Modelling and Control of Distributed Parameter Systems: The port-Hamiltonian Approach

Transfer Functions

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Introduction

The aim of this part is to define transfer function for systems described by partial differential equations.
We derive these transfer functions via a very simple calculation.
For port-Hamiltonian systems we show that the energy/power balance induces properties on the transfer function.

Transfer function for an o.d.e.

Consider the simple system described by the ordinary differential equation

$$\dot{y}(t) + 5y(t) = 3u(t),$$

the transfer function of this system is given by

$$G(s) = \frac{3}{s+5}.$$

How do you come to this?

- Laplace transform, or
- Exponential solutions.

Exponential solutions

One way for obtaining the transfer function of

$$\dot{y}(t) + 5y(t) = 3u(t)$$

is to take $u(t) = e^{st}$, $s \in \mathbb{C}$, and to try to find a solution of the same format, i.e., $y(t) = \alpha e^{st}$. Substituting this in the differential equation, gives

$$s\alpha e^{st} + 5\alpha e^{st} = 3e^{st}.$$

Since e^{st} is non-zero, we may divide by it, and we find

$$s\alpha + 5\alpha = 3.$$

If $s \neq -5$, this is solvable;

$$\alpha = \frac{3}{s+5}.$$

Exponential solutions

So if we want to find an exponential solution

$$u(t) = e^{st}, \quad y(t) = \alpha e^{st},$$

of the o.d.e.

$$\dot{y}(t) + 5y(t) = 3u(t),$$

we find that:

- It is possible for all $s \in \mathbb{C}$ except for $s = -5$.
- The α equals
$$\alpha = \frac{3}{s+5}.$$
- We call this the transfer function at s.

Transfer function via exponential solutions

Definition

Given an (abstract) differential equation in the variables $(u(t), z(t), y(t))$, where $u(t)$, $z(t)$, and $y(t)$ take their values in the (Hilbert) spaces U, Z, and Y, respectively.
Let $s \in \mathbb{C}$. If for every $u_0 \in U$, there exists a unique solution of the form $(u_0 e^{st}, z_0 e^{st}, y_0 e^{st})$, and the mapping $u_0 \mapsto y_0$ is linear and bounded, then this mapping is called the transfer function at s, and will be denoted by $G(s)$. □

We call a solution of the form $(u_0 e^{st}, z_0 e^{st}, y_0 e^{st})$ an exponential solution.

Intermezzo

We already defined the class of bounded, linear operators from the Hilbert space X to X. This set we denoted by $\mathcal{L}(X)$.

Definition Let Z and W be Hilbert spaces, we define Q to be a bounded, linear operator from Z to W if

- Linear: $Q(\alpha z_1 + \beta z_2) = \alpha Q z_1 + \beta Q z_2$ for all $z_1, z_2 \in Z$, $\alpha, \beta \in \mathbb{R}$, and
- Bounded: There exists a $q \geq 0$ such that for all $z \in Z$

$$\|Qz\| \leq q\|z\|.$$

Note that the first norm is the norm of W, whereas the second norm is that of Z.

The set of all bounded, linear operators from Z to W is denoted by $\mathcal{L}(Z, W)$. □

Transfer function for state linear systems

Consider the abstract differential equation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

with B, C, and D bounded (linear) operators.
Let $s \in \mathbb{C}$, and $u_0 \in U$. We try to find a solution of the form $(u(t), x(t), y(t)) = (u_0 e^{st}, x_0 e^{st}, y_0 e^{st})$.
Substituting, this in the abstract differential equation gives

$$\begin{aligned}sx_0 e^{st} &= Ax_0 e^{st} + Bu_0 e^{st}\\ y_0 e^{st} &= Cx_0 e^{st} + Du_0 e^{st}.\end{aligned}$$

Since e^{st} is never zero, this is equivalent to:

$$\begin{aligned}(sI - A)z_0 &= Bu_0 \\ y_0 &= Cz_0 + Du_0.\end{aligned}$$

If $sI - A$ is (boundedly) invertible, then we find

$$y_0 = C(sI - A)^{-1}Bu_0 + Du_0.$$

This clearly defines a bounded linear mapping from u_0 to y_0, and so the transfer function at s is given by

$$G(s) = C(sI - A)^{-1}B + D.$$

This holds for all
$s \in \rho(A) := \{s \in \mathbb{C} \mid (sI - A)^{-1}$ exists as bounded operator$\}$.

Example

We take a vibrating string with no force at the boundary. We apply a force on it uniformly at one half, and we measure the average position in the other half;

$$\begin{aligned}
\frac{\partial^2 w}{\partial t^2}(\zeta,t) &= c^2\frac{\partial^2 w}{\partial \zeta^2}(\zeta,t) + \mathbb{1}_{[\frac{1}{2}L,L]}(\zeta)u(t) \\
\frac{\partial w}{\partial \zeta}(0,t) &= \frac{\partial w}{\partial \zeta}(L,t) = 0 \\
y(t) &= \int_0^{\frac{1}{2}L} w(\zeta,t)d\zeta.
\end{aligned}$$

To obtain the transfer function, we could follow two approaches.

Transfer function, Method 1.

The p.d.e. can be written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

with a certain x, A, B, and C.
Now we know that

$$G(s) = C(sI - A)^{-1}B.$$

So (among others) we must calculate $q := (sI - A)^{-1}B$. In other words, find $q \in D(A)$ such that $(sI - A)q = B$.
This will turn out to be (almost) the same as the equations which need to be solve in method 2.

Transfer function, Method 2.

We try to find an exponential solution of the p.d.e. This gives the following equations

$$\begin{aligned}
s^2 x_0(\zeta)e^{st} &= c^2 \frac{d^2 x_0}{d\zeta^2}(\zeta)e^{st} + \mathbb{1}_{[\frac{1}{2}L,L]}(\zeta)u_0 e^{st} \\
\frac{dx_0}{d\zeta}(0)e^{st} &= \frac{dx_0}{d\zeta}(L)e^{st} = 0 \\
y_0 e^{st} &= \int_0^{\frac{1}{2}L} x_0(\zeta)e^{st}d\zeta.
\end{aligned}$$

Transfer function, Method 2.

Hence

$$
\begin{aligned}
s^2 x_0(\zeta) &= c^2 \frac{d^2 x_0}{d\zeta^2}(\zeta) + \mathbb{1}_{[\frac{1}{2}L, L]}(\zeta) u_0 \\
\frac{dx_0}{d\zeta}(0) &= \frac{dx_0}{d\zeta}(L) = 0 \\
y_0 &= \int_0^{\frac{1}{2}L} x_0(\zeta) d\zeta.
\end{aligned}
$$

The first two lines represent an o.d.e. with boundary conditions.

Transfer function, Method 2.

The solution of

$$\begin{aligned} s^2 x_0(\zeta) &= c^2 \frac{d^2 x_0}{d\zeta^2}(\zeta) + \mathbb{1}_{[\frac{1}{2}L,L]}(\zeta) u_0 \\ \frac{dx_0}{d\zeta}(0) &= \frac{dx_0}{d\zeta}(L) = 0 \end{aligned}$$

is given as

$$\begin{aligned} x_0(\zeta) &= \cosh(\frac{s}{c}\zeta) x_0(0) - \\ & \frac{1}{sc} \int_0^\zeta \sinh(\frac{s}{c}(\zeta - \tau))\, \mathbb{1}_{[1/2L,L]}(\tau) u_0 d\tau \end{aligned}$$

with

$$x_0(0) = \frac{\sinh(\frac{s}{c}\frac{L}{2}) u_0}{s^2 \sinh(\frac{s}{c}L)} = \frac{u_0}{2s^2 \cosh(\frac{s}{c}\frac{L}{2})}.$$

Transfer function

Using this we find that

$$\begin{aligned} y_0 &= \int_0^{\frac{1}{2}L} x_0(\zeta)d\zeta \\ &= \frac{c\sinh(\frac{s}{c}\frac{L}{2})u_0}{2s^3\cosh(\frac{s}{c}\frac{L}{2})}. \end{aligned}$$

Hence the transfer function is given by

$$G(s) = \frac{c\tanh(\frac{s}{c}\frac{L}{2})}{2s^3}.$$

Transfer function, remark

If you write the solution of the o.d.e.

$$\begin{aligned} s^2 x_0(\zeta) &= c^2 \frac{d^2 x_0}{d\zeta^2}(\zeta) + \mathbb{1}_{[\frac{1}{2}L, L]}(\zeta) u_0 \\ \frac{dx_0}{d\zeta}(0) &= \frac{dx_0}{d\zeta}(L) = 0 \end{aligned}$$

as a Fourier cosine series, then you find another expression for the transfer function. Namely,

$$G(s) = \frac{L}{4s^2} - 2L \sum_{n=1}^{\infty} \frac{\sin(n\pi\frac{1}{2})^2}{n^2\pi^2(s^2L^2 + n^2\pi^2c^2)}.$$

However, the transfer function is unique, and so we find that

$$\frac{c\tanh(\frac{s}{c}\frac{L}{2})}{2s^3} = G(s) = \frac{L}{4s^2} - 2L \sum_{n=1}^{\infty} \frac{\sin(n\pi\frac{1}{2})^2}{n^2\pi^2(s^2L^2 + n^2\pi^2c^2)}.$$

□

Transfer functions

So we have seen that working with exponential solutions, directly on the p.d.e., works very well.
Note that it is (almost) the same as the engineering trick of replacing derivative with respect to time by an s.
We can do that for systems with control and observation at the boundary.

Transfer function, boundary control and observation

Example

Consider the system with boundary control and observation

$$\begin{aligned} \frac{\partial w}{\partial t}(\zeta, t) &= \frac{\partial w}{\partial \zeta}(\zeta, t) \\ w(1,t) &= u(t) \\ y(t) &= w(0,t). \end{aligned}$$

Substituting exponential functions for all signals, gives

$$\begin{aligned} s x_0(\zeta) e^{st} &= \frac{dx_0}{d\zeta}(\zeta) e^{st} \\ x_0(1) e^{st} &= u_0 e^{st} \\ y_0 e^{st} &= x_0(0) e^{st}. \end{aligned}$$

Thus

Example of transfer function with boundary control and observation

$$\begin{aligned} sx_0(\zeta) &= \frac{dx_0}{d\zeta}(\zeta) \\ x_0(1) &= u_0 \\ y_0 &= x_0(0). \end{aligned}$$

This is an ordinary differential equation with given (end) condition, u_0 and unknown (initial) condition, y_0.
The solution equals $x_0(\zeta) = e^{s(\zeta-1)}u_0$. Thus $y_0 = e^{-s}u_0$.
The transfer function equals

$$G(s) = e^{-s} \qquad s \in \mathbb{C}.$$

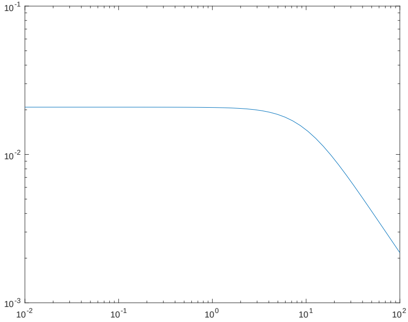
□

Bode and Nyquist plots

Similar like for rational function, we can draw the Bode and Nyquist plot of general transfer functions

For instance the Bode magnitude plot of

$$G(s) = \frac{\tanh(\sqrt{s}/2)}{2s\sqrt{s}} - \frac{1}{4s}$$



Transfer functions for pH systems

Consider the port-Hamiltonian system with input and outputs

$$\frac{\partial x}{\partial t}(\zeta,t) = \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)[\mathcal{H}(\zeta)x(\zeta,t)]$$
$$u(t) = W_{B,1}\begin{bmatrix}\mathcal{H}(b)x(b,t)\\ \mathcal{H}(a)x(a,t)\end{bmatrix}, \quad 0 = W_{B,2}\begin{bmatrix}\mathcal{H}(b)x(b,t)\\ \mathcal{H}(a)x(a,t)\end{bmatrix},$$
$$y(t) = W_C\begin{bmatrix}\mathcal{H}(b)x(b,t)\\ \mathcal{H}(a)x(a,t)\end{bmatrix}.$$

Assume that the energy balance can be expressed in the inputs and outputs. That is

$$\dot{H}(t) = \left[u(t)^\top, y(t)^\top\right] Q \begin{bmatrix}u(t)\\ y(t)\end{bmatrix}$$

with Q a symmetric matrix.

Transfer functions for pH systems

Since exponential solutions are solutions, the power balance

$$\dot{H}(t) = \left[u(t)^\top, y(t)^\top\right] Q \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$$

also holds for these.
Remark: Since the s in the exponential solution may be complex, we have to write the power balance for complex valued solutions.
The (complex) power balance equals

$$\dot{H}(t) = \left[u(t)^*, y(t)^*\right] Q \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}.$$

Transfer functions for pH systems

Hence for the exponential solution the power balance can be written as

$$
\begin{aligned}
\dot{H}(t) &= \left[u(t)^*, y(t)^*\right] Q \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \\
&= \left[u_0^* e^{\bar{s}t}, y_0^* e^{\bar{s}t},\right] Q \begin{bmatrix} u_0 e^{st} \\ y_0 e^{st} \end{bmatrix} \\
&= \left[u_0^*, y_0^*\right] Q \begin{bmatrix} u_0 \\ y_0 \end{bmatrix} e^{2\mathrm{Re}(s)t} \\
&= \left[u_0^*, u_0^* G(s)^*\right] Q \begin{bmatrix} u_0 \\ G(s) u_0 \end{bmatrix} e^{2\mathrm{Re}(s)t}.
\end{aligned}
$$

Transfer functions for pH systems

Since

$$H(t) = \|x(t)\|_X^2 = \langle x(t), x(t) \rangle_X,$$

we find for the exponential solution that

$$H(t) = \langle x_0 e^{st}, x_0 e^{st} \rangle_X = \langle x_0, x_0 \rangle_X e^{2\mathrm{Re}(s)t} = \|x_0\|_X^2 e^{2\mathrm{Re}(s)t}.$$

Combining the two results gives that

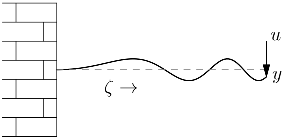
$$2\mathrm{Re}(s)\|x_0\|_X^2 e^{2\mathrm{Re}(s)t} = \dot{H}(t) = \left[u_0^*, u_0^* G(s)^*\right] Q \begin{bmatrix} u_0 \\ G(s)u_0 \end{bmatrix} e^{2\mathrm{Re}(s)t}.$$

Or equivalently:

$$2\mathrm{Re}(s)\|x_0\|_X^2 = \dot{H}(t) = \left[u_0^*, u_0^* G(s)^*\right] Q \begin{bmatrix} u_0 \\ G(s)u_0 \end{bmatrix}.$$

Transfer functions for pH systems

Example: Wave equation



$$\begin{aligned}
\frac{\partial^2 w}{\partial t^2}(\zeta,t) &= \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right] \\
u(t) &= T(1)\frac{\partial w}{\partial \zeta}(1,t), \quad 0 = \frac{\partial w}{\partial t}(0,t) \\
y(t) &= \frac{\partial w}{\partial t}(1,t)
\end{aligned}$$

To calculate the expression of the transfer function can be hard/impossible. However, the power balance equals

$$\dot{H}(t) = u(t)y(t) = \left[u(t)^*, y(t)^*\right] \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}.$$

Transfer function for the vibrating string system

From the general result we find

$$\begin{aligned} 2\mathrm{Re}(s)\|x_0\|_X^2 &= \left[u_0^*, u_0^* G(s)^*\right] \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ G(s)u_0 \end{bmatrix} \\ &= \mathrm{Re}(G(s))|u_0|^2. \end{aligned}$$

Since $\|x_0\|_X^2 \geq 0$ and $|u_0|^2 > 0$, we find that for $\mathrm{Re}(s) > 0$ there holds

$$\mathrm{Re}(G(s)) \geq 0$$

Thus G is positive real. □