IMPACTED RANDOM SYSTEMS: A DESCRIPTION BY POLYNOMIAL CHAOS EXPANSION (PCE)

N. Baldanzini, B. Bhattacharyya, D. Brizard, E. Jacquelin, M. Pierini

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

UNIVERSITÀ DEGLI STUDI FIRENZE

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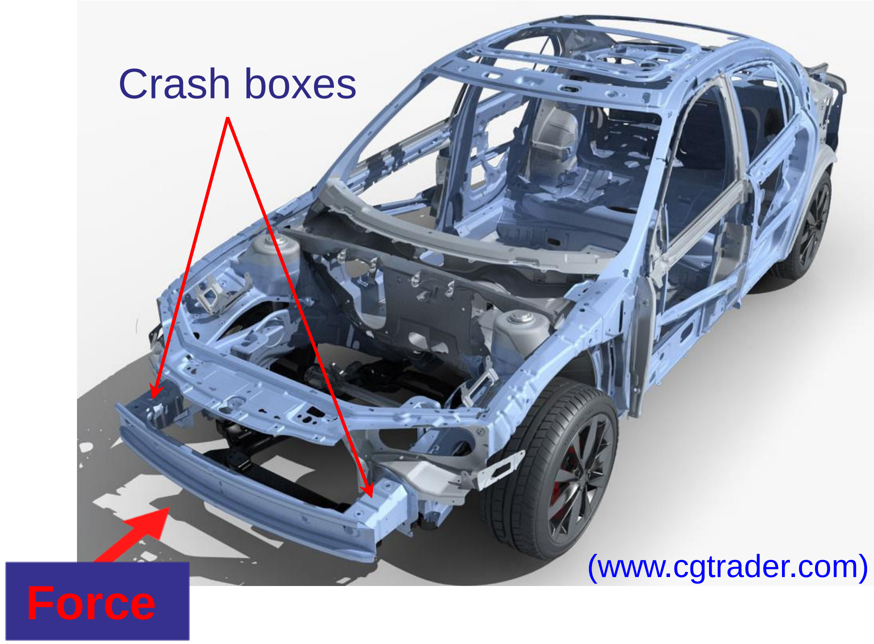
More than 25000 road accidents in 2018 within EU region. (European Commission, 2019)

Important parameters

- Material property
- Direction of impact
- Geometry of crash box
- Velocity of car
- Total mass



- ➢ Crash problem, impact loading: nonlinear dynamic problem
- ➢ Possible variation all the parameters = uncertainties

Uncertainty propagation of an impact problem

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INTRODUCTION: UNCERTAINTY PROPAGATION

modelling input uncertain parameters → Uncertainty propagation → Uncertainty Quantification: statistics of output

- Random variable/field
- Expert's knowledge

- Computational cost
  - Monte Carlo simulation
  - Metamodels

- Mean, std
- Probability density function (PDF)

Sensitivity analysis

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POLYNOMIAL CHAOS EXPANSION: ASSUMPTIONS ON UNCERTAIN PARAMETERS

$$\mathcal{A} = \{a_i\}_{i=1\cdots r}$$

- Assumptions on uncertain input parameters
  - Known statistical law (normal, uniform, etc.)
  - Independent variables
  - May be reduced to standard deviates $\xi_i \in \Xi$
    - $\mathcal{N}(0,\ 1)$
    - $\mathcal{U}_{[-1;\ 1]}$

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- Discretization of random quantity $\mathbf{Y}$

$$Y(t,\,\Xi)\;\Leftarrow\;Y^p(t,\,\Xi)\;=\;\sum_{j=1}^{n} a_j(t)\,\Phi_j(\Xi)\;=\;\sum_{J\in\mathbb{N},/|J|\leq p} a_J(t)\,\Phi_J(\Xi)\;=\;{}^T[\Phi(\boldsymbol{\Xi})][a(t)]$$

with

- $\Xi = (\Xi) = (\xi_1, \cdots, \xi_r)$
- n: number of terms in the expansion
- Multivariate Polynomial chaos $\Phi_J(\Xi)$ (Wiener 1938):
  - $\Phi_J(\Xi) = \prod_{j=1}^{r} \phi_{J_j}(\xi_j)$
  - J_i= degree(ϕ_{J_j})
  - $|\,J\,|$= degree(Φ_J) $= \sum_{i=1}^{r} J_i$
  - p: maximal degree of all Φ_J = degree(Φ_n)= PCE degree

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POLYNOMIAL CHAOS EXPANSION: PRESENTATION

$$Y^p(t,\ \Xi) = \sum_{j=1}^{n} a_j(t)\ \Phi_j(\Xi) = \sum_{J\in\mathbb{N},/|J|\leq p} a_J(t)\ \Phi_J(\Xi) = {}^T[\Phi(\Xi)][a(t)]$$

$$\Phi_J(\Xi) = \prod_{j=1}^{r} \phi_{J_j}(\xi_j) \Longleftrightarrow [\Phi(\Xi)]$$

- Known univariate orthogonal polynomial $\phi_{J_j}(\xi_j)$
  - Hermite polynomial (normal variate)
  - Legendre polynomial (uniform variate)

$$\{a_j(t)\}_{j=1\cdots n} \Longleftrightarrow [a(t)]$$

- Unknown vectors
  - Non-intrusive approach: *e.g.* Regression
  - Intrusive approach: Galerkin projection (effective only for linear problem)

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POLYNOMIAL CHAOS EXPANSION: NON INTRUSIVE APPROACH

- NI-PCE: data-driven approach: N samples of the output $\{(\boldsymbol{\Xi}_j),\ Y(t,\boldsymbol{\Xi}_j)\}_{j=1,\cdots,N}$

$$\begin{bmatrix} Y(t,\boldsymbol{\Xi}_1) \\ \cdots \\ Y(t,\boldsymbol{\Xi}_N) \end{bmatrix} \simeq \begin{bmatrix} {}^T[\Phi(\boldsymbol{\Xi}_1)] \\ \vdots \\ {}^T[\Phi(\boldsymbol{\Xi}_N)] \end{bmatrix} \begin{bmatrix} a_1(t) \\ \cdots \\ a_n(t) \end{bmatrix}$$

$N \times n$ system of equations

=> regression approach + …

- The coefficients must be computed at each time-step

- Finding a time domain basis
  - Time discretization

$$t^d = [t_1 \quad \cdots \quad t_{n_t}]$$

  - "Correlation" matrix

$$C = \; ^T Y(t^d, \Xi^s) \; Y(t^d, \Xi^s)$$

with $$Y(t^d, \Xi^s) = \begin{bmatrix} Y(t_1, \Xi_1) & \cdots & Y(t_{n_t}, \Xi_1) \\ \vdots & \cdots & \vdots \\ Y(t_1, \Xi_N) & \cdots & Y(t_{n_t}, \Xi_N) \end{bmatrix} \in \mathbb{R}^{N \times n_t}$$

- POD: Eigenvalue decomposition of C (or SVD of $Y(t^d, \Xi^s)$)

$$\forall i = 1, \ldots, n_t, \; CV_i = \lambda_i V_i$$

V_i = i-th POD vector (POV)
λ_i = i-th eigenvalue ("Energy")

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POLYNOMIAL CHAOS EXPANSION: PROPER ORTHOGONAL DECOMPOSITION (POD)

- Finding a time domain basis
  - POD expansion

$$Y\left(t^d, \Xi\right) = \sum_{i=1}^{n_t} b_i\left(\Xi\right)\ {}^T\boldsymbol{V}_i\left(t^d\right) \approx \sum_{i=1}^{n_b} b_i\left(\Xi\right)\ {}^T\boldsymbol{V}_i\left(t^d\right)$$

In the following $\sum_{i=1}^{n_b} \lambda_i \approx 99.99\,\%\ \sum_{i=1}^{n_t} \lambda_i$

  - POD coefficient

$$b_i\left(\Xi\right) = Y\left(t^d, \Xi\right) \boldsymbol{V}_i(t^d)$$

  - POD-PCE coefficient

$$b_i^p\left(\Xi\right) = \sum_{j=1}^{n} a_{j,i}\ \Phi_J\left(\Xi\right)$$

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POLYNOMIAL CHAOS EXPANSION: PROPER ORTHOGONAL DECOMPOSITION (POD)

- Decoupling time domain and randomness:
  - POD-PCE expansion

$$Y\left(t^d, \Xi\right) \approx Y^{p,n_b}\left(t^d, \Xi\right) = \sum_{i=1}^{n_b} \sum_{j=1}^{n} a_{j,i} \Phi_j\left(\Xi\right) \boldsymbol{V}_i\left(t^d\right)$$

with $a_{J,i}$ estimated by the non-intrusive approach (*e.g.* regression)

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- Impact law: Hert'z law

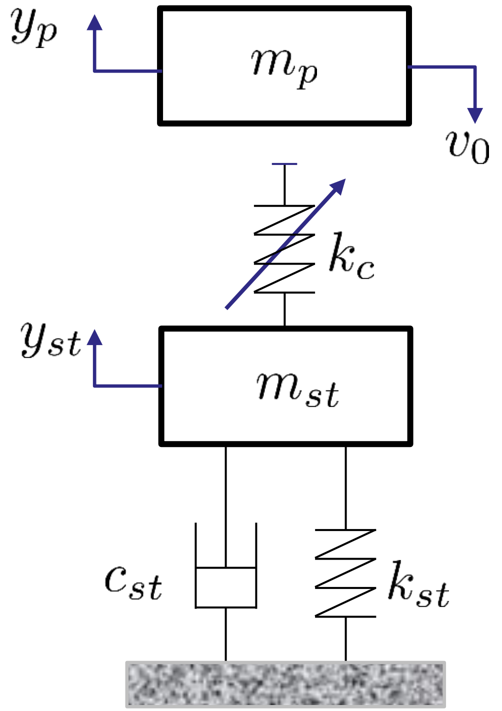
$$f_c = k_c(y_{st} - y_p)^{\frac{3}{2}}; \quad y_{st} \geq y_p$$
$$= 0 \quad ; \quad y_{st} < y_p$$

- Motion equation

$$m_p\ddot{y}_p - f_c = 0$$
$$m_{st}\ddot{y}_{st} + c_{st}\dot{y}_{st} + k_{st}y_{st} + f_c = 0$$

with

$$y_{st}(0) = 0 \qquad \dot{y}_{st}(0) = 0$$
$$y_p(0) = 0 \qquad \dot{y}_p(0) = -v_0$$



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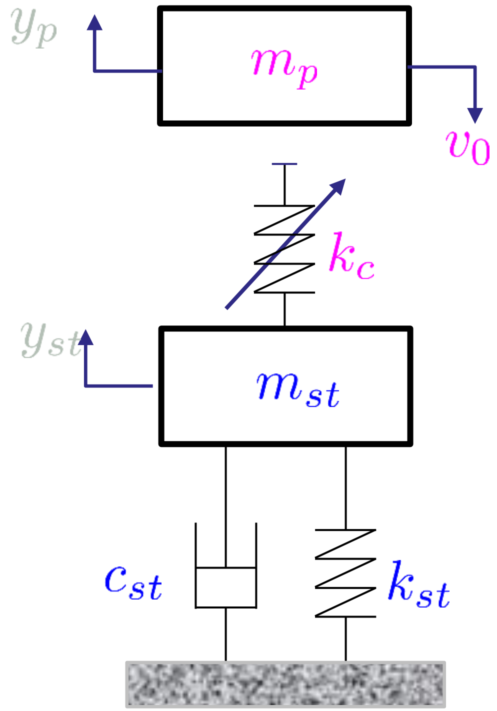
- Deterministic input data
  - Stiffness
    - Multiple impacts $k_{st} = 2.4\,\mathrm{MN\,m^{-1}}$
  - Mass
  - Damping ratio

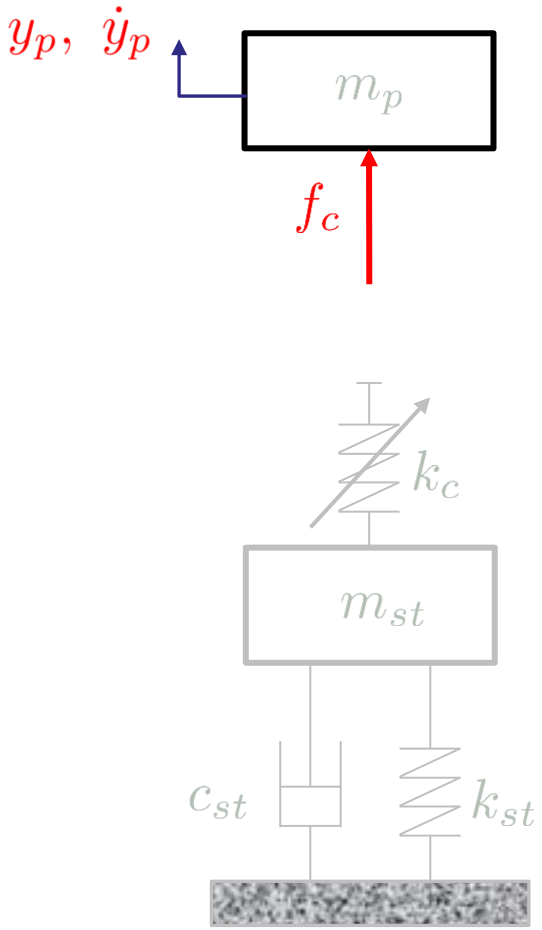
$m_{st} = 60\,\mathrm{g}$

$\zeta_{st} = 0.5\%$

- Random input data
  - $m_p,\ k_c,\ v_0$ $\quad p_i = \overline{p}_i(1 + \delta_{p_i}\ \xi_i)$
  - ξ_i: uniform distribution: mean=0; std=1
  - $\delta_{pi} = 10\ \%$

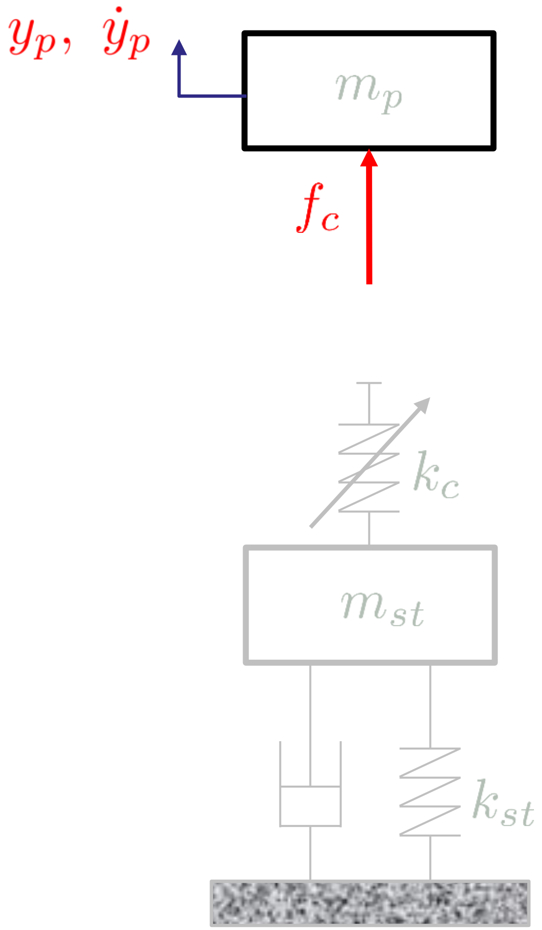


- Metamodels for
  - Impact force f_c
  - Projectile displacement y_p
  - Projectile velocity $\dot{y}_p$



RANDOM IMPACT OSCILLATOR: METAMODEL; POD-PCE CHARACTERISTICS

- POD-PCE model
  - PCE degree: two cases:
    - $p = 2$ $(n = 10)$
    - $p = 3$ $(n = 20)$
  - $N = 50$ samples
  - n_b: chosen to keep 99.99% of the "energy«
- Time discretization
  - $n_t = 3001$
  - $\Delta t = 1\mu s$



RANDOM IMPACT OSCILLATOR: METAMODEL; POD-PCE CHARACTERISTICS

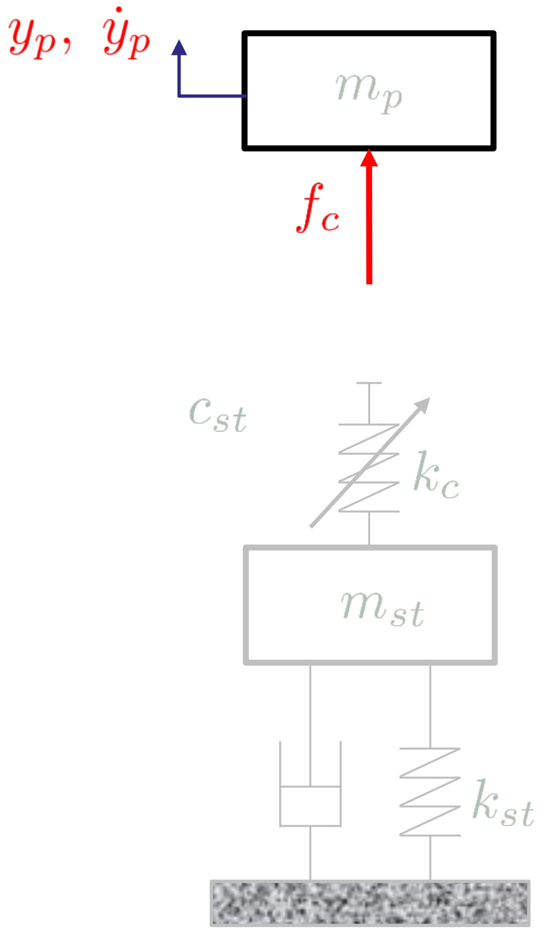
- Comparison:
  - Reference model: Monte Carlo simulation (MCS)
  - POD-PCE estimation
  - $N_{MCS} = 10^4$ samples
  - Relative error

$$\epsilon_k = \frac{\sum_{i=1}^{n_t} \left[Y(t_i, \Xi_k) - Y^p(t_i, \Xi_k)\right]^2}{\sum_{i=1}^{n_t} \left[Y(t_i, \Xi_k) - \bar{Y}(\Xi_k)\right]^2}$$

$$\bar{Y}(\Xi_k) = \frac{1}{n_t} \sum_{i=1}^{n_t} Y(t_i, \Xi_k)$$

  - Mean relative error

$$\bar{\epsilon} = \frac{1}{N_{\mathrm{MCS}}} \sum_{i=1}^{N_{\mathrm{MCS}}} \epsilon_i$$



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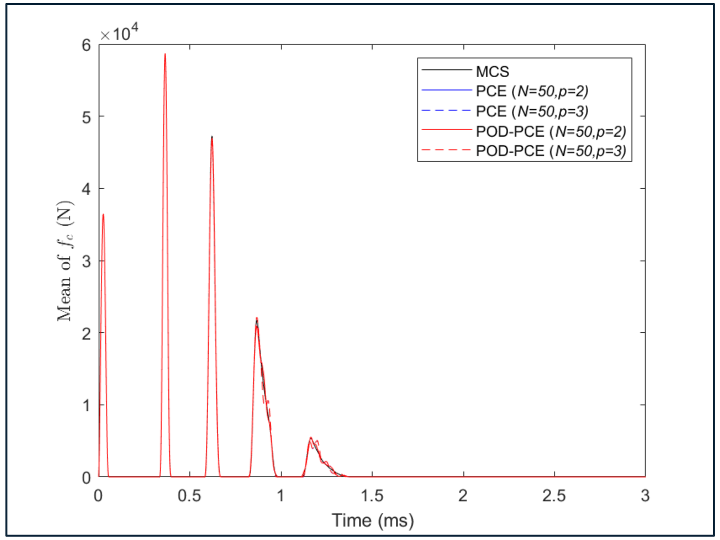
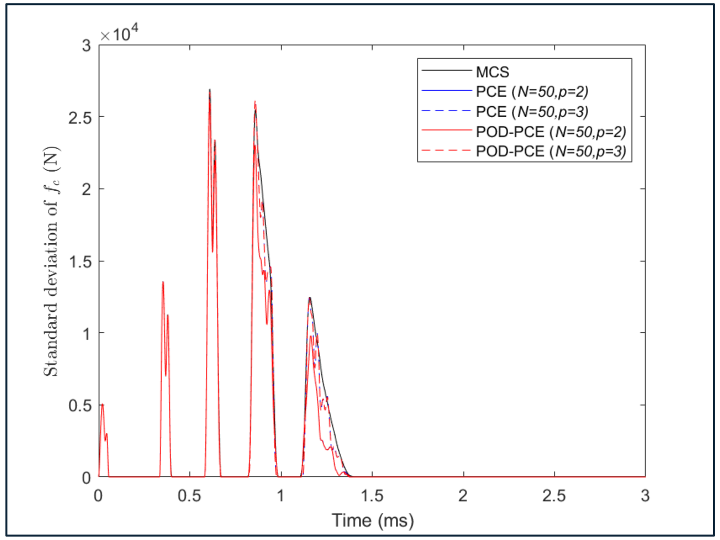
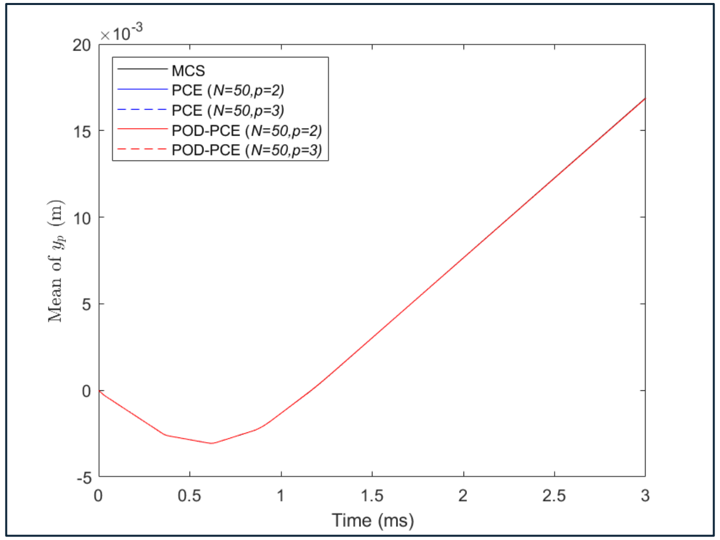
RANDOM IMPACT OSCILLATOR: METAMODEL; MEAN & STANDARD DEVIATION OF THE QOI

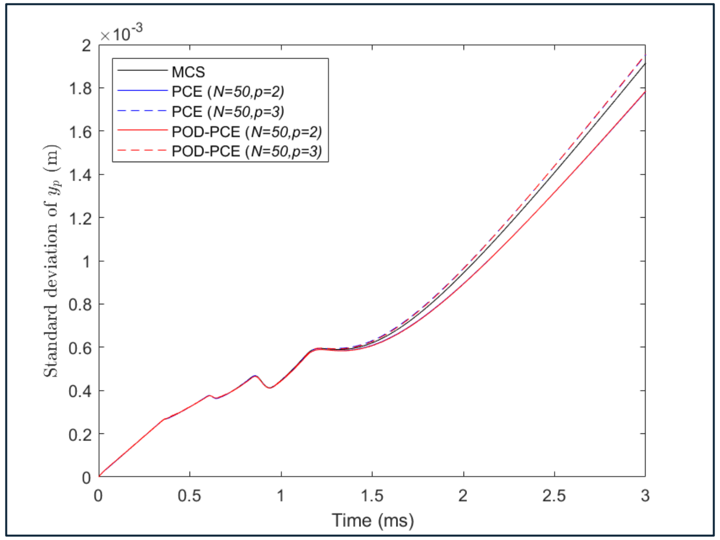
Contact force



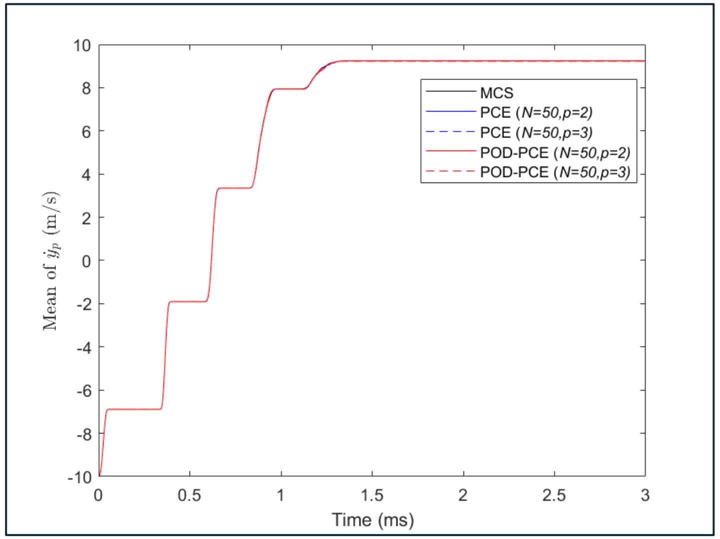


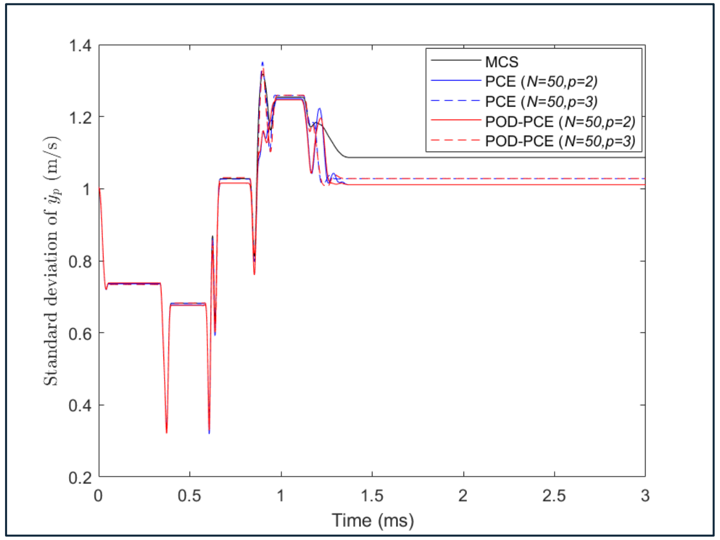
Projectile displacement





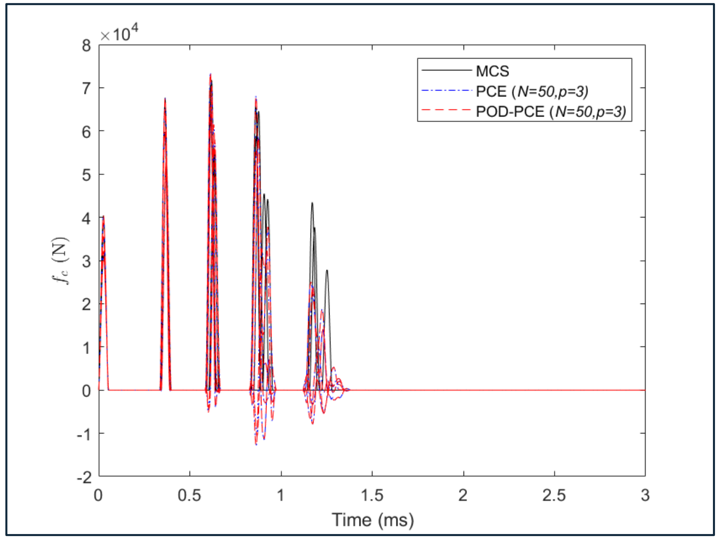
Projectile velocity



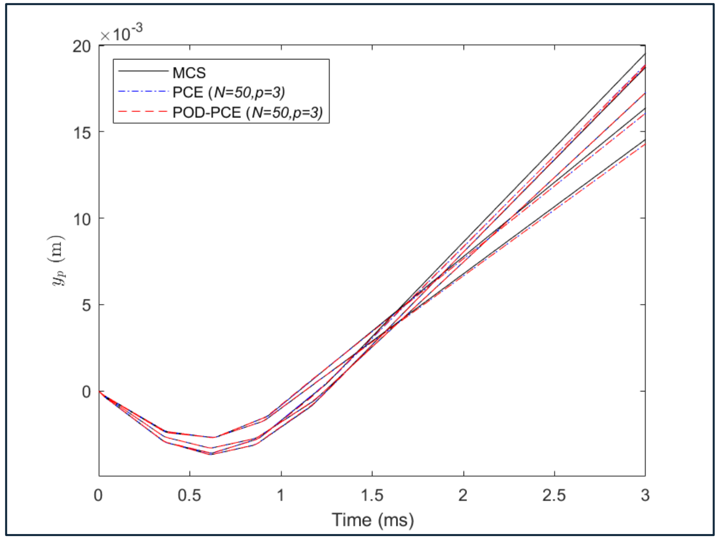


RANDOM IMPACT OSCILLATOR: METAMODEL; PREDICTION OF 5 RESPONSES

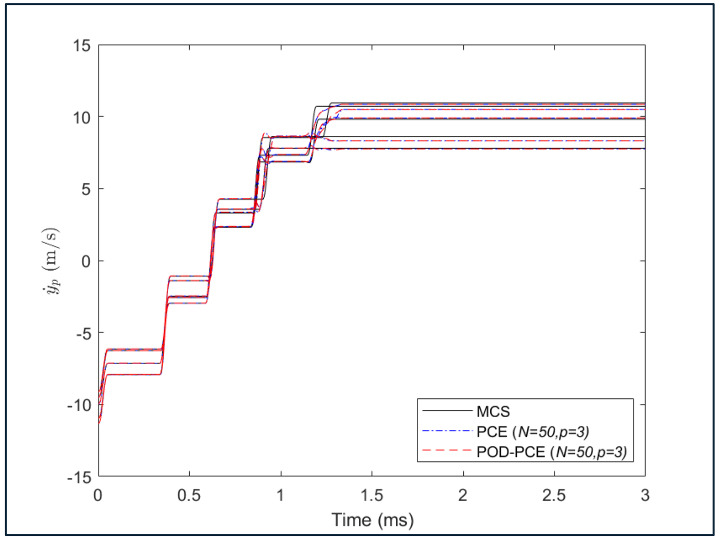
Contact force



Projectile displacement



Projectile velocity



		Contact force			Projectile displacement			Projectile velocity		
Method	p	n_b	n_{tot}	$\bar{\epsilon}$	n_b	n_{tot}	$\bar{\epsilon}$	n_b	n_{tot}	$\bar{\epsilon}$
PCE	2	-	30010	1.14×10^{-1}	-	30010	2.52×10^{-3}	-	30010	3.91×10^{-3}
PCE	3	-	60020	1.05×10^{-1}	-	60020	1.49×10^{-3}	-	60020	2.21×10^{-3}
POD-PCE	2	31	310	1.13×10^{-1}	3	30	2.52×10^{-3}	9	90	3.91×10^{-3}
POD-PCE	3	31	620	1.05×10^{-1}	3	60	1.49×10^{-3}	9	180	2.16×10^{-3}

RANDOM IMPACT OSCILLATOR: METAMODEL; COMMENTS

- Time domain basis and randomness basis with POD-PCE.
- Good accuracy → few POM.
- Good accuracy using low degree polynomials.
- POD-PCE model → 1% coefficients as compared to PCE model for contact force.
- Non-physical negative forces predicted due to the consequence of PCE model.
- "Discontinuity" of velocity predicted well
- 50 samples: too much?

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RANDOM IMPACT OSCILLATOR: METAMODEL; COMMENTS

- 50 samples: too much?

⇨ Sparse POD-PCE model

  - Least Angle Regression (LAR): L1-regression (Brad Efron et al., Least Angle Regression, the Annals of statistics, vol 32(4), 2004)
  - Variational Bayesian inference with automatic relevance determination (Jan Drugowitsch, Variational Bayesian inference for linear and logistic regression, arXiv 1310.5438, 2019-v4)

- Same errors with 5 samples!

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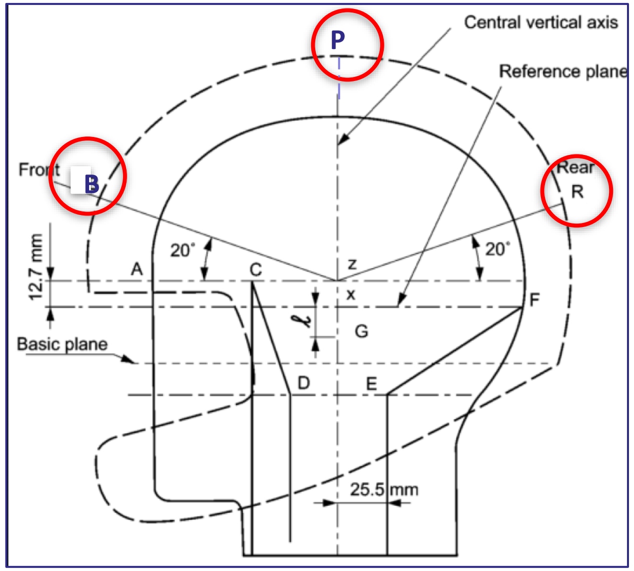
MOTORCYCLE HELMET: HOMOLOGATION

- ECE 22.05 standard on helmet homologation: Impact tests
  - Test conditions
    - Several anvils: flat and "kerbstone"
    - 1 impact velocity target: 7.5 m/s
    - Several impact points
  - Measurements: headform acceleration
  - Criteria of success:
    - Maximum of acceleration:
    - $a_M < 275\ g$
    - Head Injury Criterion:

$$\text{HIC} < 2400$$

$$\text{HIC} = \max_{t_1,t_2}\left\{\left[\frac{1}{t_2-t_1}\int_{t_1}^{t_2} a(t)dt\right]^{2.5}(t_2-t_1)\right\}$$

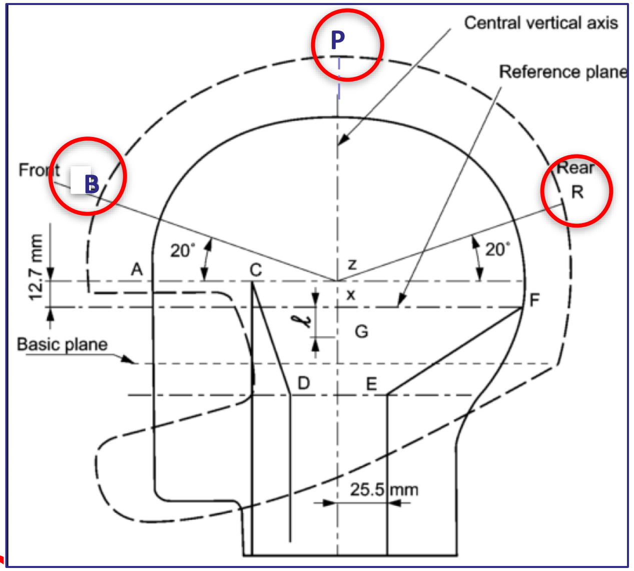
$[\ a\ ]=g$, $[\ t\ ]=s$



ECE 22.05, 2002

COST 327 – Motorcycle safety helmet
Final report, 2001

- ECE 22.05 standard on helmet homologation: Impact tests
  - Uncertainties
    - Velocity $[7.5, 7.5 + 0.15]$ m/s
  - Impact point location
    - B: disc of radius 10 mm
    - P: disc of radius 50 mm



ECE 22.05, 2002

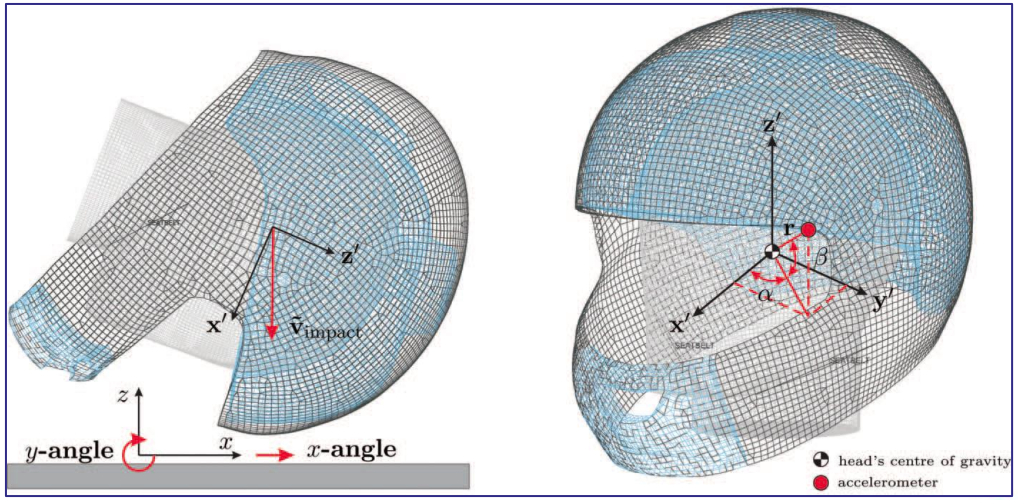
- Considering the uncertainties, what is the probability to pass the impact test?
  - Answering the question requires a model:
    - Problem of a validated model
    - Numerous simulations

=> metamodel?

COST 327 – Motorcycle safety helmet
Final report, 2001

- Unifi's helmet model
  - Based on a medium-size commercial helmet
  - Material properties
    - Experiments
    - EC project APROSYS
  - LS-dyna
  - ~20 min/simulation

=> metamodel is necessary



International Journal of Crashworthiness (2011), 16(5), 523-536. A. Pratellesi et al.

MOTORCYCLE HELMET: UNCERTAINTY MODEL

- Uncertainties: uniform random variables
  - Velocity: $v \sim \mathcal{U}[7.5, 7.65]$ m/s
  - Impact point location
    - B: disc of radius 10 mm => $\theta_x,\ \theta_y \sim \mathcal{U}[-3,\ +3]^\circ$
    - P: disc of radius 50 mm => $\theta_x,\ \theta_y \sim \mathcal{U}[-15,\ +15]^\circ$
- Samples
  - 1 model simulation: ~15 min

    $\Rightarrow$ budget = 60 simulations
  - LHS (Latin Hypercube Sampling): 60 samples
  - 60 outputs
    - 60 (maximum of) acceleration
    - 60 HIC

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MOTORCYCLE HELMET: METAMODEL

- $p = 10$

  => 286 terms

  => sparce PCE

- 60 model evaluations
  - 50 (randomly drawn): PCE identification
  - 10 (the others): PCE validation
    - calculation of an error ($x = a_M$ or HIC) $\epsilon_x = \frac{\|x_{\text{actual}} - x_{\text{metamodel}}\|_2}{\|x_{\text{actual}}\|_2}$
    - 50 repetitions:
      - calculation of a mean error $\bar{\epsilon}_x$

	x	a_M	hic
Point B	$\bar{\epsilon}_x$ (%)	0.24	0.20
Point P	$\bar{\epsilon}_x$ (%)	1.31	1.69

=> metamodel validated

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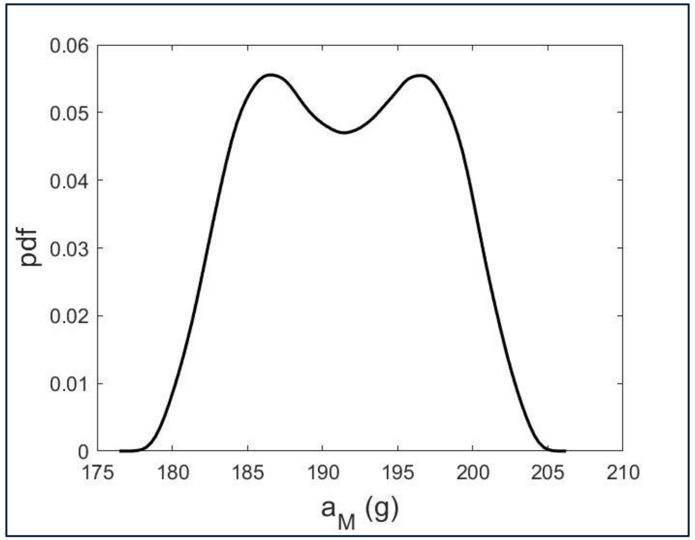
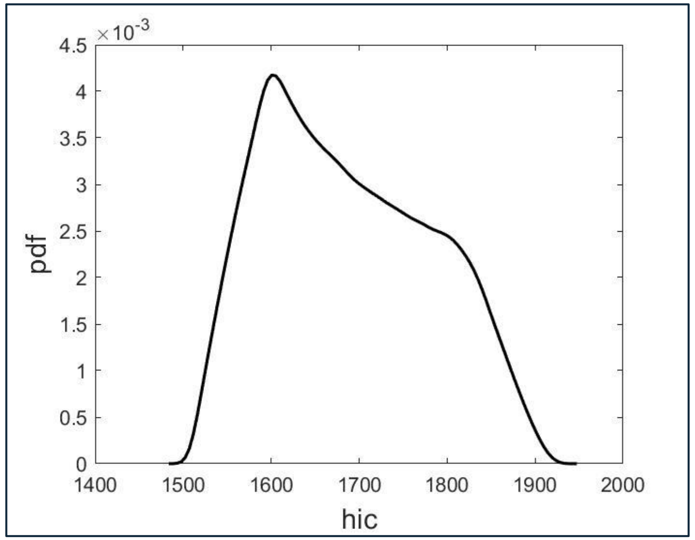
- Probability density function (pdf)
  - Estimated with the metamodel
  - Monte Carlo simulation
  - 10 000 samples (Latin Hypercube Sampling)

- Point B

$\Pr(a_M > 275\text{ g}) \sim 0$

$\Pr(\text{HIC} > 2400) \sim 0$





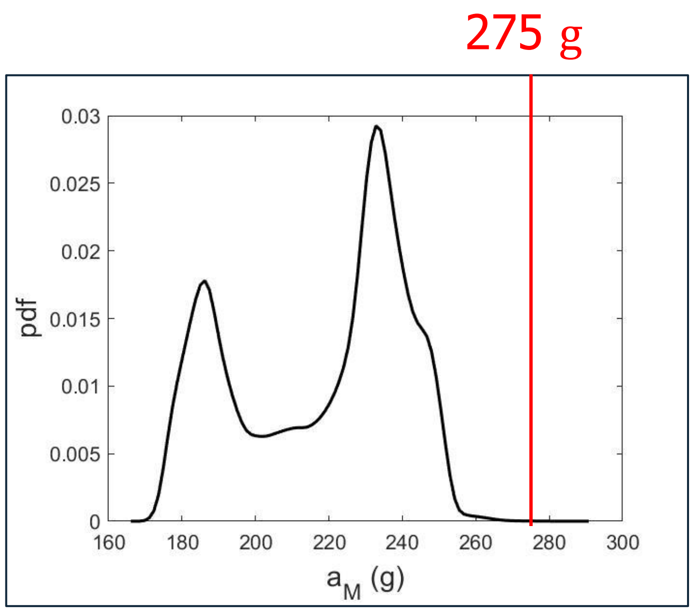
MOTORCYCLE HELMET: METAMODEL: RESULTS

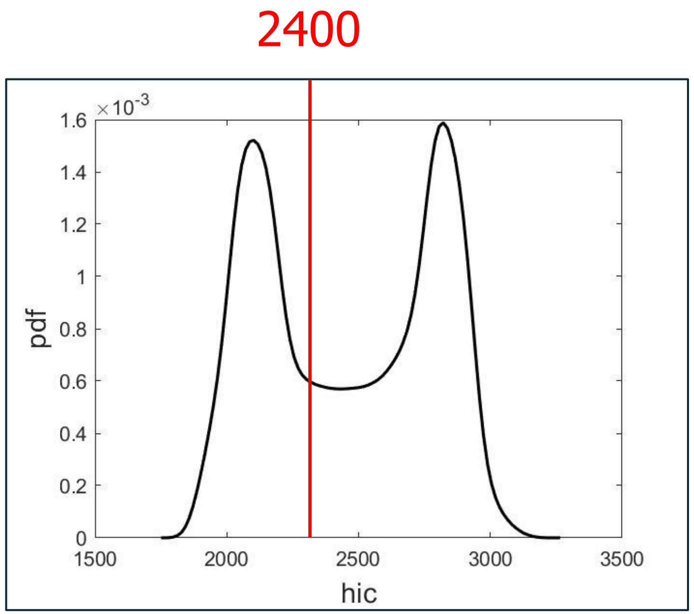
- Probability density function (pdf)
  - Estimated with the metamodel
  - Monte Carlo simulation
  - 10 000 samples (Latin Hypercube Sampling)

- Point P

$\Pr(a_M > 275\ \text{g}) \sim 0$

$\Pr(\text{HIC} > 2400) \sim 0{,}55$





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CONCLUSIONS

- Effective sparse POD-PCE for uncertainty propagation for impact problem
- But, physical conditions should be added (positiveness of an impact force)
- Optimal PCE degree?
- Optimal number of samples?

=> adaptative procedure

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IMPACTED RANDOM SYSTEMS: A DESCRIPTION BY POLYNOMIAL CHAOS EXPANSION (PCE)

Thank you for your attention