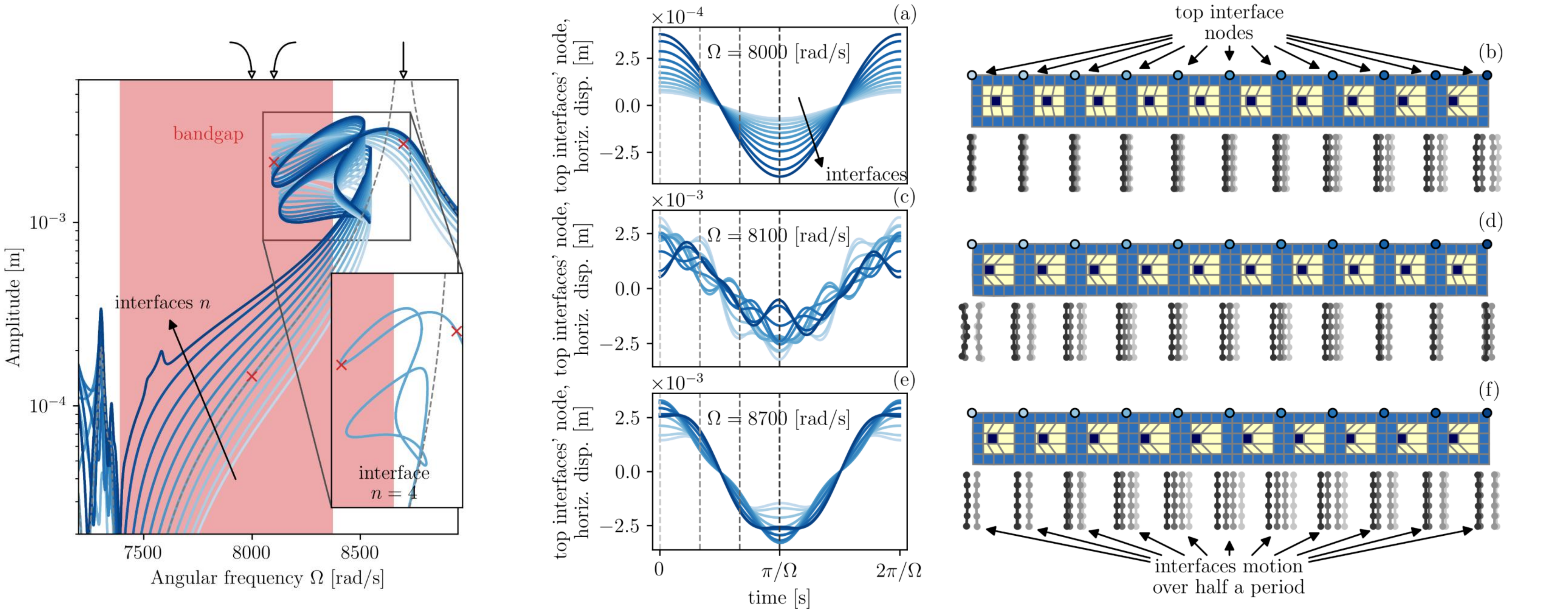
Harmonic convergence



Higher harmonics



Operational shapes



Peak to peak

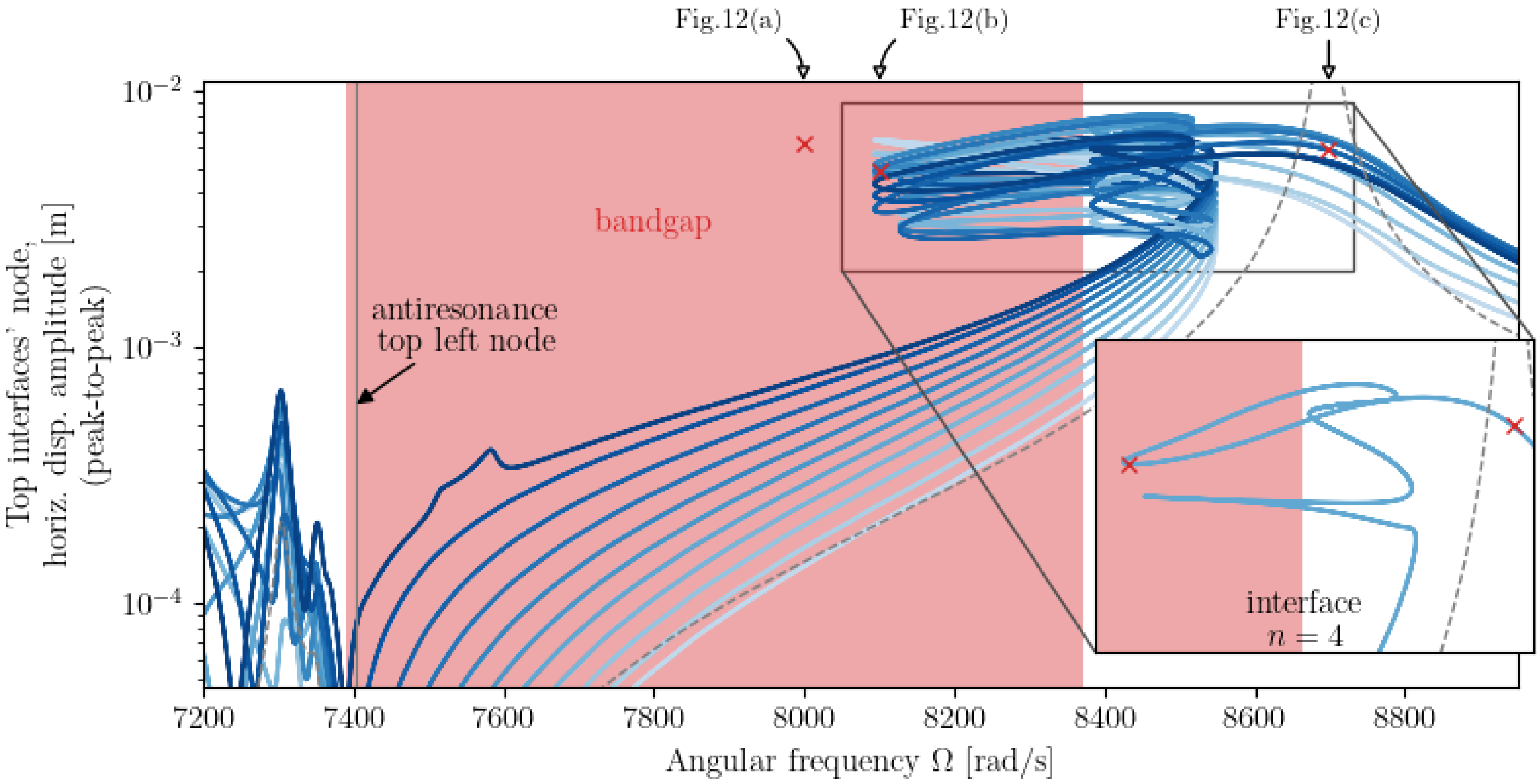


Fig.12(a)
Fig.12(b)
Fig.12(c)
bandgap
antiresonance
top left node
interface
$n = 4$
10^{-2}
10^{-3}
10^{-4}
Top interfaces' node,
horiz. disp. amplitude [m]
(peak-to-peak)
7200
7400
7600
7800
8000
8200
8400
8600
8800
Angular frequency Ω [rad/s]