Tutorial 2

1. Consider the following p.d.e.

$$\frac{\partial w}{\partial t}(\zeta,t) = c\frac{\partial w}{\partial \zeta}(\zeta,t), \qquad \zeta \in [0,\ell], \quad t \geq 0, \tag{1}$$

   with boundary conditions

$$w(0,t) = 0 \qquad t \geq 0. \tag{2}$$

   We take as units for t, ζ and w, seconds, meter, and gram, respectively.

   (a) Determine the dimension of c.

   (b) Write the p.d.e. (1) with boundary condition (2) as the abstract differential equation $\dot{x}(t) = Ax(t)$. Determine A and its domain $D(A)$.
   As state space you may take $X = L^2(0,\ell)$.

   (c) We assume that $c < 0$. Show that your abstract differential equation possesses a unique solution for every initial condition $w_0(\zeta) \in L^2(0,\ell)$. The solution is contractive, i.e., $\|x(t)\| \leq \|w_0\|$.

   (d) In the previous item you had to solve $(A - I)z = f$. Comment on the units.

   (e) Show that $w(\zeta,t) = f(\zeta + ct)$ satisfies (1). Here f is a smooth function.
   May we add ζ and ct?

   (f) How must we choose f such that the initial and the boundary condition are satisfied?

   (g) What will be your weak solution?

2. Show that for a pH system the following power balance equation holds:

$$\dot{H}(t) = \frac{1}{2}\left[(\mathcal{H}x)^T(\zeta,t)P_1(\mathcal{H}x)(\zeta,t)\right]_a^b$$

3. Consider once more the p.d.e. (1). We add now the following boundary condition

$$\alpha w(0,t) + \beta w(\ell,t) = 0 \qquad t \geq 0 \tag{3}$$

   with $\alpha, \beta \in \mathbb{R}$.

   (a) Show that (1), (3) is a port-Hamiltonian system.

(b) Determine all pairs $(\alpha, \beta) \in \mathbb{R}^2$ for which there exists a unique weak solution not increasing in the energy.

(c) Determine all pairs $(\alpha, \beta) \in \mathbb{R}^2$ for which there exists a unique weak solution with constant energy.

4. We consider the model of the vibrating string and study the d'Alembert solution. Not to complicate things further, we take $\zeta \in \mathbb{R}$ and so there are no boundary conditions. Thus we consider

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = c^2 \frac{\partial^2 w}{\partial \zeta^2}(\zeta, t), \qquad \zeta \in \mathbb{R}, \quad t \geq 0, \tag{4}$$

(a) Show that

$$w(\zeta, t) = f(\zeta + ct) + g(\zeta - ct) \tag{5}$$

is a solution of (4). Here f and g are arbitrary smooth functions.

(b) How are f and g related to the initial conditions, $w(\zeta, 0)$ and $\frac{\partial w}{\partial t}(\zeta, 0)$?

(c) Assume that $\frac{\partial w}{\partial t}(\zeta, 0) = 0$ for all ζ and $w(\zeta, 0) = \cos(\zeta)$ for $\zeta \in [-\pi/2, \pi/2]$ and zero elsewhere. Sketch the solution (5) for a $t > 0$.