

Modelling and Control of Distributed Parameter Systems: The port-Hamiltonian Approach

Stability

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Introduction

As before, we write our p.d.e. as the abstract differential equation on the state space X

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0.$$

We assume that this equation has a unique weak solution for every initial condition $x_0 \in X$.

Definition The system $\dot{x}(t) = Ax(t)$ is exponentially stable, if there exists a $M \geq 1$, $\omega < 0$ such that for all $x_0 \in X$ the following holds

$$\|x(t)\| \leq Me^{\omega t} \|x_0\|, \quad t \geq 0.$$

Exponential stability for pH-systems

We return to our homogeneous pH system. That is, we consider

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} [\mathcal{H}(\zeta)x(\zeta, t)] + P_0 [\mathcal{H}(\zeta)x(\zeta, t)] \quad (1)$$

with the boundary condition

$$W_B \begin{bmatrix} (\mathcal{H}x)(b, t) \\ (\mathcal{H}x)(a, t) \end{bmatrix} = 0, \quad (2)$$

As before/always we assume that the following hold:

- ▶ P_1 is an invertible, symmetric real $n \times n$ matrix;
- ▶ P_0 is an anti-symmetric real $n \times n$ matrix;
- ▶ For all $\zeta \in [a, b]$ the $n \times n$ matrix $\mathcal{H}(\zeta)$ is real, symmetric, and $mI \leq \mathcal{H}(\zeta) \leq MI$, for some $M, m > 0$ independent of ζ ;
- ▶ W_B be a full rank real matrix of size $n \times 2n$.

Exponential stability for pH-systems

Theorem Consider the operator A associated with (1) and (2). Furthermore, we assume that next to the standard conditions the following is satisfied;

- ▶ \mathcal{H} is continuously differentiable on the interval $[a, b]$.

Then, if for some positive constant k one of the following conditions is satisfied for all $x_0 \in D(A)$

$$\langle Ax_0, x_0 \rangle_X + \langle x_0, Ax_0 \rangle_X \leq -k \|(\mathcal{H}x_0)(b)\|^2 \quad (3)$$

$$\langle Ax_0, x_0 \rangle_X + \langle x_0, Ax_0 \rangle_X \leq -k \|(\mathcal{H}x_0)(a)\|^2, \quad (4)$$

then the system is exponentially stable. □

Exponential stability for pH-systems

Since

$$\dot{H}(t)|_{t=0} = \langle Ax_0, x_0 \rangle_X + \langle x_0, Ax_0 \rangle_X,$$

we can formulate the condition

$$\langle Ax_0, x_0 \rangle_X + \langle x_0, Ax_0 \rangle_X \leq -k \|(\mathcal{H}x_0)(b)\|^2$$

equivalently as

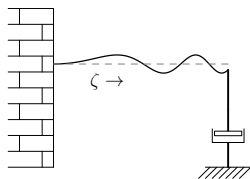
$$\dot{H}(t)|_{t=0} \leq -k \|(\mathcal{H}x_0)(b)\|^2,$$

for all initial conditions $x_0 \in D(A)$.

Similarly, the condition at $\zeta = a$.

Exponential stability for pH systems

Example: Damped wave equation



$$\begin{aligned}\frac{\partial^2 w}{\partial t^2}(\zeta, t) &= \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left[T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) \right] \\ \frac{\partial w}{\partial t}(1, t) &= -\alpha \cdot T(1) \frac{\partial w}{\partial \zeta}(1, t), \\ 0 &= \frac{\partial w}{\partial t}(0, t)\end{aligned}$$

Here is α a positive constant. We assume that ρ and T are continuous differentiable.

We check the contraction property. The condition on P 's, \mathcal{H} , and number of boundary conditions are satisfied (check) and so we calculate the power balance (check)

$$\dot{H}(t) = -\alpha \left[T(1) \frac{\partial w}{\partial \zeta}(1, t) \right]^2 \leq 0.$$

So we have a contractive solution for every initial condition in X .

Exponential stability for pH systems

To conclude exponential stability, we need that

$$\dot{H}(t) \leq -k \|(\mathcal{H}x)(1, t)\|^2 = -k \left[\left(T(1) \frac{\partial w}{\partial \zeta}(1, t) \right)^2 + \frac{\partial w}{\partial t}(1, t)^2 \right]$$

Since we have that $\frac{\partial w}{\partial t}(1, t) = -\alpha \cdot T(1) \frac{\partial w}{\partial \zeta}(1, t)$, we find that

$$\begin{aligned} \left(T(1) \frac{\partial w}{\partial \zeta}(1, t) \right)^2 + \frac{\partial w}{\partial t}(1, t)^2 &= \left(T(1) \frac{\partial w}{\partial \zeta}(1, t) \right)^2 + \alpha^2 \left(T(1) \frac{\partial w}{\partial \zeta}(1, t) \right)^2 \\ &= (1 + \alpha^2) \left(T(1) \frac{\partial w}{\partial \zeta}(1, t) \right)^2. \end{aligned}$$

Combining this with the result on the previous slide, gives

Exponential stability for pH systems

$$\begin{aligned}\dot{H}(t) &= -\alpha \left[T(1) \frac{\partial w}{\partial \zeta}(1, t) \right]^2 \\ &= -\alpha \cdot \frac{1}{1 + \alpha^2} \left[\left(T(1) \frac{\partial w}{\partial \zeta}(1, t) \right)^2 + \frac{\partial w}{\partial \zeta}(1, t)^2 \right] \\ &= \frac{-\alpha}{1 + \alpha^2} \|(\mathcal{H}x)(1, t)\|^2.\end{aligned}$$

Thus we can conclude exponential stability.

