

Modeling of Distributed Parameter Systems: The Port Hamiltonian Approach

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Outline

Context and motivation

Infinite dimensional Port Hamiltonian systems (PHS)

Control by interconnection and energy shaping

Observer design

Irreversible boundary controlled port Hamiltonian Systems

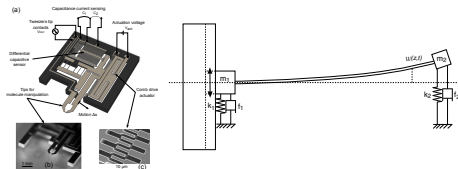
Conclusions and future works



Context : control of flexible structures

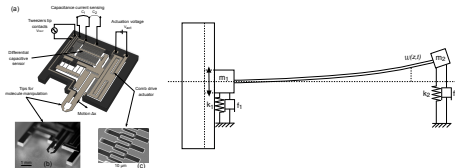


► Boundary controlled systems

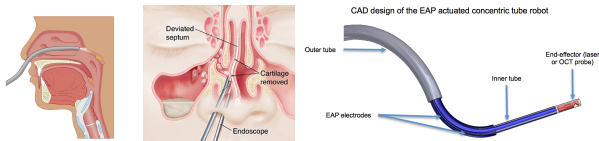


Context : control of flexible structures

► Boundary controlled systems



► In-domain control of distributed parameter systems



- Exploration, imaging, diagnosis.
- Mini invasive surgery.
- Toward miniaturized and *smart* endoscopes.

Context : port Hamiltonian systems



- ▶ Port Hamiltonian systems:
 - ▶ The state variables are chosen as the energy variables.
 - ▶ The links between the energy function and the system dynamics is made explicit through symmetries.
 - ▶ The boundary port variables are power conjugated.
- ▶ Energy shaping consists in using the physical properties of the system to derive efficient control laws with guaranteed performances (step further stabilization).
- ▶ "Easy" to extend to non linear or systems defined on higher dimensional spaces.



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Infinite dimensional Port Hamiltonian systems (PHS)

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$$\frac{\partial}{\partial t} \begin{pmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{G} \\ -\mathcal{G}^* & -R \end{pmatrix} \begin{pmatrix} \mathcal{H}_1(\zeta)x_1(\zeta, t) \\ \mathcal{H}_2(\zeta)x_2(\zeta, t) \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} u_d(\zeta, t) \quad (1)$$

$$y_d(\zeta, t) = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} \mathcal{H}_1(\zeta)x_1(\zeta, t) \\ \mathcal{H}_2(\zeta)x_2(\zeta, t) \end{pmatrix} \quad (2)$$

$$u_{\partial} = \mathcal{B} \begin{pmatrix} \mathcal{H}_1(\zeta)x_1(\zeta, t) \\ \mathcal{H}_2(\zeta)x_2(\zeta, t) \end{pmatrix}, \quad y_{\partial} = \mathcal{C} \begin{pmatrix} \mathcal{H}_1(\zeta)x_1(\zeta, t) \\ \mathcal{H}_2(\zeta)x_2(\zeta, t) \end{pmatrix} \quad (3)$$

where $x = [x_1^T, x_2^T]^T \in X := L^2([a, b], \mathbb{R}^n) \times L^2([a, b], \mathbb{R}^n)$, $\mathcal{H} = \text{diag}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{H}(\zeta) = \mathcal{H}^T(\zeta)$ and $\mathcal{H}(\zeta) \geq \eta$ with $\eta > 0$ for all $\zeta \in [a, b]$, $R \in \mathbb{R}^{(n,n)}$, $R = R^T > 0$, $\mathcal{B}(\cdot)$ and $\mathcal{C}(\cdot)$ are some boundary input and boundary output mapping operators. Furthermore

$$\mathcal{G} = \sum_{i=0}^N G_i \frac{\partial^i}{\partial \zeta^i}, \quad \text{and} \quad \mathcal{G}^* = \sum_{i=0}^N (-1)^i G_i^T \frac{\partial^i}{\partial \zeta^i}$$

pmatrix with $G_i \in \mathbb{R}^{(n,n)}$.

Infinite dimensional Port Hamiltonian systems (PHS)

For a sake of compactness we shall use the following notation

$$P_i = \begin{pmatrix} 0 & G_i \\ (-1)^{i+1} G_i^T & 0 \end{pmatrix}, R_0 = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix} \quad (4)$$

and the formulation of (1)

$$\frac{\partial x}{\partial t}(\zeta, t) = \sum_{i=0}^N P_i \frac{\partial^i}{\partial \zeta^i} (\mathcal{H}(\zeta)x(\zeta, t)) - R_0 \mathcal{H}(\zeta)x(\zeta, t) + \begin{pmatrix} 0 \\ I \end{pmatrix} u_d(\zeta, t) \quad (5)$$

$$y_d(\zeta, t) = (0 \quad I) \mathcal{H}(\zeta)x(\zeta, t) \quad (6)$$

$$u_{\partial} = \mathcal{B}(\mathcal{H}(\zeta)x(\zeta, t)), y_{\partial} = \mathcal{C}(\mathcal{H}(\zeta)x(\zeta, t)) \quad (7)$$

The total energy of the system $H(x)$ is defined by

$$H(x) = \frac{1}{2} \int_a^b \left(x^T(\zeta, t) \mathcal{H}(\zeta)x(\zeta, t) \right) d\zeta$$

Boundary controlled port Hamiltonian systems

Mixed in-domain / boundary controlled port Hamiltonian systems (IDBC-PHS)

A mixed in-domain / boundary controlled port Hamiltonian system is an infinite dimensional system of the form (5-7) where

$$u_{\partial} = W_B \begin{pmatrix} \mathcal{H}(b)x(b, t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{H}x)}{\partial \zeta^{N-1}}(b, t) \\ \mathcal{H}(a)x(a, t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{H}x)}{\partial \zeta^{N-1}}(a, t) \end{pmatrix} \mathcal{H}, \text{ and } y_{\partial} = W_C \begin{pmatrix} \mathcal{H}(b)x(b, t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{H}x)}{\partial \zeta^{N-1}}(b, t) \\ \mathcal{H}(a)x(a, t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{H}x)}{\partial \zeta^{N-1}}(a, t) \end{pmatrix} \quad (8)$$

with

$$W_B = \left(\frac{1}{\sqrt{2}} (\Xi_2 + \Xi_1 P_e) \quad \frac{1}{\sqrt{2}} (\Xi_2 - \Xi_1 P_e) \right), \quad (9)$$

$$W_C = \left(\frac{1}{\sqrt{2}} (\Xi_1 + \Xi_2 P_e) \quad \frac{1}{\sqrt{2}} (\Xi_1 - \Xi_2 P_e) \right), \quad (10)$$

Boundary controlled port Hamiltonian systems

where

$$P_e = \begin{pmatrix} P_1 & \cdots & (-1)^{N-1} P_N \\ \vdots & \ddots & 0 \\ (-1)^{N-1} P_N & 0 & 0 \end{pmatrix} \quad (11)$$

and Ξ_1 and Ξ_2 in $\mathbb{R}^{k \times k}$ satisfy

$$\Xi_2^T \Xi_1 + \Xi_1^T \Xi_2 = 0, \text{ and } \Xi_2^T \Xi_2 + \Xi_1^T \Xi_1 = I \quad (12)$$

The energy balance associated to the system reads

$$\frac{dH}{dt} = \int_a^b y_d^T u_d d\zeta - \int_a^b \left(x_2^T(\zeta, t) \mathcal{H}_2^T(\zeta) R \mathcal{H}_2(\zeta) x_2(\zeta, t) \right) d\zeta + y_\partial^T u_\partial \quad (13)$$

$$\leq \int_a^b y_d^T u_d d\zeta + y_\partial^T u_\partial \quad (14)$$

Boundary controlled port Hamiltonian systems



Existence of solution [Le Gorrec et al., 2005]

The operator

$$\mathcal{J} = \sum_{i=0}^N P_i \frac{\partial^i}{\partial \zeta^i} (\mathcal{H}(\zeta)x(\zeta, t)) - R_0 \mathcal{H}(\zeta)x(\zeta, t)$$

with domain

$$D(\mathcal{J}) = \left\{ \mathcal{H} \in H^N(a, b; \mathbb{R}^n) \mid \begin{pmatrix} \mathcal{H}(b)x(b, t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{H}x)}{\partial \zeta^{N-1}}(b, t) \\ \mathcal{H}(a)x(a, t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{H}x)}{\partial \zeta^{N-1}}(a, t) \end{pmatrix} \in \text{Ker} W_B \right\}$$

where W_B is defined by (9) and Ξ_1 and Ξ_2 satisfy (12), generates a contraction semigroup on X . Furthermore the system (5-7) with (9-10) and (12) defines a boundary control system.





The general formulation (1) allows to model a large class of systems.

For example:

- ▶ The 1D wave equation where $n = 1$, $N = 1$, $G_0 = 0$, $G_1 = 1$.
- ▶ The Euler Bernoulli beam equation. In this case $n = 1$, $N = 2$, $G_0 = 0$, $G_1 = 0$, $G_2 = 1$.
- ▶ The Timoshenko beam equation. In this case $n = 2$, $N = 1$, and

$$G_0 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, G_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In what follows we focus on first order differential operators



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Control by interconnection

The system is interconnected with a dynamic controller in a power preserving way.

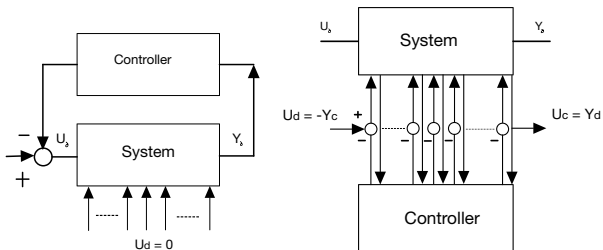


Figure: Control by interconnection. Boundary control (left), in domain control (right).

The closed loop energy is equal to the sum of the open loop energy and the controller energy.



Energy shaping

Objectives

Modification of the closed loop system's properties (energy shaping) + stabilization (damping injection).



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$$H_{cl}(x, x_c) = H(x) + H_c(x_c)$$



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We first look for structural invariants $C(x, x_c)$ i.e. $\frac{dC}{dt} = 0$

$$C(x, x_c) = x_c + F(x) = \kappa$$

where F is a smooth function.



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where F is a smooth function. In this case the closed loop energy function reads

$$H_{cl}(x, x_c) = H_{cl}(x) = H(x) + H_c(\kappa - F(x))$$

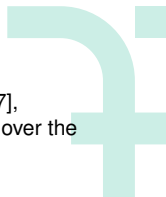
Asymptotic stability of the closed loop system in x^* is achieved using damping injection such that

$$\frac{dH_{cl}}{dt} < 0, \forall x \neq x^*.$$



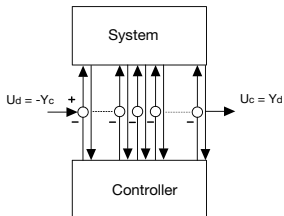
Energy shaping

- **Boundary control case** : Asymptotic stabilisation [Macchelli et al., 2017], Exponential stabilisation [Macchelli et al., 2020] \Rightarrow Control = integrals over the spatial domain.



Energy shaping

- **Boundary control case** : Asymptotic stabilisation [Macchelli et al., 2017], Exponential stabilisation [Macchelli et al., 2020] \Rightarrow Control = integrals over the spatial domain.
- **In this talk** : we consider in domain control



and the system is connected to the controller in a power preserving way:

$$\begin{pmatrix} u_d(\zeta, t) \\ y_d(\zeta, t) \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u_c(\zeta, t) \\ y_c(\zeta, t) \end{pmatrix} + \begin{pmatrix} u'(\zeta, t) \\ 0 \end{pmatrix}, \quad (15)$$

Control by interconnection : ideal case

- **Ideal case** : the control acts at each point ζ of the spatial domain. The controller is of the form

$$\begin{cases} \frac{\partial x_C}{\partial t}(\zeta, t) = \mathcal{J}_c \mathcal{Q}_c x_C(\zeta, t) + \mathcal{B}_c u_C(\zeta, t) \\ y_C(\zeta, t) = \mathcal{B}_c^* \mathcal{Q}_c x_C(\zeta, t) + \mathcal{S}_c u_C(\zeta, t) \end{cases} \quad (16)$$

where $\mathcal{Q}_c(\zeta) = \mathcal{Q}_c^T(\zeta)$ and $\mathcal{Q}_c(\zeta) \geq \eta_c$ with $\eta_c > 0$ for all $\zeta \in [a, b]$, \mathcal{S}_c and $\mathcal{S}_c(\zeta) = \mathcal{S}_c^T(\zeta)$ and $\mathcal{S}_c(\zeta) \geq \eta_s$ with $\eta_s > 0$ for all $\zeta \in [a, b]$ and:

$$\mathcal{B}_c = B_{c0} + B_{c1} \frac{\partial}{\partial \zeta}, \text{ and } \mathcal{J}_c = J_{c0} + J_{c1} \frac{\partial}{\partial \zeta} \quad (17)$$

with $B_{c0}, B_{c1} \in \mathbb{R}^{(n_c, 1)}$, $J_{c0} = -J_{c0}^T$, $J_{c1} = J_{c1}^T \in \mathbb{R}^{(n_c, n_c)}$.



Control by interconnection : ideal case

The closed loop system reads :

$$\frac{\partial x_e}{\partial t} = \begin{pmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \\ \frac{\partial x_c}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{G} & 0 \\ -\mathcal{G}^* & -(\mathcal{S}_c + R) & -\mathcal{B}_c^* \\ 0 & \mathcal{B}_c & \mathcal{J}_c \end{pmatrix} \begin{pmatrix} \mathcal{H}_1 x_1 \\ \mathcal{H}_2 x_2 \\ \mathcal{Q}_c x_c \end{pmatrix} \quad (18)$$

Structural invariants

The closed loop system (18) admits structural invariants of the form

$$\kappa_0 = C(x_e) = \int_a^b \Psi^T x_e d\zeta \quad (19)$$

with $\Psi = (\psi_1, \psi_2, \psi_3)$ if and only if

$$-\mathcal{G}\psi_2(\zeta) = 0 = -\mathcal{B}_c\psi_2(\zeta) + \mathcal{J}_c^*\psi_3(\zeta) \quad (20)$$

$$(\mathcal{S}_c + R)\psi_2(\zeta) = 0 \quad (21)$$

$$\mathcal{G}\psi_1(\zeta) + \mathcal{B}_c^*\psi_3(\zeta) = 0 \quad (22)$$

$$\begin{pmatrix} 0 & \mathcal{G}_1 & 0 \\ -\mathcal{G}_1^T & 0 & -\mathcal{B}_{c1} \\ 0 & \mathcal{B}_{c1}^T & \mathcal{J}_{c1} \end{pmatrix} \begin{pmatrix} \psi_1(\zeta) \\ \psi_2(\zeta) \\ \psi_3(\zeta) \end{pmatrix} \Big|_{a,b} = 0 \quad (23)$$

Energy shaping : ideal case

Energy shaping [Trenchant et al., 2017]

Choosing $\mathcal{B}_c = \mathcal{G}$ and $\mathcal{J}_c = 0$ the closed loop system (18) admits as structural invariants the function $C(x_e)$ defined by (19) and

$$\Psi = (\Psi_1, 0, \Psi_1)$$

In this case the hyperbolic system (1) connected to the dynamic controller (29) of the form

$$\begin{cases} \frac{\partial x_C}{\partial t}(\zeta, t) = \mathcal{G}u_C(\zeta, t) \\ y_C(\zeta, t) = \mathcal{G}^* \mathcal{Q}_c x_C(\zeta, t) + \mathcal{S}_c u_C(\zeta, t) \end{cases} \quad (24)$$

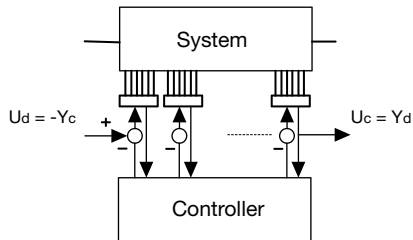
is equivalent to the system

$$\frac{\partial}{\partial t} \begin{pmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{G} \\ -\mathcal{G}^* & -(R + \mathcal{S}_c) \end{pmatrix} \begin{pmatrix} (\mathcal{H}_1(\zeta) + \mathcal{Q}_c(\zeta)) x_1(\zeta, t) \\ \mathcal{H}_2(\zeta) x_2(\zeta, t) \end{pmatrix} \quad (25)$$

$$u_\partial = \mathcal{B} \begin{pmatrix} (\mathcal{H}_1(\zeta) + \mathcal{Q}_c(\zeta)) x_1(\zeta, t) \\ \mathcal{H}_2(\zeta) x_2(\zeta, t) \end{pmatrix}, \quad y_\partial = \mathcal{C} \begin{pmatrix} (\mathcal{H}_1(\zeta) + \mathcal{Q}_c(\zeta)) x_1(\zeta, t) \\ \mathcal{H}_2(\zeta) x_2(\zeta, t) \end{pmatrix} \quad (26)$$

Control by interconnection

- **Non ideal case** : the distributed parameter system is actuated through piecewise constant elements.



Early lumping approach

The system is first discretized using a structure preserving method (mixed finite element method [Golo et al., 2004]) such that the approximation of (1) is again a PHS with n elements:

$$\begin{pmatrix} \dot{x}_{1d} \\ \dot{x}_{2d} \end{pmatrix} = (J_n - R_n) \begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix} + B_b u_b + \begin{pmatrix} 0 \\ B_{0d} \end{pmatrix} \mathbf{u}_d, \quad (27a)$$

$$y_b = B_b^T \begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix} + D_b u_b, \quad (27b)$$

$$\mathbf{y}_d = \begin{pmatrix} 0 & B_{0d}^T \end{pmatrix} \begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix}, \quad (27c)$$

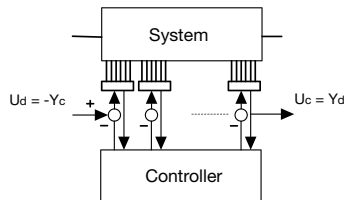
where $x_{id} = (x_i^1 \ \cdots \ x_i^n)^T \in \mathbb{R}^{np \times 1}$ for $i \in \{1, \dots, 2p\}$,

$$J_n = \begin{pmatrix} 0 & J_i \\ -J_i^T & 0 \end{pmatrix} \quad \text{and} \quad R_n = \begin{pmatrix} 0 & 0 \\ 0 & R_d \end{pmatrix},$$

The discretized energy reads:

$$H_d(x_{1d}, x_{2d}) = \frac{1}{2} \left(x_{1d}^T Q_1 x_{1d} + x_{2d}^T Q_2 x_{2d} \right). \quad (28)$$

Control by interconnection



The controller is designed as finite dimensional PHS of the form:

$$\begin{cases} \dot{x}_c = (J_c - R_c) Q_c x_c + B_c u_c, \\ y_c = B_c^T Q_c x_c + D_c u_c, \end{cases} \quad (29)$$

interconnected in a power preserving way through the relation

$$\begin{pmatrix} u_d \\ u_c \end{pmatrix} = \begin{pmatrix} 0 & -M \\ M^T & 0 \end{pmatrix} \begin{pmatrix} y_d \\ y_c \end{pmatrix}, \text{ where } M = \mathbb{I}_m \otimes \mathbf{1}_{k \times 1} \in \mathbb{R}^{n \times m}, \quad (30)$$



Control by interconnection

The closed loop system is given by

$$\dot{x}_{cl} = (J_{cl} - R_{cl}) Q_{cl} x_{cl}, \quad (31)$$

where $x_{cl} = (x_{1d}^T, x_{2d}^T, x_c^T)^T$, $Q_{cl} = \text{diag}(Q_1, Q_2, Q_c)$,

$$J_{cl} = \begin{pmatrix} O & J_i & 0 \\ -J_i^T & 0 & -B_{0d} M B_c^T \\ 0 & B_c M^T B_{0d}^T & J_c \end{pmatrix}, \quad R_{cl} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & R_d + B_{0d} M D_c M^T B_{0d}^T & 0 \\ 0 & 0 & R_c \end{pmatrix}.$$

The Hamiltonian of the controller (29) is:

$$H_c(x_c) = \frac{1}{2} x_c^T Q_c x_c. \quad (32)$$

Therefore, the closed loop Hamiltonian function reads:

$$H_{cld}(x_{1d}, x_{2d}, x_c) = H_d(x_{1d}, x_{2d}) + H_c(x_c). \quad (33)$$



Energy shaping

Approximate energy shaping [Liu et al., 2021]

Choosing $J_c = 0$, and $R_c = 0$, the closed loop system (31) admits:

$$C(x_{1d}, x_c) = B_c M^T B_{0d}^T J_i^{-1} x_{1d} - x_c \quad (34)$$

as structural invariant, *i.e.* $\dot{C}(x_{1d}, x_c) = 0$ along the closed loop trajectories. If $x_{1d}(0)$ and $x_c(0)$ satisfy $C(x_{1d}(0), x_c(0)) = 0$, the controller is a proportional-integral control, and the control law (30) is equivalent to the state feedback:

$$\mathbf{u}_d = -B_c^T Q_c B_c M^T B_{0d}^T J_i^{-1} x_{1d} - D_c M^T B_{0d}^T Q_2 x_{2d}. \quad (35)$$

Therefore, the closed loop system yields:

$$\begin{pmatrix} \dot{x}_{1d} \\ \dot{x}_{2d} \end{pmatrix} = \begin{pmatrix} 0 & J_i \\ -J_i^T & -(R_d + B_{0d} M D_c M^T B_{0d}^T) \end{pmatrix} \begin{pmatrix} \tilde{Q}_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix}, \quad (36)$$

where: $\tilde{Q}_1 = Q_1 + J_i^{-T} B_{0d} M B_c^T Q_c B_c M^T B_{0d}^T J_i^{-1}$ is the new closed loop energy matrix associated to x_{1d} .

Problem

The energy related to first n elements of x_{1d} in closed loop is shaped in an optimal way if and only if $X = B_c^T Q_c B_c$ sym. sem. def. pos. minimizes

$f(X) = \|AXA^T - Q_m\|_F$, where $A = (J_i)^{-T} B_{0d} M \in \mathbb{R}^{n \times m}$, $SR_0^{m \times m}$ and $Q_m = (\tilde{Q}_{1d} - Q_1)_{n \times n}$.

Solution

$f(X)$ is convex and the minimization of $f(X)$ is equivalent to the minimization of $f^2(X)$, which has a unique minimum given by

$$\hat{X} = V \Sigma_0^{-1} U_1^T Q_m U_1 \Sigma_0^{-1} V^T \quad (37)$$

with V , Σ_0 and U_1 the matrices of the singular value decomposition (SVD) of the matrix A i.e. $A = U \Sigma V^T = \begin{pmatrix} U_1 & U_2 \\ & 0 \end{pmatrix} \begin{pmatrix} \Sigma_0 \\ 0 \end{pmatrix} V^T$, where $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{m \times m}$ are unitary matrices, $U_1 \in \mathbb{R}^{n \times m}$, $U_2 \in \mathbb{R}^{n \times q}$, $q = n - m$, and $\Sigma_0 = \Sigma_0^T \geq 0$ is the diagonal matrix of singular values of A .

Energy shaping



- ▶ The choice of the controller matrices B_c and Q_c is not unique. It only has to satisfy the condition (37). It is done in order to modify the shape of the closed loop energy function of the system in the x_1 coordinate.
- ▶ The choice of the controller matrix D_c follows a similar procedure, with the optimization of the difference between the approximate dissipation and the desired one.
- ▶ D_c allows to add **local dissipation** but more interestingly it may be used to add **global dissipative effect equivalent to diffusion**.
- ▶ The quality of the shaping (controller bandwidth) depends on the number of patches.



Stability analysis

The controller is now connected to the infinite dimensional system leading to :

$$\dot{\mathcal{X}} = \underbrace{\begin{pmatrix} (\mathcal{J} - \mathcal{R} - \mathcal{B}D_c\mathcal{B}^*) & -\mathcal{B}B_c^T \\ B_c\mathcal{B}^* & 0 \end{pmatrix}}_{\mathcal{A}_{cl}} \begin{pmatrix} \mathcal{H} & 0 \\ 0 & Q_c \end{pmatrix} \mathcal{X}, \quad (38)$$

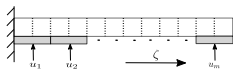
where $\mathcal{X} = (x^T \quad x_c^T)^T \in X_s$ where $X_s = L_2([0, L], \mathbb{R}^{2p}) \times \mathbb{R}^m$.

Existence of solution, stability analysis

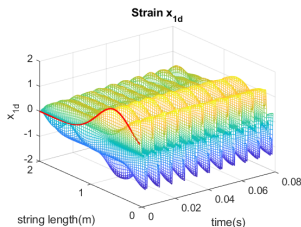
- ▶ The operator \mathcal{A}_{cl} defined in (38) generates a contraction semigroup on $X_s = L_2([0, L], \mathbb{R}^{2p}) \times \mathbb{R}^m$.
- ▶ The operator \mathcal{A}_{cl} has a compact resolvent.
- ▶ Asymptotic stability: For any $\mathcal{X}(0) \in L_2([0, L], \mathbb{R}^{2n}) \times \mathbb{R}^m$, the unique solution of (38) tends to zero asymptotically, and the closed loop system (38) is globally asymptotically stable.

Energy shaping : application

We consider the control of a weakly damped vibrating string using m homogeneously distributed patches (n discretization elements).



We consider the case with m patches, *i.e.* $m = 10$, $n = 50$ and $k = 5$. The initial conditions are set to a spatial distribution $x_1(\zeta, 0) = \mathcal{N}(1.5, 0.113)$ for the strain distribution and to zero for velocity distribution.



Control by interconnection

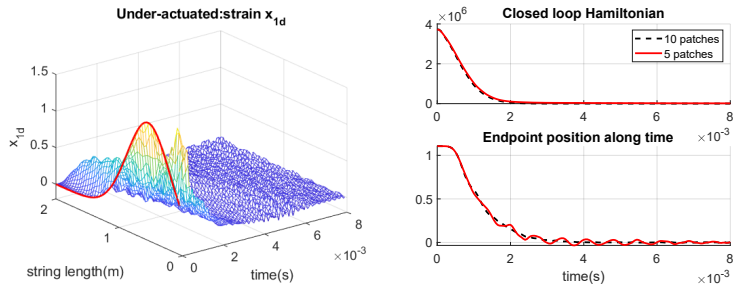


Figure: Closed loop evolution of the angular strain for $m = 10$ (a), Hamiltonian function and endpoint position (b) in the under-actuated case for $m = 10$, $m = 5$.

Control by interconnection

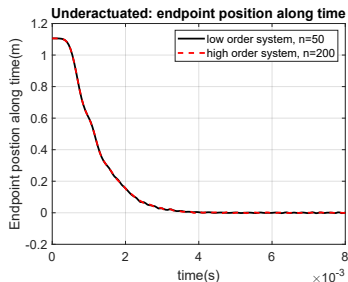
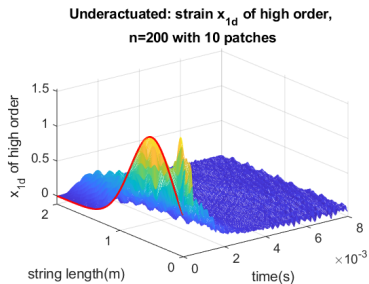


Figure: Closed loop evolution of the angular strain of the high order system (a), and comparison of the endpoint position of the low order and high order systems using the same controller (b).



Control by interconnection

Achievable performances

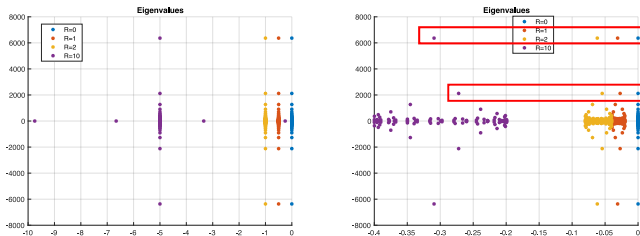


Figure: Control by interconnection. Full actuation (left), partial actuation (right).

Outline

Context and motivation

Infinite dimensional Port Hamiltonian systems (PHS)

Control by interconnection and energy shaping

Observer design

Irreversible boundary controlled port Hamiltonian Systems

Conclusions and future works



Observer design

In many cases the power conjugated variable is not (completely) measurable. In this case one has to use an **observer**.

$$\mathcal{U} \begin{cases} \frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} (\mathcal{H}(\zeta)x(\zeta, t)) + P_0 \mathcal{H}(\zeta)x(\zeta, t), \\ W_B \begin{pmatrix} \mathbf{f}_\partial(t) \\ \mathbf{e}_\partial(t) \end{pmatrix} = u(t), \quad x(\zeta, 0) = x_0(\zeta), \\ y(t) = W_C \begin{pmatrix} \mathbf{f}_\partial(t) \\ \mathbf{e}_\partial(t) \end{pmatrix}, \\ y_m(t) = \mathcal{C}_m x(\zeta, t), \end{cases} \quad (39)$$

$$\hat{\mathcal{U}} \begin{cases} \frac{\partial \hat{x}}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} (\mathcal{H}\hat{x}(\zeta, t)) + P_0 (\mathcal{H}\hat{x}(\zeta, t)), \\ W_B \begin{pmatrix} \hat{\mathbf{f}}_\partial(t) \\ \hat{\mathbf{e}}_\partial(t) \end{pmatrix} = \hat{u}(t), \quad \hat{x}(\zeta, 0) = \hat{x}_0(\zeta) \\ \hat{y}(t) = W_C \begin{pmatrix} \hat{\mathbf{f}}_\partial(t) \\ \hat{\mathbf{e}}_\partial(t) \end{pmatrix}, \\ \hat{y}_m(t) = \mathcal{C}_m \hat{x}(\zeta, t), \end{cases} \quad (40)$$

Since the system $\hat{\mathcal{U}}$ in (40) is virtual, the input $\hat{u}(t)$ is designed with all the available information, i.e. $\hat{u}(t) = f(u(t), y_m(t), \hat{x}(\zeta, t))$, where $u(t)$ and $y_m(t)$ are considered known from (39) and $f(\cdot)$ is a function to be designed.

Observer design

Defining

$$\tilde{x}(\zeta, t) := x(\zeta, t) - \hat{x}(\zeta, t). \quad (41)$$

Then, from (39) and (40), we obtain the error dynamics equations as follows:

$$\tilde{\mathcal{U}} \begin{cases} \frac{\partial \tilde{x}}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} (\mathcal{H} \tilde{x}(\zeta, t)) + P_0 (\mathcal{H} \tilde{x}(\zeta, t)), \\ W_{\mathcal{B}} \begin{pmatrix} \tilde{f}_{\partial}(t) \\ \tilde{e}_{\partial}(t) \end{pmatrix} = \tilde{u}(t), & \tilde{x}(\zeta, 0) = \tilde{x}_0(\zeta), \\ \tilde{y}(t) = W_{\mathcal{C}} \begin{pmatrix} \tilde{f}_{\partial}(t) \\ \tilde{e}_{\partial}(t) \end{pmatrix}. \end{cases} \quad (42)$$

We define the Hamiltonian of the error system as:

$$\tilde{H}(t) = \frac{1}{2} \|\tilde{x}(t)\|_{\mathcal{H}}^2 = \frac{1}{2} \int_a^b \tilde{x}(\zeta, t)^T \mathcal{H}(\zeta) \tilde{x}(\zeta, t) d\zeta. \quad (43)$$

Since $W_{\mathcal{B}}$ and $W_{\mathcal{C}}$ are such that $W_{\mathcal{C}} \Sigma W_{\mathcal{B}}^T = I$, the time derivative of $\tilde{H}(t)$ satisfies

$$\dot{\tilde{H}}(t) = \tilde{u}(t)^T \tilde{y}(t). \quad (44)$$

Observer design

Full sensing case

Consider the BC-PHS (39) with $y_m(t) = y(t)$. The state of the observer (40) with

$$\hat{u}(t) = u(t) + L(y_m(t) - \hat{y}_m(t)), \quad (45)$$

converges exponentially to the state of the BC-PHS (39) if $0 < L + L^T \in \mathbb{R}^{n \times n}$.

Partial sensing case

Consider the BC-PHS (39) with $y_m(t) = C_m y(t)$ and $C_m = (I_p \ 0_{p \times n-p}) \in \mathbb{R}^{p \times n}$, $0 < p < n$. The states of the observer (40) with

$$\hat{u}(t) = u(t) + C_m^T L (y_m(t) - \hat{y}_m(t)) \quad \text{and} \quad L \in \mathbb{R}^{p \times p} \quad (46)$$

converges exponentially to the state of the BC-PHS (39) if L is such that $C_m^T L^T C_m + C_m^T L C_m \geq 0$, and one of the following conditions is satisfied ($\gamma > 0$)

$$\begin{aligned} \|\mathcal{H}(b)\tilde{x}(b, t)\|_{\mathbb{R}}^2 &\leq \gamma \tilde{y}(t)^T C_m^T L C_m \tilde{y}(t) \quad \text{or} \\ \|\mathcal{H}(a)\tilde{x}(a, t)\|_{\mathbb{R}}^2 &\leq \gamma \tilde{y}(t)^T C_m^T L C_m \tilde{y}(t), \end{aligned} \quad (47)$$

Position measurement

Consider the BC-PHS (39). Assume that the measurement is on the following form:

$$y_m(t) = \int_0^t C_m y(\tau) d\tau + y_m(0), \text{ with } C_m = \begin{pmatrix} 0_{p \times n-p} & I_p \end{pmatrix}. \quad (48)$$

Assume that the BC-PHS is approximately observable with respect to the output $C_m y(t)$. The state of the observer (40) with

$$\begin{aligned} \hat{u}(t) &= u(t) + C_m^T L_1 (y_m(t) - \hat{y}_m(t) + \theta(t)), \\ \dot{\theta}(t) &= -L_2 (y_m(t) - \hat{y}_m(t) + \theta(t)), \quad \theta(0) = \theta_0. \end{aligned} \quad (49)$$

converges asymptotically to the state of the BC-PHS (39) if $L_1, L_2 \in \mathbb{R}^{p \times p}$ are both positive definite matrices.

Implementation on the elastic string example

We consider now

- ▶ The position of the end point *i.e.* $\omega(b, t)$, is measured .
- ▶ The state is reconstructed using a Luenberger PH finite dimensional observer (the control uses $\hat{\omega}(b, t)$ and $\hat{v}(b, t)$) \Rightarrow the closed loop stability is guaranteed [Toledo et al., 2020].



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Irreversible systems

We consider a 1-D isentropic fluid in Lagrangian coordinates, also known as p -system, with $[a, b] \ni z$, $a, b \in \mathbb{R}$, $a < b$. We choose as state variables

- ▶ the specific volume $\phi(t, z)$,
- ▶ the velocity $v(t, z)$ of the fluid.

System of two conservation laws :

$$\begin{aligned}\frac{\partial \phi}{\partial t}(t, z) &= \frac{\partial v}{\partial z}(t, z) \\ \frac{\partial v}{\partial t}(t, z) &= -\frac{\partial p}{\partial z}(t, z) - \frac{\partial \tau}{\partial z}(t, z)\end{aligned}$$

where $p(\phi)$ is the pressure of the fluid, $\tau = -\hat{\mu} \frac{\partial v}{\partial z}$ with $\hat{\mu}$ the viscous damping coefficient. The total energy of the system is given by the sum of the kinetic energy and internal energy:

$$H(v, \phi) = \int_a^b \left(\frac{1}{2} v^2 + u(\phi) \right) dz$$

Irreversible systems

$$\begin{pmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial z} \begin{pmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta v} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial}{\partial z} \left(\hat{\mu} \frac{\partial \cdot}{\partial z} \right) \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta v} \end{pmatrix}, \quad (50)$$

Or alternatively, splitting

$$\frac{\partial}{\partial z} \left(\hat{\mu} \frac{\partial v}{\partial z} \right) = \frac{\partial}{\partial z} e_r, \quad \text{with } e_r = \hat{\mu} f_r, \quad \text{and } f_r = \frac{\partial v}{\partial z}$$

One can write

$$\begin{pmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial v}{\partial t} \\ f_r \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\tilde{P}_1} \frac{\partial}{\partial z} \begin{pmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta v} \\ e_r \end{pmatrix}, \quad \text{with } e_r = \hat{\mu} f_r \quad (51)$$

Irreversible systems

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- ▶ There is still an underlying Dirac structure and we can define from \tilde{P}_1 some boundary port variables such that $\frac{dH}{dt} \leq f_{\partial}^T e_{\partial}$
- ▶ **This is a DAE system and the thermal domain is not represented.**

The non-isentropic fluid: the irreversible case

We can account for the **thermal domain** by considering Gibbs' equation

$$du = -pd\phi + Tds$$

where s denotes the entropy density and T the temperature. The total energy of the system is still the sum of the kinetic and the internal energy but now depends on s

$$H(v, \phi, s) = \int_a^b \left(\frac{1}{2}v^2 + u(\phi, s) \right) dz$$

From the conservation of the total energy and Gibbs' equation $\frac{\partial u}{\partial s} = T$ we get

$$\frac{\partial s}{\partial t}(t, z) = \frac{\hat{\mu}}{T} \left(\frac{\partial v}{\partial z} \right)^2(t, z)$$

The non-isentropic fluid: the irreversible case

The system of balance equations may be written as the quasi-Hamiltonian system

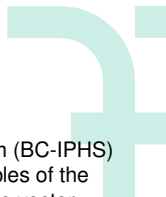
$$\begin{pmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial v}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial(\cdot)}{\partial z} & 0 \\ \frac{\partial(\cdot)}{\partial z} & 0 & \frac{\partial}{\partial z} \left(\frac{\hat{\mu}}{T} \left(\frac{\partial v}{\partial z} \right) (\cdot) \right) \\ 0 & \frac{\hat{\mu}}{T} \left(\frac{\partial v}{\partial z} \right) \frac{\partial(\cdot)}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta v} \\ \frac{\delta H}{\delta s} \end{pmatrix}$$

With

$$\frac{dH}{dt} = y^T \nu$$

and

$$\frac{dS}{dt} = \int_a^b \sigma dz + y_s^T \nu_s$$



We introduce the Boundary Controlled Irreversible Port Hamiltonian System (BC-IPHS) defined on a 1D spatial domain $z \in [a, b]$, $a, b \in \mathbb{R}$, $a < b$. The state variables of the system are the $n + 1$ *extensive variables*. The following partition of the state vector $\mathbf{x} \in \mathbb{R}^{n+1}$ shall be considered: the first n variables by $x = [q_1, \dots, q_n]^\top \in \mathbb{R}^n$ and the entropy density by $s \in \mathbb{R}$. Gibbs' equation is equivalent to the existence of an energy functional

$$H(x, s) = \int_a^b h(x(z), s(z)) dz \quad (52)$$

where $h(x, s)$ is the energy density function. The total entropy functional is denoted by

$$S(t) = \int_a^b s(z, t) dz \quad (53)$$



IPHS : General formulation

An infinite dimensional IPHS undergoing m irreversible processes is defined by

$$\frac{\partial}{\partial t} \begin{pmatrix} x(t, z) \\ s(t, z) \end{pmatrix} = \begin{pmatrix} P_0 & G_0 \mathbf{R}_0 \\ -\mathbf{R}_0^\top G_0^\top & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta x}(t, z) \\ \frac{\delta H}{\delta s}(t, z) \end{pmatrix} + \begin{pmatrix} P_1 \frac{\partial(\cdot)}{\partial z} & \frac{\partial(G_1 \mathbf{R}_1 \cdot)}{\partial z} \\ \mathbf{R}_1^\top G_1^\top \frac{\partial(\cdot)}{\partial z} & g_s \mathbf{r}_s \frac{\partial(\cdot)}{\partial z} + \frac{\partial(g_s \mathbf{r}_s \cdot)}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta x}(t, z) \\ \frac{\delta H}{\delta s}(t, z) \end{pmatrix} \quad (54)$$

where $P_0 = -P_0^\top \in \mathbb{R}^{n \times n}$, $P_1 = P_1^\top \in \mathbb{R}^{n \times n}$, $G_0 \in \mathbb{R}^{n \times m}$, $G_1 \in \mathbb{R}^{n \times m}$ with $m \leq n$ with $\mathbf{R}_l \left(\mathbf{x}, \frac{\delta H}{\delta \mathbf{x}} \right) \in \mathbb{R}^{m \times 1}$, $l = 0, 1$, defined by

$$R_{0,i} = \gamma_{0,i} \left(x, z, \frac{\delta H}{\delta x} \right) \{S|G_0(:, i)|H\}$$

$$R_{1,i} = \gamma_{1,i} \left(x, z, \frac{\delta H}{\delta x} \right) \left\{ S|G_1(:, i) \frac{\partial}{\partial z} |H \right\}$$

and

$$r_s = \gamma_s \left(x, z, \frac{\delta H}{\delta x} \right) \{S|H\}$$

and $\gamma_{k,i} \left(x, z, \frac{\delta H}{\delta x} \right) > 0$, $k = 0, 1$; $i \in \{1, \dots, m\}$, $\gamma_s \left(x, z, \frac{\delta H}{\delta x} \right) > 0$ and $g_s(x)$,



For any two functionals H_1 and H_2 of the type (52) and for any matrix differential operator \mathcal{G} we define the pseudo-brackets

$$\begin{aligned} \{H_1|\mathcal{G}|H_2\} &= \begin{pmatrix} \frac{\delta H_1}{\delta x} \\ \frac{\delta H_1}{\delta s} \end{pmatrix} \begin{pmatrix} 0 & \mathcal{G} \\ -\mathcal{G}^* & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H_2}{\delta x} \\ \frac{\delta H_2}{\delta s} \end{pmatrix}, \\ \{H_1|H_2\} &= \frac{\delta H_1}{\delta s}{}^\top \begin{pmatrix} \partial & \delta H_2 \\ \partial z & \delta s \end{pmatrix} \end{aligned} \tag{55}$$

where \mathcal{G}^* denotes the formal adjoint operator of \mathcal{G} .



Definition 1

A Boundary Controlled IPHS (BC-IPHS) is an infinite dimensional IPHS

$$\frac{\partial}{\partial t} \begin{pmatrix} x(t, z) \\ s(t, z) \end{pmatrix} = \begin{pmatrix} P_0 & G_0 \mathbf{R}_0 \\ -\mathbf{R}_0^\top G_0^\top & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \mathbf{x}}(t, z) \\ \frac{\delta H}{\delta s}(t, z) \end{pmatrix} + \begin{pmatrix} P_1 \frac{\partial(\cdot)}{\partial z} & \frac{\partial(G_1 \mathbf{R}_1 \cdot)}{\partial z} \\ \mathbf{R}_1^\top G_1^\top \frac{\partial(\cdot)}{\partial z} & g_s \mathbf{r}_s \frac{\partial(\cdot)}{\partial z} + \frac{\partial(g_s \mathbf{r}_s \cdot)}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \mathbf{x}}(t, z) \\ \frac{\delta H}{\delta s}(t, z) \end{pmatrix} \quad (56)$$

Augmented with the boundary port variables

$$v(t) = W_B \begin{pmatrix} e(t, b) \\ e(t, a) \end{pmatrix}, \quad y(t) = W_C \begin{pmatrix} e(t, b) \\ e(t, a) \end{pmatrix} \quad (57)$$

as linear functions of the modified effort variable

$$e(t, z) = \begin{pmatrix} \frac{\delta H}{\delta \mathbf{x}}(t, z) \\ \mathbf{R}(\mathbf{x}, \frac{\delta H}{\delta \mathbf{x}}) \frac{\delta H}{\delta s}(t, z) \end{pmatrix}, \quad \text{with } \mathbf{R} \left(\mathbf{x}, \frac{\delta H}{\delta \mathbf{x}} \right) = \left(1 \quad \mathbf{R}_1(\mathbf{x}, \frac{\delta H}{\delta \mathbf{x}}) \quad \mathbf{r}_s(\mathbf{x}, \frac{\delta H}{\delta \mathbf{x}}) \right)^\top \quad (58)$$



Furthermore

$$\begin{aligned}W_B &= \left(\frac{1}{\sqrt{2}} (\Xi_2 + \Xi_1 P_{ep}) M_p \quad \frac{1}{\sqrt{2}} (\Xi_2 - \Xi_1 P_{ep}) M_p \right), \\W_C &= \left(\frac{1}{\sqrt{2}} (\Xi_1 + \Xi_2 P_{ep}) M_p \quad \frac{1}{\sqrt{2}} (\Xi_1 - \Xi_2 P_{ep}) M_p \right),\end{aligned}$$

where $M_p = (M^\top M)^{-1} M^\top$, $P_{ep} = M^\top P_e M$ and $M \in \mathbb{R}^{(n+m+2) \times k}$ is spanning the columns of $P_e \in \mathbb{R}^{n+m+2}$ of rank k , defined by

$$P_e = \begin{pmatrix} P_1 & 0 & G_1 & 0 \\ 0 & 0 & 0 & g_s \\ G_1^\top & 0 & 0 & 0 \\ 0 & g_s & 0 & 0 \end{pmatrix} \quad (59)$$

and where Ξ_1 and Ξ_2 in $\mathbb{R}^{k \times k}$ satisfy $\Xi_2^\top \Xi_1 + \Xi_1^\top \Xi_2 = 0$ and $\Xi_2^\top \Xi_2 + \Xi_1^\top \Xi_1 = I$.



IPHS : General formulation

First law of Thermodynamics

The total energy balance is

$$\dot{H} = y(t)^\top v(t)$$

which leads, when the input is set to zero, to $\dot{H} = 0$ in accordance with the first law of Thermodynamics.



First law of Thermodynamics

The total energy balance is

$$\dot{H} = y(t)^\top v(t)$$

which leads, when the input is set to zero, to $\dot{H} = 0$ in accordance with the first law of Thermodynamics.

Second law of Thermodynamics

The total entropy balance is given by

$$\dot{S} = \int_a^b \sigma_t dz + y_S^\top v_s$$

where y_s and v_s are the entropy conjugated input/output and σ_t is the total internal entropy production. This leads, when the input is set to zero, to $\dot{S} = \int_a^b \sigma_t dz \geq 0$ in accordance with the second law of Thermodynamics.

Control design : Heat equation

Balance equation on u ($z \in [0, L]$)

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial z} \left(-\lambda \frac{\partial T}{\partial z} \right)$$

where λ denotes the heat conduction coefficient. From Gibbs' equation $du = Tds$

$$\frac{\partial s}{\partial t} = \frac{\lambda}{T^2} \frac{\partial T}{\partial z} \frac{\partial}{\partial z} \left(\frac{\delta U}{\delta s} \right) + \frac{\partial}{\partial z} \left(\frac{\lambda}{T^2} \frac{\partial T}{\partial z} \left(\frac{\delta U}{\delta s} \right) \right) \quad (60)$$

which is equivalent to (56) where $P_0 = 0$, $P_1 = 0$, $G_0 = 0$, $G_1 = 0$, $g_s = 1$ and

$r_s = \gamma_s \{S|U\}$ with $\gamma_s = \frac{\lambda}{T^2}$ and $\{S|U\} = \frac{\partial T}{\partial z}$. In this case $P_e = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $n = 1$

and $m = 1$. Choosing $\Xi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $\Xi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$ we have

$$v(t) = \begin{pmatrix} \left(\frac{\lambda_s}{T} \frac{\partial T}{\partial z} \right) (t, L) \\ - \left(\frac{\lambda_s}{T} \frac{\partial T}{\partial z} \right) (t, 0) \end{pmatrix}, \quad y(t) = \begin{pmatrix} T(t, L) \\ T(t, 0) \end{pmatrix}, \quad (61)$$

respectively the entropy flux and the temperature at each boundary.

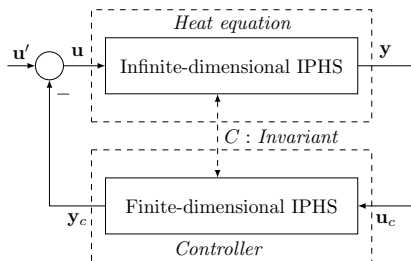


Figure: Cbl of the heat equation

We consider reflective BC at 0 and control at L.

$$\mathbf{u}_c = \mathbf{y}, \quad \begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} = -\mathbf{y}_c + \begin{pmatrix} \mathbf{u}' \\ 0 \end{pmatrix} \quad (62)$$

A nonlinear finite dimensional boundary controller is designed by extending the control by interconnection (Cbl) for BC-PHS [Macchelli et al., 2017] to BC-IPHS. The objective is to characterize the conditions for the existence of closed-loop invariant functions, which are then used to shape the closed-loop energy function and assign the closed-loop entropy

Control design

The controller is looked for on the form

$$\dot{x}_c = 0e_c + G_c(x_c, \mathbf{u}_c)\mathbf{u}_c, \quad \mathbf{y}_c = G_c^\top(x_c, \mathbf{u}_c)e_c, \quad (63)$$

where $e_c = \partial_{x_c} H_c$.

Denoting by \mathcal{C} and \mathcal{B} the boundary operators such that the output and input in (61) can be expressed as $\mathbf{y} = \mathcal{C}e_s$ and $\begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} = \mathcal{B}e_s$, respectively, we obtain the coupled PDE-ODE system that follows

$$\begin{aligned} \underbrace{\begin{pmatrix} \partial_t s \\ \dot{x}_c \end{pmatrix}}_{\dot{\mathbf{x}}_{cl}} &= \underbrace{\begin{pmatrix} r_s \partial_z(\cdot) + \partial_z(r_s \cdot) & 0 \\ G_c(x_c, \mathbf{u}_c)\mathcal{C} & 0 \end{pmatrix}}_{\mathcal{J}_{cl}} \underbrace{\begin{pmatrix} e_s \\ e_c \end{pmatrix}}_{\mathbf{e}_{cl}} \\ \underbrace{\begin{pmatrix} \mathbf{u}' \\ 0 \end{pmatrix}}_{\mathbf{u}'_{cl}} &= \underbrace{\begin{pmatrix} \mathcal{B} & G_c^\top(x_c, \mathbf{u}_c) \end{pmatrix}}_{W_{B_{cl}}} \mathbf{e}_{cl} \end{aligned} \quad (64)$$

where $\mathbf{e}_{cl} \in \mathbb{E}_{cl}$ denote the co-states of the closed-loop system, with $\mathbb{E}_{cl} = \mathbb{T} \times \mathbb{R}$ the corresponding co-state space and the inner product

$$\langle \mathbf{f}^1, \mathbf{f}^2 \rangle_{\mathbb{E}_{cl}} = \int_0^L f_1^1(z) f_1^2(z) dz + f_2^1 f_2^2$$

for all $\mathbf{f}^i = [f_1^i(z) \quad f_2^i]^\top \in \mathbb{E}_{cl}$.



The operator \mathcal{J}_{cl} satisfies

$$\langle \mathbf{f}^1, \mathcal{J}_{cl} \mathbf{f}^2 \rangle_{\mathbb{E}_{cl}} = \langle -\mathcal{J}_{cl} \mathbf{f}^1, \mathbf{f}^2 \rangle_{\mathbb{E}_{cl}} + [\mathcal{C} \mathbf{f}_1^1]^\top W_{B_{cl}} \mathbf{f}^2 + [W_{B_{cl}} \mathbf{f}^1]^\top \mathcal{C} \mathbf{f}_1^2. \quad (65)$$

Setting $W_{B_{cl}} \mathbf{f}^i = 0, \forall \mathbf{f}^i \in \mathbb{E}_{cl}$ we obtain that $\langle \mathbf{f}^1, \mathcal{J}_{cl} \mathbf{f}^2 \rangle_{\mathbb{E}_{cl}} = \langle -\mathcal{J}_{cl} \mathbf{f}^1, \mathbf{f}^2 \rangle_{\mathbb{E}_{cl}}$, i.e., \mathcal{J}_{cl} is formally skew-adjoint on the space \mathbb{E}_{cl} .





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Setting $W_{B_{cl}} \mathbf{f}^i = 0, \forall \mathbf{f}^i \in \mathbb{E}_{cl}$ we obtain that $\langle \mathbf{f}^1, \mathcal{J}_{cl} \mathbf{f}^2 \rangle_{\mathbb{E}_{cl}} = \langle -\mathcal{J}_{cl} \mathbf{f}^1, \mathbf{f}^2 \rangle_{\mathbb{E}_{cl}}$, i.e., \mathcal{J}_{cl} is formally skew-adjoint on the space \mathbb{E}_{cl} .

Casimir invariants

Consider the boundary control system (64) with $\mathbf{u}' = 0$. A function $C : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ is an invariant of (64) if $\dot{C} = 0$ along the trajectories (64) for any \mathbf{e}_{cl} .



Proposition

We consider $C(s, x_c)$ of the form

$$C(s, x_c) = \Gamma x_c + \int_0^L f(s(z)) dz = \kappa \quad (66)$$

where κ is a constant and $f(s) \in H^1([0, L], \mathbb{R})$ is a continuous function. Then (66) is an invariant if

$$\langle \mathcal{J}_{cl} \epsilon, \mathbf{e}_{cl} \rangle = 0 \quad (67)$$

$$(W_B \quad G_c^\top(x_c, \mathbf{u}_c)) \epsilon = 0 \quad (68)$$

where

$$\epsilon = \begin{pmatrix} \epsilon_s \\ \epsilon_c \end{pmatrix} = \begin{pmatrix} \delta_s C \\ \partial_{x_c} C \end{pmatrix} = \begin{pmatrix} \partial_s f(s) \\ \Gamma \end{pmatrix} \quad (69)$$



Proposition

The function $C(s, x_c) = \Gamma x_c + \int_0^L f(s(z))dz$ is an invariant if $f(s) = \alpha u(s) + c_1$ where c_1 is a function that does not depend on s . The state of the control system (63) is then given by the state feedback

$$x_c = -\frac{\alpha}{\Gamma} \int_0^L u(s)dz + \bar{k}/\Gamma = -\frac{\alpha}{\Gamma} \mathcal{H}(s) + \bar{k}/\Gamma \quad (70)$$

with $\bar{k} = \left(k + \int_0^L c_1 dz \right)$ and the controller energy function is

$$H_c = \frac{\Gamma}{\alpha} x_c + k_c = -\mathcal{H}(s) + k' \quad (71)$$

where $k' = \frac{\bar{k}}{\alpha} + k_c$, with k_c a constant. Furthermore, the energy function of the closed-loop system is constant and equal to k' .



Control design

Although the energy H_{cl} of the closed-loop system is constant, the state of the controller provides a measure of the internal energy of the heat equation.

- ▶ Setting $\alpha = -1$, $\Gamma = 1$ and $\bar{k} = 0$ the state of the controller is

$$x_c(t) = \int_0^L u(z, t) dz = \mathcal{H}(t), \quad x_c(0) = \mathcal{H}(0)$$

i.e., x_c is a measure of the total energy of (60).

- ▶ Setting $\alpha = -1$, $\Gamma = 1$ and $\bar{k} = -\mathcal{H}(0)$,

$$x_c(t) = \int_0^L (u(z, t) - u_0) dz = \Delta\mathcal{H}, \quad x_c(0) = 0$$

with $\Delta\mathcal{H} = \mathcal{H}(t) - \mathcal{H}(0)$. In this case x_c is a measure of the energy supplied by the controller to the process.

- ▶ Setting $\alpha = -1$, $\Gamma = \mathcal{H}(0)$ and $\bar{k} = -\mathcal{H}(0)$ the controller state characterizes the normalized total energy of (60), $x_c(t) = \mathcal{H}/\mathcal{H}(0)$ if $x_c(0) = 1$, and the normalized energy supplied by the controller, $x_c(t) = \Delta\mathcal{H}/\mathcal{H}_0$ if $x_c(0) = 0$.

Proposition

The boundary controller (63) with

$$G_c^\top = \frac{\alpha}{\Gamma T|_L} \begin{pmatrix} g(x_c, \mathbf{y}) \\ 0 \end{pmatrix} \quad (72)$$

where

$$g(x_c, \mathbf{y}) = \phi_L(x_c)(T - T^*)|_L + \phi_0(x_c)(T - T^*)|_0$$

with $\phi_0(x_c) \geq 0$ and $\phi_L(x_c) > \phi_0(x_c) \left(\frac{L^2 \phi_0(x_c)}{2k} - 1 \right)$, exponentially stabilizes (60) at the desired equilibrium profile T^* .

The proof follows from considering the following closed loop Lyapunov function

$$\mathcal{V} = \int_0^L \frac{1}{2} (T - T^*)^2 dz$$

where $T^* \in \mathcal{J}^*$ is the reachable dynamic equilibrium profile of the temperature.

Proposition

We consider now that the system is fully actuated. The boundary controller (63) with

$$G_c^\top = \frac{\alpha}{\Gamma} \left(\begin{pmatrix} -\frac{km^*}{T|_L} \\ -\frac{km^*}{T|_0} \end{pmatrix} + \frac{1}{c_v} \Phi(x_c) \begin{pmatrix} T(T - T^*)|_L \\ -T(T - T^*)|_0 \end{pmatrix} \right) \quad (73)$$

where $\Phi(x_c) = \Phi(x_c)^\top \geq 0$, exponentially stabilizes the closed loop system at the desired equilibrium profile T^* .

The proof follows from considering the following closed loop Lyapunov function

$$\mathcal{V} = \int_0^L \frac{1}{2} (T - T^*)^2 dz$$

where $T^* \in \mathcal{J}^*$ is the reachable dynamic equilibrium profile of the temperature.

Heat equation



We consider a copper bar of length $L = 0.1m$ and a cross-sectional area of $10^{-4}m^2$.

$$T_0 = T(\xi, 0) = 200\xi + 330, \forall \xi \in [0, 0.1]$$

the desired temperature equilibrium profile is defined as

$$T^* = 325, \forall \xi \in [0, 0.1]$$

The boundary controller acts on the entropy flux at the boundary $\xi = L$, i.e., $u = -q_s|L$.
To move T_0 to T^* we use the boundary controller described in Proposition 3.



Heat equation

Control design

Using the controller (63) with $\alpha = -1$, $\Gamma = \mathcal{H}_0$ and $\bar{k} = 0$, the controller state variable represents a measurement of the normalized total energy if $x_c(0) = 1$, i.e.,

$$x_c = \frac{1}{\mathcal{H}_0} \int_0^L u(z, t) dz.$$

Assuming temperature measurements at both boundaries we select

$$\phi_L(x_c) = 5x_c \quad \text{and} \quad \phi_0(x_c) = 10x_c,$$

Control law

$$\begin{aligned} \mathbf{u} &= - \frac{x_c}{T|_L} (5(T - T^*)|_L + 10(T - T^*)|_0) \\ &= - \frac{5(T - T^*)|_L + 10(T - T^*)|_0}{\mathcal{H}_0 T|_L} \int_0^L u(z, t) dz \end{aligned} \quad (74)$$

Heat equation

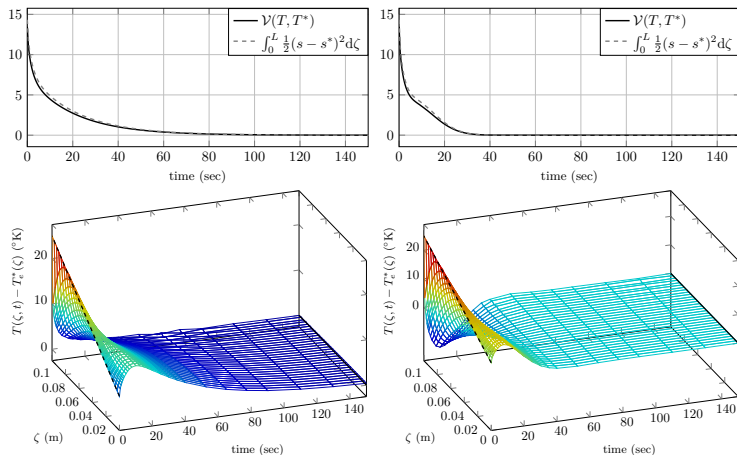


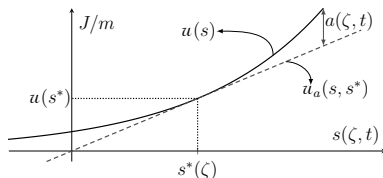
Figure: left: $\phi_L = 5x_c, \phi_0 = 0$, right: $\phi_L = 5x_c, \phi_0 = 10x_c$

Control design : The heat equation

Idea

- Use the Thermodynamic availability function as closed loop Lyapunov function.

$$\mathcal{A} = \int_0^L (u(s) - u_a(s)) dz$$



- Use Entropy Assignment to guarantee the convergence of trajectories.

Availability Based Interconnection

The boundary control feedback $v = \beta(\mathbf{y}) + v'$, with v' an auxiliary boundary input, maps (60), (61) into the target system

$$\partial_t s = \bar{r}_s \partial_z (\delta_s \mathcal{H}) + \partial_z (\bar{r}_s \delta_s \mathcal{H}) \quad (75)$$

$$\tilde{\mathbf{u}} = \Xi v' \quad (76)$$

where $\mathcal{H} = U$ and

$$\Xi = \begin{pmatrix} \left. \frac{\delta_s \mathcal{A}}{T} \right|_L & 0 \\ 0 & \left. \frac{\delta_s \mathcal{A}}{T} \right|_0 \end{pmatrix} \text{ and } v' = \begin{pmatrix} \lambda \left(\left. \frac{\partial_z (\delta_s \mathcal{A})}{T} \right|_L \right) \\ \lambda \left(\left. \frac{\partial_z (\delta_s \mathcal{A})}{T} \right|_0 \right) \end{pmatrix} \quad (77)$$

and $\bar{r}_s = \gamma_s \{ \mathcal{S} | \mathcal{A} \}$, if the following matching conditions are satisfied

$$\gamma_s \{ \mathcal{S} | \mathcal{H}_a \} \partial_z (\delta_s \mathcal{H}) + \partial_z (\gamma_s \{ \mathcal{S} | \mathcal{H}_a \} \delta_s \mathcal{H}) = 0 \quad (78)$$

$$\beta(\mathbf{y}) + \begin{pmatrix} \lambda \left(\left. \frac{\partial_z (\delta_s \mathcal{H}_a)}{T} \right|_L \right) \\ \lambda \left(\left. \frac{\partial_z (\delta_s \mathcal{H}_a)}{T} \right|_0 \right) \end{pmatrix} = 0 \quad (79)$$

Entropy Assignment

Let's consider the irreversible port-Hamiltonian system (60)-(76) with boundary control law

$$\tilde{\mathbf{u}} = -\Gamma \mathbf{y} \quad (80)$$

with $\Gamma = \Xi \Phi \Xi^\top$, and $\Phi = \Phi^\top > 0$, then the system is asymptotically stable. If Φ is defined by

$$\Phi = \begin{pmatrix} \frac{\phi_L}{T|_L} & 0 \\ 0 & \frac{\phi_0}{T|_0} \end{pmatrix} \quad (81)$$

where ϕ_L and ϕ_0 are strictly positive, the target temperature profile $T_e^* = m^* z + b^*$, $\forall z \in [0, L]$, is achievable from any initial condition T_0 . At the end the control is

$$\mathbf{u} = \beta(\mathbf{y}) - \Phi \Xi^\top \mathbf{y} \quad (82)$$



Control design



- ▶ Initial condition $T_0 = 303.15, \forall z \in [0, 0.1]$
- ▶ Target profile $T_e^* = 150z + 313.15, z \in [0, 0.1]$

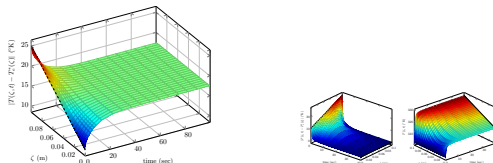


Figure: Behavior of the absolute error of temperature response with respect to desired equilibrium profile, using ABI (left) control and ABI-EA (right) control.

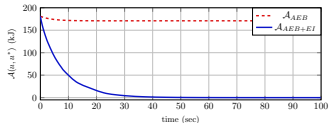


Figure: Available energy.



Outline

Context and motivation

Infinite dimensional Port Hamiltonian systems (PHS)

Control by interconnection and energy shaping

Observer design

Irreversible boundary controlled port Hamiltonian Systems

Conclusions and future works



Conclusions and future works

Conclusion

- ▶ Extension of energy shaping control design technique to in domain controlled DPS and to a class of boundary controlled IPHS.
- ▶ We proposed first ideas on observer design.
- ▶ In the in domain case energy shaping is achieved in an optimal way considering an early lumping approach and closed loop stability was proven.
- ▶ In the IPHS case we did not pay attention to existence of solutions.

Conclusions and future works

Conclusion

- ▶ Extension of energy shaping control design technique to in domain controlled DPS and to a class of boundary controlled IPHS.
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- ▶ In the in domain case energy shaping is achieved in an optimal way considering an early lumping approach and closed loop stability was proven.
- ▶ In the IPHS case we did not pay attention to existence of solutions.

Future works

- ▶ Study of the impact of the distribution of the patches on the achievable performances.
- ▶ Control design for a class of non linear PDE systems.
- ▶ Extension to 2D DPS.
- ▶ Extension to a larger class of irreversible PHS.



Thank you for your attention !





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