Modelling and Control of Distributed Parameter Systems: The port-Hamiltonian Approach

Inputs and Outputs

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Introduction

Next we formulate and study partial differential equations with a control and observation term.
Before we do so, we first reconsider the finite-dimensional case.
Let the ordinary differential equation be given

$$\ddot{y}(t) + 4\dot{y}(t) + 8y(t) = -3u(t),$$

where u is the input, and y is the output of this system.
With the state $x(t) = \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix}$ this ODE can be written in the state space form

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -8 & -4 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ -3 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t),$$

Introduction

We can rewrite the o.d.e. $\ddot{y}(t) + 4\dot{y}(t) + 8y(t) = -3u(t)$, as

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t).$$

It is well-known that this inhomogeneous state-space equation possesses the unique solution given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$
$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau,$$

where x_0 is the initial condition.
Question: Can we do the same for p.d.e's?

Inputs

Similar as writing a homogeneous PDE into an abstract differential equation, we can write an inhomogeneous PDE into an abstract differential equation with an input term. We show this in an example first.

Inputs, example

Example Consider the controlled p.d.e. on the spatial interval $[0, 1]$ in with constants $c > 0$ and $b_0 \neq 0$

$$\begin{aligned} \frac{\partial w}{\partial t}(\zeta, t) &= c\frac{\partial w}{\partial \zeta}(\zeta, t) + b_0 u(t) \\ w(1, t) &= 0. \end{aligned}$$

For $u = 0$, we have seen in that the p.d.e. can be written on the state space $X = L^2(0, 1)$ as $\dot{x}(t) = Ax(t)$ with

$$\begin{aligned} Ax &= c\frac{dx}{d\zeta}, \\ D(A) &= \left\{ x \in L^2(0, 1) \mid \frac{dx}{d\zeta} \in L^2(0, 1) \text{ and } x(1) = 0 \right\}. \end{aligned}$$

Inputs

So

$$\underbrace{\frac{\partial w}{\partial t}(\zeta,t)}_{\dot{x}(t)} = \underbrace{c\frac{\partial w}{\partial \zeta}(\zeta,t)}_{Ax(t)} + \beta u(t)$$

$$\dot{x}(t) = Ax(t) \quad + Bu(t)$$

As for the transformation from o.d.e.'s to state space equations, the input is added to the right hand-side.
So Bu is defined as

$$(Bu)(\zeta) = b_0\, \mathbb{1}(\zeta) \cdot u,$$

where $\mathbb{1}(\zeta)$ is the function identically equal to one.
So B is the mapping, which maps the scalar u to the function $b_0\, \mathbb{1}(\zeta) \cdot u$ (u times the constant-b_0 function).

Intermezzo

We already defined the class of bounded, linear operators from the Hilbert space X to X. This set we denoted by $\mathcal{L}(X)$.
Defnition Let Z and W be Hilbert spaces, we define Q to be a bounded, linear operator from Z to W if

- Linear: $Q(\alpha z_1 + \beta z_2) = \alpha Q z_1 + \beta Q z_2$ for all $z_1, z_2 \in Z$, $\alpha, \beta \in \mathbb{R}$, and
- Bounded: There exists a $q \geq 0$ such that for all $z \in Z$

$$\|Qz\| \leq q\|z\|.$$

Note that the first norm is the norm of W, whereas the second norm is that of Z.
The set of all bounded, linear operators from Z to W is denoted by $\mathcal{L}(Z, W)$. □

Bounded, linear B

Since the constant functions are elements of the state space X, we see that B maps the scalar $u \in \mathbb{R} = U$ (the input values) into the state space.

To see if B is in $\mathcal{L}(U, X)$ we have to check two conditions.

- Linear: For all $u_1, u_2 \in U$, $\alpha, \beta \in \mathbb{R}$, there holds

$$\begin{aligned} B(\alpha u_1 + \beta u_2) &= b_0 \, \mathbb{1}(\zeta) \cdot (\alpha u_1 + \beta u_2) \\ &= \alpha b_0 \, \mathbb{1}(\zeta) \cdot u_1 + \beta b_0 \, \mathbb{1}(\zeta) \cdot u_2 = \alpha B u_1 + \beta B u_2. \end{aligned}$$

- Bounded:

$$\|Bu\|^2 = \int_0^1 [b_0 \, \mathbb{1}(\zeta) \cdot u]^2 d\zeta = b_0^2 \cdot u^2.$$

So $B \in \mathcal{L}(U, X)$ □

Bounded inputs operators

A useful result is the following; If $U = \mathbb{R}^m$ and Bu can be written as

$$Bu = \sum_{j=1}^{m} b_j u_j$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

and $b_j \in X$ for $j = 1, \cdots, m$, then $B \in \mathcal{L}(U, X)$.

Second example, inputs

Before we study existence of weak and classical solutions for the inhomogeneous equation, we treat another example first.
Example Consider the freely hanging string with two distributed forces working on it

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2}(\zeta,t) &= c^2 \frac{\partial^2 w}{\partial \zeta^2}(\zeta,t) + \mathbb{1}_{[\frac{1}{8},\frac{3}{8}]}(\zeta)u_1(t) + \mathbb{1}_{[\frac{4}{7},\frac{6}{7}]}(\zeta)u_2(t), \\ \frac{\partial w}{\partial \zeta}(0,t) &= 0 = \frac{\partial w}{\partial \zeta}(1,t), \qquad t \geq 0, \\ w(\zeta,0) &= w_0(\zeta), \quad \frac{\partial w}{\partial t}(\zeta,0) = w_1(\zeta). \end{aligned}$$

Here by $\mathbb{1}_{[a,b]}$ we mean the function which is identically one when $\zeta \in [a,b]$ and zero elsewhere. So we can put a force onto the string at two places. Namely, uniformly in the interval $[\frac{1}{8}, \frac{3}{8}]$ and uniformly in the interval $[\frac{4}{7}, \frac{6}{7}]$. This can be done independently of each other.

Second example, inputs

This is a port-Hamiltonian system with constant coefficients. As state we choose

$$x(t) = \begin{bmatrix} \frac{\partial w}{\partial t}(\cdot,t) \\ \frac{\partial w}{\partial \zeta}(\cdot,t) \end{bmatrix}.$$

$$\begin{aligned}
\dot{x}(t) &= \frac{\partial}{\partial t} \begin{bmatrix} \frac{\partial w}{\partial t}(\cdot,t) \\ \frac{\partial w}{\partial \zeta}(\cdot,t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial^2 w}{\partial t^2}(\cdot,t) \\ \frac{\partial^2 w}{\partial t \partial \zeta}(\cdot,t) \end{bmatrix} = \begin{bmatrix} c^2 \frac{\partial^2 w}{\partial \zeta^2}(\cdot,t) \\ \frac{\partial^2 w}{\partial \zeta \partial t}(\cdot,t) \end{bmatrix} + \begin{bmatrix} \mathbb{1}_{[\frac{1}{8},\frac{3}{8}]}(\cdot) u_1(t) + \mathbb{1}_{[\frac{4}{7},\frac{6}{7}]}(\cdot) u_2(t) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} 1 & 0 \\ 0 & c^2 \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial t}(\cdot,t) \\ \frac{\partial w}{\partial \zeta}(\cdot,t) \end{bmatrix} \right) + \\
&\qquad \begin{bmatrix} \mathbb{1}_{[\frac{1}{8},\frac{3}{8}]}(\cdot) u_1(t) + \mathbb{1}_{[\frac{4}{7},\frac{6}{7}]}(\cdot) u_2(t) \\ 0 \end{bmatrix}.
\end{aligned}$$

Second example, inputs

So we can write the p.d.e. with inputs as:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} 1 & 0 \\ 0 & c^2 \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial t}(\cdot, t) \\ \frac{\partial w}{\partial \zeta}(\cdot, t) \end{bmatrix} \right) + \begin{bmatrix} \mathbb{1}_{[\frac{1}{8}, \frac{3}{8}]}(\cdot) u_1(t) + \mathbb{1}_{[\frac{4}{7}, \frac{6}{7}]}(\cdot) u_2(t) \\ 0 \end{bmatrix}.$$

$$= Ax(t) + ??$$

with

$$Ax = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} 1 & 0 \\ 0 & c^2 \end{bmatrix} \right) x$$

$$D(A) = \{x \in L^2((0,1); \mathbb{R}^2) \mid x \in H^1((0,1); \mathbb{R}^2), x_2(1) = 0 = x_2(0)\}.$$

We have two inputs. So if we define

Second example, inputs

$$(Bu)(\zeta) = b_1(\zeta)u_1 + b_2(\zeta)u_2,$$

with $u = \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right] \in \mathbb{R}^2$, and

$$b_1(\zeta) = \begin{bmatrix} \mathbb{1}_{[\frac{1}{8},\frac{3}{8}]}(\zeta) \\ 0 \end{bmatrix}, \quad b_2(\zeta) = \begin{bmatrix} \mathbb{1}_{[\frac{4}{7},\frac{6}{7}]}(\zeta) \\ 0 \end{bmatrix}.$$

Then

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} 1 & 0 \\ 0 & c^2 \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial t}(\cdot, t) \\ \frac{\partial w}{\partial \zeta}(\cdot, t) \end{bmatrix} \right) + \\ &\qquad \begin{bmatrix} \mathbb{1}_{[\frac{1}{8},\frac{3}{8}]}(\cdot)u_1(t) + \mathbb{1}_{[\frac{4}{7},\frac{6}{7}]}(\cdot)u_2(t) \\ 0 \end{bmatrix}. \\ &= Ax(t) + Bu(t). \end{aligned}$$

Since $b_1, b_2 \in X = L^2((0,1);\mathbb{R}^2)$, we have that $B \in \mathcal{L}(\mathbb{R}^2; X)$. □

Existence of solutions

The following theorem shows that if we have existence and uniqueness of solutions when $u = 0$, i.e, the homogeneous situation, then that implies existence and uniqueness for a very large class of inputs.
By $L^1_{\mathrm{loc}}([0,\infty);U)$ we denote the set of all functions from $[0,\infty)$ to U which satisfy $\int_0^{t_1} \|u(t)\| dt < \infty$ for all $t_1 > 0$. Finally, by $C^1([0,\infty);U)$ we denote the set of continuously differentiable functions from $[0,\infty)$ to U.

Existence of solutions

Theorem Consider on the state space X the inhomogeneous abstract differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0. \tag{1}$$

Assume that the following holds;

- The homogeneous equation $\dot{x}(t) = Ax(t), x(0) = x_0$ has for every $x_0 \in X$ a unique weak solution in X;
- For the input operator there holds $B \in \mathcal{L}(U, X)$.

Under these conditions the inhomogeneous equation (1) has for every $x_0 \in X$ and every $u \in L^1_{\mathrm{loc}}([0, \infty); U)$ a unique weak solution.
Furthermore, when $u \in C^1([0, \infty); U)$ and $x_0 \in D(A)$, then this weak solution is the unique classical solution of (1).

Remarks

The weak solution is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds$$

with $T(t)$ the C_0-semigroup associated to $A, D(A)$.
In the examples in this part we applied a control within the spatial domain. However, we could have applied a control at the boundary. When doing so, we cannot rewrite this system in our standard form $\dot{x}(t) = Ax(t) + Bu(t)$.
This is general the case when controlling a p.d.e. via its boundary. Thus systems with control at the boundary form a new class of systems, and are introduced later. We first add outputs to the input-state equation treated in this section.

Outputs

In the previous part we have added an input function to our system. Now additionally an output is added. As often, we begin with an example. Therefore we take our vibrating string example and add a measurement.

$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = c^2 \frac{\partial^2 w}{\partial \zeta^2}(\zeta,t) + \mathbb{1}_{[\frac{1}{8},\frac{3}{8}]}(\zeta)u_1(t) + \mathbb{1}_{[\frac{4}{7},\frac{6}{7}]}(\zeta)u_2(t),$$

$$\frac{\partial w}{\partial \zeta}(0,t) = 0 = \frac{\partial w}{\partial \zeta}(1,t), \qquad t \geq 0,$$

$$w(\zeta,0) = w_0(\zeta), \frac{\partial w}{\partial t}(\zeta,t) = w_1(\zeta)$$

$$y(t) = \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{\partial w}{\partial t}(\zeta,t)d\zeta.$$

So we can apply a force on the string at two places and we measure the average velocity on the interval $[\frac{1}{3},\frac{2}{3}]$.

Outputs, example

In this system the input space is $\mathbb{R}^2$ and the state space X equals $L^2((0,1);\mathbb{R}^2)$ and the state is

$$x(t) = \begin{bmatrix} \frac{\partial w}{\partial t}(\cdot, t) \\ \frac{\partial w}{\partial \zeta}(\cdot, t) \end{bmatrix}.$$

The state space has the inner product

$$\langle f, g \rangle = \int_0^1 f_1(\zeta) g_1(\zeta) + c^2 f_2(\zeta) g_2(\zeta) d\zeta$$

Outputs, example

Thus

$$\langle f, x(t) \rangle = \int_0^1 f_1(\zeta) \frac{\partial w}{\partial t}(\zeta, t) + c^2 f_2(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) d\zeta$$

So

$$y(t) = \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{\partial w}{\partial t}(\zeta, t) d\zeta = \langle \begin{bmatrix} \mathbb{1}_{[\frac{1}{3}, \frac{2}{3}]}(\cdot) \\ 0 \end{bmatrix}, x(t) \rangle =: Cx(t).$$

From this it follows easily that $C \in \mathcal{L}(X, \mathbb{R})$.
If the weak solution exists of the state-differential equation $\dot{x}(t) = Ax(t) + Bu(t)$, then $x(t) \in X$ for every $t \geq 0$, and thus the output is well-defined.

Theorem

Consider on the state space X, input space U and output space Y the abstract system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \tag{2}$$
$$y(t) = Cx(t) + Du(t). \tag{3}$$

Assume that the following holds;

- The equation $\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0$ has for the given $x_0 \in X$ and input function $u(t) \in L^1_{\text{loc}}((0, \infty); U)$ a unique weak solution in X;
- The output operator C is in $\mathcal{L}(X, Y)$
- The feedthrough operator D is in $\mathcal{L}(U, Y)$

Under these conditions the output equation (3) is well-defined.

Theorem, continued

The solution is given as

$$y(t) = CT(t)x_0 + \int_0^t CT(t-s)Bu(s)ds + Du(s).$$

If $D = 0$, then $y(t)$ is a continuous function. $\square$

Boundary control systems

We now consider p.d.e's with control and observation at the boundary.
We first explain the idea by means of the controlled transport equation.
Consider the following system

$$\begin{aligned} \frac{\partial w}{\partial t}(\zeta,t) &= c\frac{\partial w}{\partial \zeta}(\zeta,t), && \zeta \in [0,1],\ t \geq 0 \\ w(\zeta,0) &= x_0(\zeta), && \zeta \in [0,1] \\ w(1,t) &= u(t), && t \geq 0. \end{aligned}$$

with an input $u \in L^1_{\mathrm{loc}}(0,\infty)$ and $c > 0$.
This cannot be written as $\dot{x}(t) = Ax(t) + Bu(t)$ with a bounded B.

Boundary control systems, example

Let $u(t)$ be smooth and let $x(\cdot, t)$ be a classical solution of

$$\frac{\partial w}{\partial t}(\zeta, t) = c\frac{\partial w}{\partial \zeta}(\zeta, t), \quad x(1, t) = u(t).$$

For $v(\cdot, t)$ defined as

$$v(\zeta, t) = w(\zeta, t) - u(t),$$

we obtain the following p.d.e.

$$\begin{aligned} \frac{\partial v}{\partial t}(\zeta, t) &= c\frac{\partial v}{\partial \zeta}(\zeta, t) - \dot{u}(t), \qquad \zeta \in [0, 1],\ t \geq 0 \\ v(1, t) &= 0, \qquad t \geq 0. \end{aligned}$$

We have seen this p.d.e. (for v) can be written in the standard form

$$\dot{v}(t) = Av(t) + B\tilde{u}(t)$$

for $\tilde{u} = \dot{u}$. We know the existence of solutions of this one.

Boundary control systems, example

Now we make this more abstract.
We take as state space $X = L^2(0,1)$, and introduce the "almost A-operator"

$$\mathfrak{A}f = c\frac{df}{d\zeta},$$

with domain $D(\mathfrak{A}) = \{f \in X \mid \dot{f} \in X\}$. Furthermore, we define

$$\mathfrak{B}f = f(1).$$

with $D(\mathfrak{B}) = D(\mathfrak{A})$.
With this the boundary control p.d.e. is formulated as

$$\begin{aligned}\dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0,\\ \mathfrak{B}x(t) &= u(t).\end{aligned}$$

Boundary control systems, definition

Definition The abstract system

$$\dot{x}(t) = \mathfrak{A}x(t), \qquad x(0) = x_0,$$
$$\mathfrak{B}x(t) = u(t).$$

with $\mathfrak{A} : D(\mathfrak{A}) \subset X \mapsto X$, $u(t) \in U$, and $\mathfrak{B} : D(\mathfrak{A}) \subset X \mapsto U$ is a boundary control system if the following holds:

a. The abstract differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

has for all $x_0 \in X$ a unique weak solution in X. Here A is defined as the operator $A : D(A) \mapsto X$ with $D(A) = D(\mathfrak{A}) \cap \ker(\mathfrak{B})$

$$Ax = \mathfrak{A}x \qquad \text{for } x \in D(A).$$

b. There exists a $B \in \mathcal{L}(U, X)$ such that $Bu \in D(\mathfrak{A})$ for all $u \in U$ and

$$\mathfrak{B}Bu = u, \qquad u \in U.$$

Boundary control systems, comments

Part b. of the definition is equivalent to the fact that the range of the operator $\mathfrak{B}$ equals U. So it allows us to choose every value in U for $u(t)$. In other words, the values of inputs are not restricted, which is a logical condition for inputs.
Part a. of the definition guarantees that the system possesses a unique solution when the input term is absent, i.e., when the input is identically zero. In other words, the homogeneous equation is well-posed. This is also a logical condition, since we would like that the trivial input $(u = 0)$ is possible.

Boundary control systems, solutions

Definition We say that the function $x(t)$ is a classical solution of the boundary control system

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0, \\ \mathfrak{B}x(t) &= u(t). \end{aligned}$$

if $x(t)$ is a continuously differentiable function, $x(t) \in D(\mathfrak{A})$ for all t, and $x(t)$ satisfies the equations for all t. $\square$

For a general boundary control system, we can apply a similar trick as the one applied in the example. This is the subject of the following theorem.

Boundary control systems, Theorem

Theorem Consider the boundary control system

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0, \\ \mathfrak{B}x(t) &= u(t), \end{aligned} \tag{4}$$

satisfying the conditions of the Definition and the abstract Cauchy equation

$$\begin{aligned} \dot{v}(t) &= Av(t) - B\dot{u}(t) + \mathfrak{A}Bu(t), \\ v(0) &= v_0. \end{aligned} \tag{5}$$

Assume that $u \in C^2([0,\infty);U)$. If $v_0 = x_0 - Bu(0) \in D(A)$, then the classical solutions of (4) and (5) are related by

$$v(t) = x(t) - Bu(t).$$

Furthermore, the classical solution of (4) is unique. □

Boundary control systems, Remark

Hence by applying a simple trick, we can reformulate a p.d.e. with boundary control into a p.d.e. with internal control. The price we have to pay is that u has to be smooth. So in particular, not an arbitrary function in L^1, but a more smooth function. Namely, it should have its derivative in L^1.

The proof is quite insightful.

Boundary control systems, Proof

Suppose that $v(t)$ is a classical solution of (5). Then $v(t) \in D(A) \subset D(\mathfrak{A})$, $Bu(t) \in D(\mathfrak{B})$, and so

$$\mathfrak{B}x(t) = \mathfrak{B}[v(t) + Bu(t)] = \mathfrak{B}v(t) + \mathfrak{B}Bu(t) = u(t),$$

where we have used that $v(t) \in D(A) \subset \ker \mathfrak{B}$ and equation $\mathfrak{B}Bu = u$. Furthermore, we have

$$\begin{aligned} \dot{x}(t) &= \dot{v}(t) + B\dot{u}(t) \\ &= Av(t) - B\dot{u}(t) + \mathfrak{A}Bu(t) + B\dot{u}(t) \qquad \text{by (5)} \\ &= Av(t) + \mathfrak{A}Bu(t) \\ &= \mathfrak{A}(v(t) + Bu(t)) = \mathfrak{A}x(t). \end{aligned}$$

Thus, if $v(t)$ is a classical solution of (5), then $x(t) = v(t) + Bu(t)$ is a classical solution of (4).
The other implication is proved similarly. The uniqueness of the classical solutions of (4) follows from the uniqueness of the classical solutions of (5).

Boundary control systems, boundary outputs

If for a boundary control system the output is given by

$$y(t) = \mathfrak{C}x(t).$$

with $\mathfrak{C} : D(\mathfrak{A}) \mapsto Y$, then this output is well-defined for classical solutions.

Boundary inputs and outputs, example

Example Consider the system

$$\frac{\partial w}{\partial t}(\zeta,t) = c\frac{\partial w}{\partial \zeta}(\zeta,t), \qquad \zeta \in [0,1],\ t \geq 0$$
$$u(t) = w(1,t), \qquad t \geq 0,$$

with $c > 0$. Now we add the output equation

$$y(t) = w(0,t).$$

We can write this in the form $y(t) = \mathfrak{C}x(t)$ with

$$\mathfrak{C}f = f(0).$$

Since this is well-defined (and linear) on $D(\mathfrak{A}) = \{f \in L^2(0,1) \mid f \in H^1(0,1)\}$, our previous results give that the above system has well-defined (classical) solutions. □

Boundary control pH-systems

The port-Hamiltonian system with control and observation is given by

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= P_1 \frac{\partial}{\partial \zeta}[\mathcal{H}(\zeta)x(\zeta,t)] + P_0[\mathcal{H}x(\zeta,t)] \\ u(t) &= W_{B,1} \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix} \\ 0 &= W_{B,2} \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix} \\ y(t) &= W_C \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix}. \end{aligned}$$

with $P_1^T = P_1$, invertible, $P_0^T = -P_0$, $\mathcal{H}(\zeta)^T = \mathcal{H}(\zeta)$, $0 < mI \leq \mathcal{H}(\zeta) \leq MI$. On $W_{B,1}, W_{B,2}, W_C$ we assume:

Boundary control pH-systems

- $W_{B,1}$ is a $m \times 2n$ matrix. Hence there are m controls.
- $W_B := \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix}$ is a full rank real matrix of size $n \times 2n$.
- W_C is a $k \times 2n$ matrix. Hence there are k outputs.
- The matrix $\begin{bmatrix} W_B \\ W_C \end{bmatrix}$ has rank $n + k$. Hence you don't measure quantities that are set to zero, or are inputs.

We will write this as a boundary control system.

Boundary control pH-systems

As discussed we choose the weighted L^2-space $X = L^2((a,b);\mathbb{R}^n)$ equipped with the inner product

$$\langle f, g\rangle_X := \frac{1}{2}\int_a^b f(\zeta)^T \mathcal{H}(\zeta) g(\zeta)\, d\zeta$$

as our state space.
The input space U equals $\mathbb{R}^m$, and the output space Y equals $\mathbb{R}^k$
We are now in the position to show that this controlled port-Hamiltonian system is indeed a boundary control system.

Boundary control pH-systems

We write the controlled pH system in the abstract form

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0, \\ \mathfrak{B}x(t) &= u(t), \\ y(t) &= \mathfrak{C}x(t), \end{aligned}$$

with

$$\begin{aligned} \mathfrak{A}x &= P_1 \frac{\partial}{\partial \zeta}[\mathcal{H}x] + P_0[\mathcal{H}x], \\ D(\mathfrak{A}) &= \Big\{ x \in L^2((a,b);\mathbb{R}^n) \mid \mathcal{H}x \in H^1((a,b);\mathbb{R}^n), \\ &\qquad W_{B,2} \begin{bmatrix} \mathcal{H}(b)x(b) \\ \mathcal{H}(a)x(a) \end{bmatrix} = 0 \Big\}, \\ \mathfrak{B}x &= W_{B,1} \begin{bmatrix} \mathcal{H}(b)x(b) \\ \mathcal{H}(a)x(a) \end{bmatrix}, \text{ and} \\ \mathfrak{C}x &= W_C \begin{bmatrix} \mathcal{H}(b)x(b) \\ \mathcal{H}(a)x(a) \end{bmatrix}. \end{aligned}$$

Boundary control pH-systems

So we have written the controlled port-Hamiltonian system in the language of a boundary control system. It remains to show that the conditions of our definition are satisfied.

Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be given on the previous slide. If

$$\langle \mathfrak{A}x, x\rangle_X + \langle x, \mathfrak{A}x\rangle_X \leq 0 \quad \text{for all } x \in D(\mathfrak{A}) \cap \ker \mathfrak{B},$$

then the pH system is a boundary control system on X.
Furthermore, for the input u identically zero, the energy of the solution, i.e. $\|x(t)\|_X^2 = H(t)$, will not increase.
Furthermore, for classical solutions of the boundary control problem, the output $y(t)$ is well-defined. □

Boundary control pH-systems, power balance

So for smooth controls and initial conditions, satisfying the boundary conditions, we know that solutions of the port-Hamiltonian system exist. Since the energy/Hamiltonian plays an important within this class of systems, it is useful to have a relation between the change of energy (power) and the external signals input and output. In many examples there exists such a relation. When we have n inputs and n outputs, a general formula can be derived expressing this relation.

Boundary control pH-systems, power balance

We assume that $W_B = W_{B,1}$ or equivalently $W_{B,2} = 0$.
Furthermore we assume that we have n measurements, and define

$$P_{W_B,W_C} = \begin{pmatrix} W_B^T & W_C^T \end{pmatrix}^{-1} \begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix} \begin{pmatrix} W_B \\ W_C \end{pmatrix}^{-1}.$$

Theorem Consider our input-output pH system with W_B and W_C full rank $n \times 2n$ matrices such that $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]$ is invertible.
If the $n \times n$ right-lower submatrix of P_{W_B,W_C} is non-positive, then for every $u \in C^2((0,\infty);\mathbb{R}^n)$, $\mathcal{H}x(0) \in H^1((a,b);\mathbb{R}^n)$, and $u(0) = W_B \left[\begin{smallmatrix} \mathcal{H}x(b,0) \\ \mathcal{H}x(a,0) \end{smallmatrix}\right]$, the system has a unique (classical) solution, with $\mathcal{H}x(t) \in H^1((a,b);\mathbb{R}^n)$. The output $y(\cdot)$ is continuous, and the following balance equation is satisfied:

$$\frac{d}{dt}\|x(t)\|_X^2 = \frac{1}{2}\begin{bmatrix} u^T(t) & y^T(t) \end{bmatrix} P_{W_B,W_C} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}.$$

Boundary control pH-systems, example

As an example we once more study the controlled transport equation.
Example We consider the system

$$\begin{aligned} \frac{\partial w}{\partial t}(\zeta,t) &= \frac{\partial w}{\partial \zeta}(\zeta,t), \qquad && \zeta \in [0,1],\ t \geq 0 \\ w(\zeta,0) &= x_0(\zeta), && \zeta \in [0,1]. \end{aligned}$$

This system can be written in the pH-form by choosing $n = 1$, $P_0 = 0$, $P_1 = 1$ and $\mathcal{H} = 1$.
Since $n = 1$, we can either apply one control. By using the boundary variables, the control is written as,

$$u(t) = \begin{bmatrix} \alpha & \beta \end{bmatrix} \begin{bmatrix} w(1,t) \\ w(0,t) \end{bmatrix}.$$

Note that $W_B = (\alpha, \beta)$ has full rank if and only if $\alpha^2 + \beta^2 \neq 0$.

Boundary control pH-systems, example

We have a boundary control system with contractive weak solutions for $u \equiv 0$ if $\alpha^2 \geq \beta^2$.
Now we add the output equation

$$y(t) = \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} w(1,t) \\ w(0,t) \end{bmatrix}.$$

Since $W_C = [c \ d\,]$ must have full rank, we find that $c^2 + d^2 \neq 0$.
Furthermore, since $\begin{bmatrix} W_B \\ W_C \end{bmatrix}$ must be invertible, we find that $\alpha d - \beta c \neq 0$.
The matrix P_{W_B,W_C} is given by

$$\begin{aligned} P_{W_B,W_C} = & \frac{1}{(\alpha d - \beta c)^2} \begin{bmatrix} d & -c \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} d & -\beta \\ -c & \alpha \end{bmatrix} \\ = & \frac{1}{(\alpha d - \beta c)^2} \begin{bmatrix} d^2 - c^2 & -d\beta + c\alpha \\ -d\beta + c\alpha & \beta^2 - \alpha^2 \end{bmatrix}. \end{aligned}$$

Boundary control pH-systems, example

For the particular choice $\alpha = 1, \beta = 0$ i.e. $u(t) = x(1,t)$ and $c = 0, d = 1$, that is $y(t) = x(0,t)$, we find $P_{W_B,W_C} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, or equivalently

$$\frac{d}{dt}\|x(t)\|_X^2 = \frac{1}{2}\left(u(t)^2 - y(t)^2\right).$$

Thanks