Modeling of Distributed Parameter Systems: The Port Hamiltonian Approach

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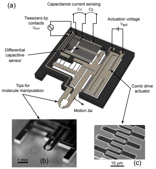
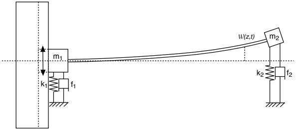
Outline

Context : control of flexible structures

- Boundary controlled systems





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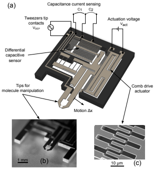
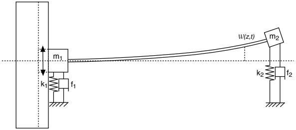
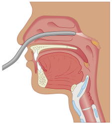
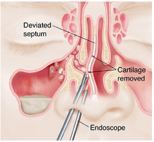
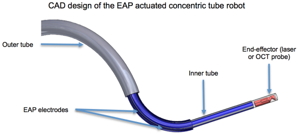




- In-domain control of distributed parameter systems







  - Exploration, imaging, diagnosis.
  - Mini invasive surgery.
  - Toward miniaturized and *smart* endoscopes.

Context : port Hamiltonian systems

- Port Hamiltonian systems:
  - The state variables are chosen as the energy variables.
  - The links between the energy function and the system dynamics is made explicit through symmetries.
  - The boundary port variables are power conjugated.

- Energy shaping consists in using the physical properties of the system to derive efficient control laws with guaranteed performances (a step further than stabilization).

- ”Easy” to extend to non linear or systems defined on higher dimensional spaces.

Outline

Port-Hamiltonian partial differential equations

Our model class are p.d.e.'s of the form

$$\frac{\partial x}{\partial t}(\zeta, t) \quad = \quad \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right) [\mathcal{H} x(\zeta, t)]$$

with

- $x(\zeta, t) \in \mathbb{R}^n$, $\zeta \in [a, b]$, $t \geq 0$
- P_1 is an invertible, symmetric real $n \times n$-matrix,
- P_0 is a skew-symmetric real $n \times n$-matrix,
- $\mathcal{H}(\zeta)$ is a symmetric, invertible $n \times n$-matrix with $mI \leq \mathcal{H}(\zeta) \leq MI$ for some $m, M > 0$.

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The energy/Hamiltonian is defined as

$$H(t) = H(x(\cdot, t)) = \frac{1}{2} \int_a^b x(\zeta, t)^T \mathcal{H}(\zeta) x(\zeta, t) d\zeta.$$

Boundary port variables

Boundary port variables

Let $\mathcal{H}x \in H^1(a,b;\mathbb{R}^n)$. Then the boundary port variables are the vectors $e_\partial, f_\partial \in \mathbb{R}^n$,

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = U \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{pmatrix} = R \begin{pmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{pmatrix}$$

$$\Leftrightarrow \ \frac{1}{2}\frac{d}{dt}\|x\|^2_{\mathcal{H}} = f_\partial^T e_\partial,$$

Where

$$U^T \Sigma U = \Sigma, \quad \Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \Sigma \in M_{2n}(\mathbb{R})$$

Input and output

Boundary controlled port Hamiltonian systems

Let W be a $n \times 2n$ real matrix. If W has full rank and satisfies $W\Sigma W^\top \geq 0$, then the system $\frac{\partial x}{\partial t} = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}(\zeta)x)(t,\zeta)) + (P_0 - G_0)\mathcal{H}(\zeta)x(t,\zeta)$with input

$$u(t) = W \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}$$

is a BCS on X. The operator $\mathcal{A}x = P_1(\partial/\partial\zeta)(\mathcal{H}x) + (P_0 - G_0)\mathcal{H}x$ with domain

$$D(\mathcal{A}) = \left\{ \mathcal{H}x \in H^1(a,b;\mathbb{R}^n) \;\middle|\; \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \in \ker W \right\}$$

generates a contraction semigroup on X.

Input and output

Let $\tilde{W}$ be a full rank matrix of size $n \times 2n$ with $\begin{bmatrix} W \\ \tilde{W} \end{bmatrix}$ invertible and let $P_{W,\tilde{W}}$ be given by

$$P_{W,\tilde{W}} = \left(\begin{pmatrix} W \\ \tilde{W} \end{pmatrix} \Sigma \begin{pmatrix} W \\ \tilde{W} \end{pmatrix}^{\top} \right)^{-1} = \begin{pmatrix} W\Sigma W^{\top} & W\Sigma\tilde{W}^{\top} \\ \tilde{W}\Sigma W^{\top} & \tilde{W}\Sigma\tilde{W}^{\top} \end{pmatrix}^{-1}.$$

Define the output of the system as the linear mapping $\mathcal{C} : \mathcal{H}^{-1}H^{1}(a,b;\mathbb{R}^{n}) \to \mathbb{R}^{n}$,

$$y = \mathcal{C}x(t) := \tilde{W} \begin{pmatrix} f_{\partial}(t) \\ e_{\partial}(t) \end{pmatrix}.$$

Then for $u \in C^{2}(0,\infty;\mathbb{R}^{k})$, $\mathcal{H}x(0) \in H^{1}(a,b;\mathbb{R}^{n})$, and $u(0) = W \begin{bmatrix} f_{\partial}(0) \\ e_{\partial}(0) \end{bmatrix}$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^{2}_{\mathcal{H}} \leq \frac{1}{2} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}^{\top} P_{W,\tilde{W}} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}$$

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$$y = \mathcal{C}x(t) := \tilde{W} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}.$$

We choose W and $\tilde{W}$ such that $\begin{pmatrix} W \\ \tilde{W} \end{pmatrix} \Sigma \begin{pmatrix} W^T & \tilde{W}^T \end{pmatrix} = \Sigma.$

In this particular case:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{H}} \leq y^T(t)u(t). \tag{1}$$

Static feedback control

Impedance passive case

If the matrices W and $\tilde{W}$ are selected such that $P_{W,\tilde{W}} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \Sigma$, then the BCS fulfills

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} \leq u^\top(t)y(t).$$

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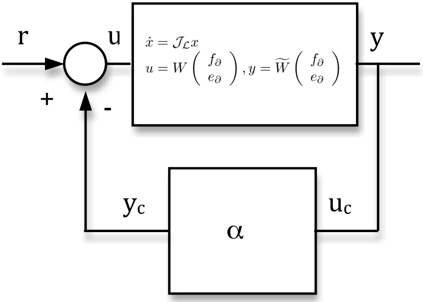


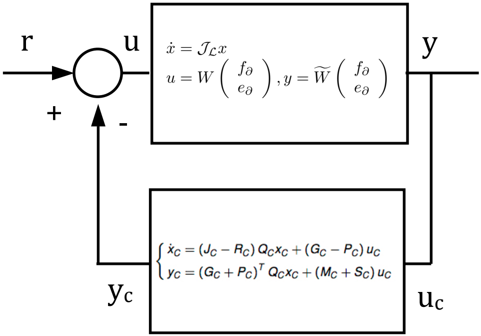
Static controller

- Asymptotic stability:
  $\alpha > 0$+(compacness condition)
- Exponential stability: α st

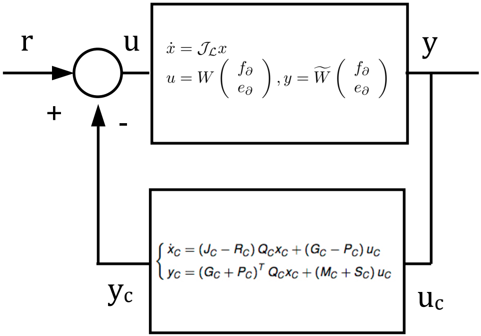
$$(dE/dt) \leq -k\|(\mathcal{L}x)(t,b)\|^2_{\mathbb{R}}$$

where $k > 0$.

Dynamic control



Dynamic control



- Can we use passivity properties to design dynamic controllers ?
- What about closed loop trajectories ?
- Can we extend the energy shaping ideas to boundary controlled port Hamiltonian systems ?

Energy shaping : The immersion/reduction approach

We consider a dynamic controller of the form

$$\begin{cases} \dot{x}_C = (J_C - R_C)\, Q_C x_C + (G_C - P_C)\, u_C \\ y_C = (G_C + P_C)^T\, Q_C x_C + (M_C + S_C)\, u_C \end{cases} \tag{2}$$

where $x_C \in \mathbb{R}^{n_C}$ and $u_C,\ y_C \in \mathbb{R}^n$, while $J_C = -J_C^T$, $M_C = -M_C^T$, $R_C = R_C^T$, and $S_C = S_C^T$, with this further condition satisfied:

$$\begin{pmatrix} R_C & P_C \\ P_C^T & S_C \end{pmatrix} \geq 0\,. \tag{3}$$

Interconnected to the boundary of the system

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u_C \\ y_C \end{pmatrix} + \begin{pmatrix} u' \\ 0 \end{pmatrix}, \tag{4}$$

where $u' \in \mathbb{R}^n$ is an additional control input.

Existence of solutions

Theorem

Let the open-loop BCS satisfy $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} = u(t)y(t)$ and consider the previous passive finite dimensional port Hamiltonian system. Then the power preserving feedback interconnection
$u = r - y_c, y = u_c$
with $r \in \mathbb{R}^n$ the new input of the system is a BCS on the extended state space $\tilde{x} \in \tilde{X} = X \times V$ with inner product $\langle \tilde{x}_1, \tilde{x}_2 \rangle_{\tilde{X}} = \langle x_1, x_2 \rangle_{\mathcal{L}} + \langle v_1, Q_c v_2 \rangle_V$. Furthermore, the operator $\mathcal{A}_e$ defined by

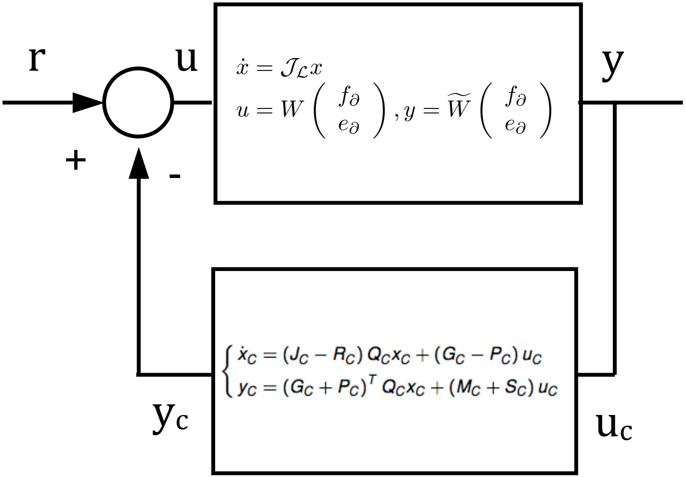
$$\mathcal{A}_e \tilde{x} = \begin{pmatrix} \mathcal{J}\mathcal{L} & 0 \\ B_c \mathcal{C} & A_c \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix},$$

$$D(\mathcal{A}_e) = \left\{ \begin{pmatrix} x \\ v \end{pmatrix} \in \begin{pmatrix} X \\ V \end{pmatrix} \middle| \mathcal{L}x \in H^N(a, b; \mathbb{R}^n), \begin{pmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \\ v \end{pmatrix} \in \ker \tilde{W}_D \right\}$$

where $$\tilde{W}_D = \left(\left(W + D_c \tilde{W} \quad C_c \right) \right)$$

generates a contraction semigroup on $\tilde{X}$.

Energy shaping : The immersion/reduction approach



Energy shaping : The immersion/reduction approach

Objectives

Modification of the closed loop system's properties (energy shaping) + stabilization (damping injection).

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$$H_{cl}(x, x_c) = H(x) + H_c(x_c)$$

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We first look for structural invariants $C(x, x_c)$ *i.e.* $\frac{dC}{dt} = 0$

$$C(x, x_c) = x_c + F(x) = \kappa$$

where F is a smooth function.

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where F is a smooth function. In this case the closed loop energy function reads

$$H_{cl}(x, x_c) = H_{cl}(x) = H(x) + H_c(\kappa - F(x))$$

Asymptotic stability of the closed loop system in x^* is achieved using damping injection such that

$$\frac{dH_{cl}}{dt} < 0, \forall x \neq x^*.$$

Energy shaping : The immersion/reduction approach

Energy shaping

- Use of the structural invariants C (such that $\dot{C} = 0$) of the form

$$C(x(t), x_c(t)) = \Gamma^T x_c(t) + \int_a^b \psi^T(\zeta) x(t,\zeta) dz = \kappa$$

  to link the controller states to the system states.
- Choice of the controller energy function to "shape" the closed loop energy function as $$H_{cl}(x(t), x_c(t)) = H(x(t)) + H_c(x_c(t)) = H(x(t)) + H_c(F(x(t)))$$
- Well known and very efficient for finite dimensional non linear systems.
- What about the linear boundary controlled infinite dimensional case ?

Structural invariants

Casimir functions

Consider the closed loop boundary control system with $u' = 0$ then,

$$C(x(t), x_c(t)) = \Gamma^T x_c(t) + \int_a^b \psi^T(\zeta) x(t,\zeta) dz$$

is a Casimir function for this system if and only if $\psi \in H^1(a,b;\mathbb{R}^n)$,

$$P_1 \frac{d\psi}{dz}(\zeta) + (P_0 + G_0)\psi(\zeta) = 0 \quad (5)$$

$$(J_C + R_C)\Gamma + (G_C + P_C)\tilde{W} R \begin{pmatrix} \psi(b) \\ \psi(a) \end{pmatrix} = 0 \quad (6)$$

$$(G_C - P_C)^T \Gamma + \left[W + (M_C - S_C)\tilde{W} \right] R \begin{pmatrix} \psi(b) \\ \psi(a) \end{pmatrix} = 0 \quad (7)$$

Energy shaping

Sketch of the proof

$C(x_e(t))$ is a Casimir function if and only if $\frac{dC}{dt} = 0$ independently to the energy function,

$$\frac{dC}{dt} = \left\langle \frac{\delta C}{\delta x_e}, \frac{dx_e}{dt} \right\rangle_{L^2} \tag{8}$$

$$= \left\langle \frac{\delta C}{\delta x_e}, \mathcal{A}_e \mathcal{H}_e x_e \right\rangle_{L^2} \tag{9}$$

$$= \left\langle \mathcal{A}_e^* \frac{\delta C}{\delta x_e}, \mathcal{H}_e x_e \right\rangle_{L^2} + BC \tag{10}$$

(11)

Energy shaping

Proposition

Under the hypothesis that the Casimir functions exist, the closed-loop dynamics (when $u = y_c + u'$) is given by :

$$\frac{\partial x}{\partial t}(t,\zeta) = P_1 \frac{\partial}{\partial \zeta} \frac{\delta H_{cl}}{\delta x}(x(t))(\zeta) + (P_0 - G_0) \frac{\delta H_{cl}}{\delta x}(x(t))(\zeta)$$
$$u' = W' R \begin{pmatrix} \left(\frac{\delta H_{cl}}{\delta x}(x)\right)(b) \\ \left(\frac{\delta H_{cl}}{\delta x}(x)\right)(a) \end{pmatrix} \tag{12}$$

in which δ denotes the variational derivative, while

$$H_{cl}(x(t)) = \frac{1}{2}\|x(t)\|^2_{cl} + \frac{1}{2}\left(\int_a^b \hat{\Psi}^T(\zeta) x(t,\zeta)\, dz\right)^T \times$$
$$\times\, \hat{\Gamma}^{-1} Q_C \hat{\Gamma}^{-T} \int_a^b \hat{\Psi}(\zeta)^T x(t,\zeta)\, dz \tag{13}$$

and W' is a $n \times 2n$ full rank, real matrix s.t. $W'\Sigma W'^T \geq 0$.

Extension to systems with dissipation

Proposition

The feedback law $u = \beta(x) + u'$, with u' an auxiliary boundary input, maps the original system into the target dynamical system

$$\begin{aligned} \frac{\partial x}{\partial t}(t,\zeta) &= P_1 \frac{\partial}{\partial \zeta} \frac{\delta H_d}{\delta x}(x(t))(\zeta) + (P_0 - G_0) \frac{\delta H_d}{\delta x}(x(t))(\zeta) \\ u'(t) &= W R \begin{pmatrix} \left(\frac{\delta H_d}{\delta x}(x(t))\right)(b) \\ \left(\frac{\delta H_d}{\delta x}(x(t))\right)(a) \end{pmatrix} \end{aligned} \tag{14}$$

with $H_d(x) = H(x) + H_a(x)$, provided that

$$P_1 \frac{\partial}{\partial \zeta} \frac{\delta H_a}{\delta x}(x) + (P_0 - G_0) \frac{\delta H_a}{\delta x}(x) = 0 \tag{15}$$

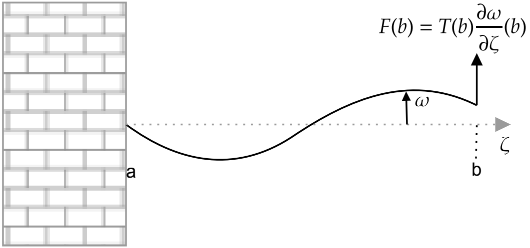
$$\beta(x) + W R \begin{pmatrix} \left(\frac{\delta H_a}{\delta x}(x)\right)(b) \\ \left(\frac{\delta H_a}{\delta x}(x)\right)(a) \end{pmatrix} = 0. \tag{16}$$

Energy shaping

With the dynamic extension or state feedback we have been able to shape a part of the closed loop energy function. It remains to prove that the closed loop system is asymptotically stable.

- We have to consider additional damping injection.
- Exponential stabilisation is not possible as ”exponential stability of the controller + direct feedforward term” are necessary $\rightarrow$ no Casimir function.

Example: Ic



State variables : deformation and linear momentum density

$$\varepsilon(t,\zeta) = \frac{\partial \omega}{\partial \zeta}(t,\zeta), \quad p(t,\zeta) = \mu(\zeta) v(t,\zeta) \tag{17}$$

Material's deformation is considered linear (Hooke's law) :

$$\mu(\zeta)\frac{\partial^2 \omega}{\partial t^2}(t,\zeta) = \frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial \omega}{\partial \zeta}(t,\zeta)\right) - D\frac{\partial \omega}{\partial t}(t,\zeta)\mathrm{d}\zeta$$

The energy is given by (kinetic+potential):

$$H(p(t,\zeta),\varepsilon(t,\zeta)) = \frac{1}{2}\int_0^L \left(\frac{p^2(t,\zeta)}{\mu(\zeta)} + T(\zeta)\varepsilon^2(t,\zeta)\right)\mathrm{d}\zeta$$

Example: longitudinal (axial) vibration of a beam

From:

$$H(p(t,\zeta),\varepsilon(t,\zeta)) = \frac{1}{2}\int_0^L \left(\frac{p^2(t,\zeta)}{\mu(\zeta)} + T(\zeta)\varepsilon^2(t,\zeta)\right) \mathrm{d}\zeta$$

We define the co-energy variables:

$$\frac{\delta H}{\delta \varepsilon}(\varepsilon(t,\zeta)) = T(\zeta)\varepsilon(t,\zeta) = \sigma(t,\zeta)$$

$$\frac{\delta H}{\delta p}(p(t,\zeta)) = \frac{p(t,\zeta)}{\mu(\zeta)} = \frac{\partial \omega}{\partial t}(t,\zeta) = v(t,\zeta)$$

Then:

$$\frac{\partial}{\partial t}\left(\mu(\zeta)\frac{\partial \omega}{\partial t}(t,\zeta)\right) = \frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial \omega}{\partial \zeta}(t,\zeta)\right) - D\frac{\partial \omega}{\partial t}(t,\zeta)$$

with

$$\frac{\partial}{\partial t}\left(\frac{\partial \omega}{\partial \zeta}(t,\zeta)\right) = \frac{\partial}{\partial \zeta}\left(\frac{\partial \omega}{\partial t}(t,\zeta)\right)$$

Example: longitudinal (axial) vibration of a beam

The port-Hamiltonian formulation of the system is then

$$\frac{\partial}{\partial t}\begin{pmatrix}\varepsilon(t,\zeta)\\ p(t,\zeta)\end{pmatrix} = \begin{pmatrix}0 & \frac{\partial}{\partial\zeta}\\ \frac{\partial}{\partial\zeta} & -D\end{pmatrix}\begin{pmatrix}T(\zeta) & 0\\ 0 & \frac{1}{\mu(\zeta)}\end{pmatrix}\begin{pmatrix}\varepsilon(t,\zeta)\\ p(t,\zeta)\end{pmatrix}$$

which is in the form :

$$\frac{\partial x}{\partial t}(t,\zeta) = P_1\frac{\partial}{\partial\zeta}\big(\mathcal{H}(\zeta)x(t,\zeta)\big) + (P_0 - G_0)\mathcal{H}(\zeta)x(t,\zeta) \qquad (18)$$

with $P_0 = 0$ and

$$P_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad G_0 = \begin{pmatrix}0 & 0\\ 0 & D\end{pmatrix} \qquad \mathcal{H}(\zeta) = \begin{pmatrix}T(\zeta) & 0\\ 0 & \frac{1}{\mu(\zeta)}\end{pmatrix}$$

Input and output

The boundary port variables are

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} v(L) - v(0) \\ \sigma(L) - \sigma(0) \\ \sigma(L) + \sigma(0) \\ v(L) + v(0) \end{pmatrix}$$

The boundary input and output are selected as

$$u(t) = \begin{pmatrix} v(t,0) \\ \sigma(t,L) \end{pmatrix} \qquad y(t) = \begin{pmatrix} -\sigma(t,0) \\ v(t,L) \end{pmatrix} \tag{19}$$

which can be derived choosing W and $\tilde{W}$ such that:

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad \tilde{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

The energy balance is then :

$$\frac{\mathrm{d}H}{\mathrm{d}t}(t) = -\int_0^L Dv^2(t,\zeta)\,\mathrm{d}\zeta + y^{\mathrm{T}}(t)u(t) \leq y^{\mathrm{T}}(t)u(t).$$

Lossless case : Approach based on structural invariants

We consider a dynamic controller with $n_C = 2$, $R_C = P_C = M_C = S_C = 0$, $G_C = I$ and

$$J_C = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

which implies that the closed-loop system is characterized by the following Casimir functions:

$$C_1(\xi_1(t), \varepsilon(t,\cdot)) = \xi_1(t) - \int_0^L \varepsilon(t,\zeta)\,\mathrm{d}\zeta$$
$$C_2(\xi_2(t), p(t,\cdot)) = \xi_2(t) - \int_0^L p(t,\zeta)\,\mathrm{d}\zeta.$$

The controller Hamiltonian is chosen such that

$$\hat{H}_c(\xi_1, \xi_2) = \frac{1}{2}\Xi_1\xi_1^2 + \frac{1}{2}\Xi_2\xi_2^2 \tag{20}$$

Approach based on structural invariants

The closed loop energy function is:

$$H_{cl}(\varepsilon, p) = \frac{1}{2}\int_0^L \left(\frac{p^2}{\mu(\zeta)} + T(\zeta)\varepsilon^2\right) \mathrm{d}\zeta + \\ + \frac{1}{2}\Xi_1 \left(\int_0^L \varepsilon \, \mathrm{d}\zeta\right)^2 + \frac{1}{2}\Xi_2 \left(\int_0^L p \, \mathrm{d}\zeta\right)^2 \quad (21)$$

and the control is of the form

$$u = -y_c = -G_c \delta H_c = -\begin{pmatrix} \Xi_2 & 0 \\ 0 & \Xi_1 \end{pmatrix} \begin{pmatrix} \int_0^L p \, \mathrm{d}\zeta \\ \int_0^L \varepsilon \, \mathrm{d}\zeta \end{pmatrix}$$

System with dissipation

Due to the dissipation $D \neq 0$, the energy-Casimir method cannot be applied. The closed loop energy function cannot be shaped in the p coordinate.

Admissible H_a :

$$\hat{H}_a(\xi_1, \xi_2) = \frac{1}{2}\Xi_1 \xi_1^2 + \frac{1}{2}\Xi_2 \xi_2^2$$

with

$$\begin{aligned} \xi_1(\varepsilon(t,\cdot)) &= \int_0^L \varepsilon(t,\zeta)\,\mathrm{d}\zeta \\ \xi_2(\varepsilon(t,\cdot), p(t,\cdot)) &= \int_0^L (D(L-\zeta)\varepsilon(t,\zeta) + p(t,\zeta))\,\mathrm{d}\zeta \end{aligned} \tag{22}$$

Leading to

$$u = -\begin{pmatrix} \Xi_2 & 0 \\ 0 & \Xi_1 \end{pmatrix} \begin{pmatrix} \int_0^L (D(L-\zeta)\varepsilon(t,\zeta) + p(t,\zeta))\,\mathrm{d}\zeta \\ \int_0^L \varepsilon\,\mathrm{d}\zeta \end{pmatrix}$$

Achievable performances

We consider now that $D = 0$, all parameters equal 1 (simulations are provided considering a finite volume approximation)

$$u(t) = \begin{pmatrix} v(t,0) \\ \sigma(t,L) \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{u}(t) \end{pmatrix} \qquad y(t) = \begin{pmatrix} -\sigma(t,0) \\ v(t,L) \end{pmatrix} = \begin{pmatrix} \tilde{y}(t) \\ \bar{y}(t) \end{pmatrix}$$

and we plot the position at the end point of the system.

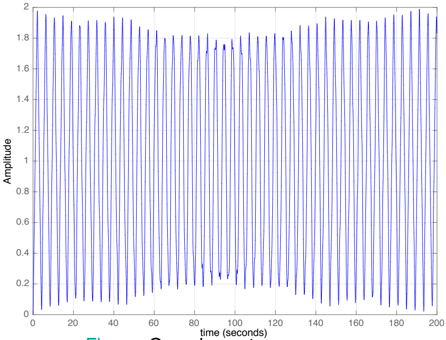


Figure: Open loop step response.

Simulation

We first consider the static feedback case *i.e.* when pure dissipation is added at the boundary:

$$u_2 = -k_d y_2$$

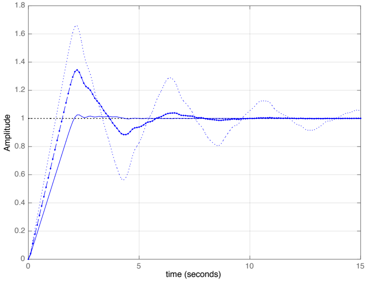


Figure: Step response of the closed loop system with pure dissipation term.

Simulation

In a second instance we consider the control law devoted to energy shaping in addition to a pure dissipation term:

$$u = -k_c\left(x_{22} - x_{01}\right) - k_d\dot{x}_{22}$$

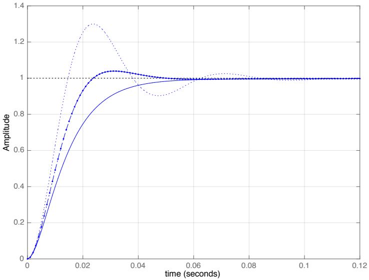


Figure: Step response of the closed loop system with state feedback.

Conclusion and future work

- A large class of boundary control systems are asymptotically (exponentially) stable if they are interconnected in a power preserving manner with an (input strictly passive and) exponentially stable finite dimensional linear controller.
- Stability established for static control of BCS has been extended to the case of dynamic boundary control.
- These results can be used for control design.

Conclusion and future work

- A large class of boundary control systems are asymptotically (exponentially) stable if they are interconnected in a power preserving manner with an (input strictly passive and) exponentially stable finite dimensional linear controller.
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Ongoing and future work

- Generalization to 2D and 3D systems.
- Extension to non-linear PDEs
- Constructive methods for control design.

Thank you for your attention !