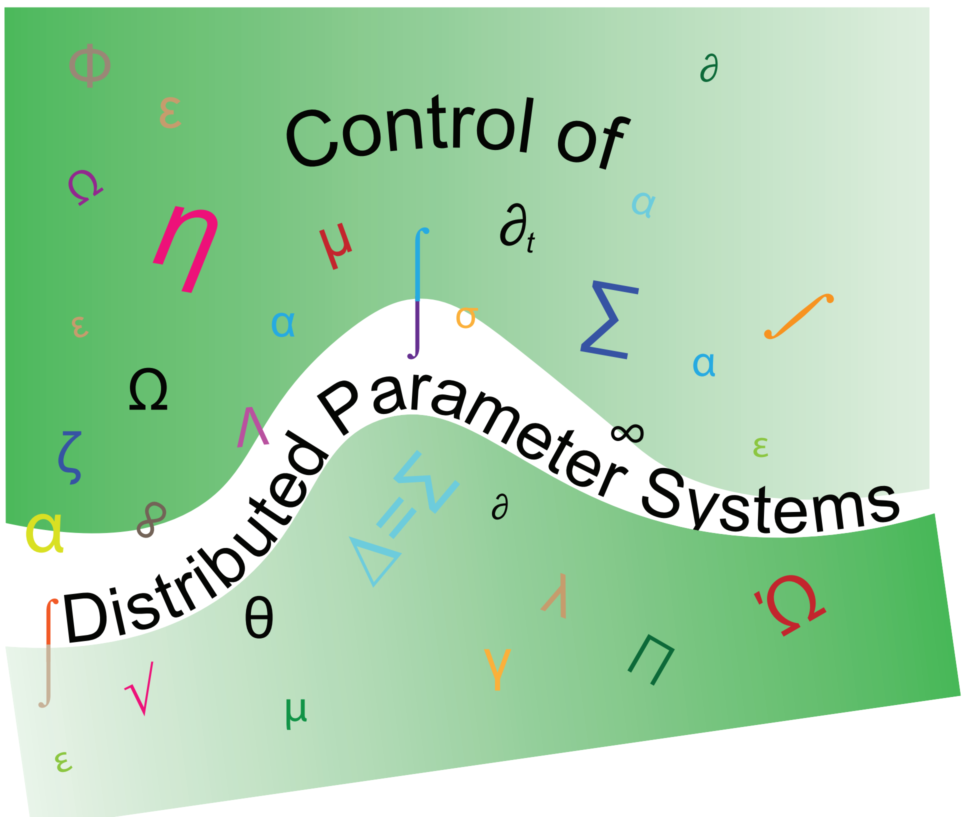

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Version March 21, 2024

Preface

The aim of these course notes is to give an engineering introduction to the theory of distributed parameter systems and its applications to port-Hamiltonian systems.

The field of distributed parameter systems (also known as infinite-dimensional systems) has become a well-established field within mathematics and systems theory. There are basically two approaches to infinite-dimensional linear systems theory: an abstract functional analytical approach and a partial differential equation (PDE) approach. There are excellent books dealing with infinite-dimensional linear systems theory, such as (in alphabetical order) Bensoussan, Da Prato, Delfour and Mitter [3], Curtain and Pritchard [6], Curtain and Zwart [7, 8], Fattorini [11], Luo, Guo and Morgul [19], Lasiecka and Triggiani [15, 16], Lions [17], Lions and Magenes [18], Staffans [23], and Tucsnak and Weiss [24].

Many physical systems can be formulated using a Hamiltonian framework. This class contains ordinary as well as partial differential equations. Each system in this class has a Hamiltonian, generally given by the energy function. In the study of Hamiltonian systems it is usually assumed that the system does not interact with its environment. However, for the purpose of control and for the interconnection of two or more Hamiltonian systems it is essential to take this interaction with the environment into account. This led to the class of port-Hamiltonian systems, see [25, 26]. The Hamiltonian/energy has been used to control a port-Hamiltonian system, see e.g. [4, 12, 21]. For port-Hamiltonian systems described by ordinary differential equations this approach is very successful, see the references mentioned above. Port-Hamiltonian systems described by partial differential equation is a subject of current research, see e.g. [1, 9, 13, 14, 20].

The material of this notes has been developed over a series of years. Javier Villegas [27] studied in his PhD-thesis a port-Hamiltonian approach to distributed parameter systems. The first setup of the book was written for a graduate course on control of distributed parameter systems for the Dutch Institute of Systems and Control (DISC) in the spring of 2009 which was attended by 25 PhD students. This material was adapted for the CIMPA-UNESCO-Marrakech School on Control and Analysis for PDE in May 2009. Over the last decade this material has been adapted to be suitable for a master course in Eindhoven. Then it was also decided to reduce the mathematics to a minimum, and concentrate on concepts and underlying ideas.[1] We hope that we have succeeded to make it into an engineering friendly guide to distributed parameter systems.

Yann Le Gorrec and Hans Zwart
February 2024

[1] For those who think that this book is very mathematical, please open [7] once

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Chapter 1
Introduction

In this chapter we shall present some of motivating examples, but we begin by expressing the aim of these lecture notes.

1.1 Aim

The aim of these lecture notes are to make the reader acquainted with control theory for systems described by partial differential equations. Modelling phenomena via partial differential equations is almost as old as modelling via ordinary differential equations. The first model described via a partial differential equation is the wave equation, which we will encounter a lot in these notes. As can be seen from this example our model class contains models in which there is a strong spatial and temporal behaviour. The first results towards controlling these systems appeared around the 1970's for the control of flexible antenna or solar panels of satellites.

Our models will almost always be linear and time-invariant. For these models we will set-up a state space theory analogue to finite dimensional state space theory. We show that these models possess a transfer function. Although these transfer functions are no longer rational, control design in the frequency domain has been done with it, and hardly differs from the classical control design.

In these notes we focus on how to use the existing theory of control for distributed parameter systems, and much less on the proofs. Hence our examples and exercises are there to show how the presented results can be used. The proofs of the results are quite hard and rely on many different branches of mathematics, normally not all thought to engineers. In the last

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section of every chapter we will give the necessary references to these proofs, and so the background material can be consulted.

In writing these notes we realised that we cannot do without certain concepts. They are included, and so there are sections on Hilbert spaces, operators, cweak solutions, spectra, etc.

1.2 Motivating examples

In order to motivate the development of a theory for linear infinite-dimensional systems, we present some simple examples of control problems that arise for spatially invariant, delay and distributed parameter (those described by partial differential equations) systems. These three special classes of infinite-dimensional systems occur most frequently in the applications.

Example 1.2.1 *Boundary control of a flexible beam*

The Euler Bernoulli beam equation model was used by Bailey and Hubbard [2] as a model of one of the arms of a satellite, consisting of a central hub with two or four flexible beams attached to it (see Figure 1.1).

Figure 1.1: Satellite composed of a central hub and two flexible solar panels.

The Euler Bernoulli beam equation model describes the transverse vibrations of a beam. To apply it, we use the schematic representation of the arm/beam attached to the satellite, see Figure 1.2. The displacement is given as $w(\zeta, t)$, where $\zeta \in [0, L]$ is position on the spatial axis, and t denote the time. Thus we assume that the beam has length L. The satellite is seen as a point mass attached at position $\xi = 0$. A piezoelectric film is bonded to the beam, which applies a bending moment to the beam if a voltage is applied to it. This voltage is the control input of the system and the angular velocity at the tip the measurement.

The displacement $w(\zeta, t)$ satisfies the PDE (Euler Bernoulli beam equa-

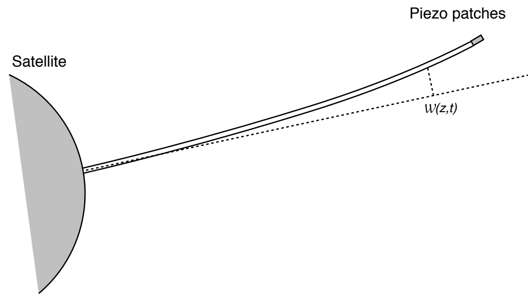

Figure 1.2: Schematic representation of the control problem.

tion model)

$$\rho \frac{\partial^2 w}{\partial t^2}(\zeta,t) + EI \frac{\partial^4 w}{\partial \zeta^4}(\zeta,t) = 0 \quad \text{for } 0 < \zeta < L, \tag{1.1}$$

where ρ is the mass density, and EI is an elasticity. Both are assumed to be constant. If we assume that the mass is so heavy compared to the beam that it will not move nor rotate we get the boundary conditions

$$w(0,t) = \frac{\partial w}{\partial \zeta}(0,t) \;=\; 0, \text{(no movement/rotation)} \tag{1.2}$$

$$\frac{\partial^3 w}{\partial \zeta^3}(0,t) = 0, \qquad \text{(no force at the tip)} \tag{1.3}$$

$$\frac{\partial^2 w}{\partial \zeta^2}(L,t) = -J \frac{\partial^3 w}{\partial t^2 \partial \zeta}(L,t) + u(t), \tag{1.4}$$

where the last models the bending moment at the tip of the beam. We add the measurement

$$y(t) = \frac{\partial^2 w}{\partial t \partial \zeta}(L,t). \tag{1.5}$$

This system is using the Euler Bernoulli beam equation model with clamped conditions at one side and bending control at the other side. We shall see later on that this equation can be reformulated as a second order (in space) equation by using the port-Hamiltonian framework. □

Example 1.2.2 *Heat equation.* In this example we consider the heat propagation in a uniform rod (cf. Figure 1.3). In absence of external source the balance equation on the internal heat energy per unit volume $h(\zeta,t)$ with $\zeta \in [0,L]$ is given by

$$\frac{\partial h}{\partial t}(\zeta,t) = -\frac{\partial q}{\partial \zeta}(\zeta,t), \tag{1.6}$$

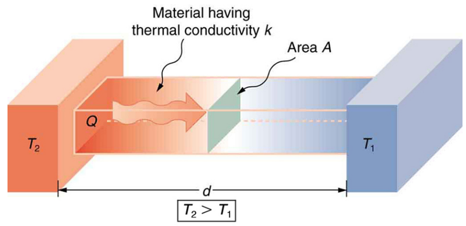

Figure 1.3: Heat conduction.

where q is the distributed heat flow. The internal heat energy density can be written as a function of the temperature $T(\zeta, t)$ as

$$h(\zeta, t) = c_v \rho T(\zeta, t), \tag{1.7}$$

where c_v is the heat capacity and ρ is the mass density. From Fourier's law, the heat flow is proportional to the negative gradient in temperature

$$q(\zeta, t) = -k \frac{\partial T}{\partial \zeta}(\zeta, t). \tag{1.8}$$

Combining the above equations gives

$$c_v \rho \frac{\partial T}{\partial t}(\zeta, t) = \frac{\partial}{\partial \zeta}\left(k \frac{\partial T}{\partial \zeta} \right)(\zeta, t), \tag{1.9}$$

in which we have allowed that c_v, ρ, and k depend on ζ. If they are constant, then we can rewrite the heat equation as

$$\frac{\partial T}{\partial t}(\zeta, t) = c \frac{\partial^2 T}{\partial \zeta^2}(\zeta, t). \tag{1.10}$$

which is its most familiar form. $\square$

Example 1.2.3 *Shallow water equation.* Fluid flows in irrigation channels or in moving fluid tanks can be represented by the shallow water equations. We consider a channel with a single reach and constant section (cf. Figure 1.4).

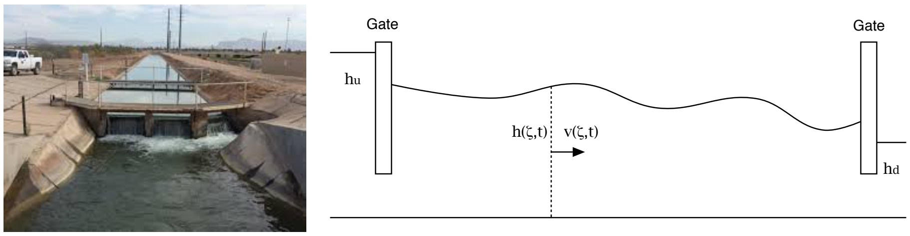

Figure 1.4: Single reach channel.

The shallow water equation can be simplified to 1-D Saint-Venant equation. This equation is derived by integrating the Navier-Stokes equation in the depth coordinate, assuming that the horizontal length scale is much greater than the vertical length scale. In this case the conservation of mass induces that the vertical velocity is neglectible. Moreover the vertical pressure gradients are hydrostatic and the horizontal gradients are due to the displacement of the pressure surface. In this case the velocity field is constant over the depth of the channel. Hence the free surface flow is driven by the boundary conditions. The model of the system is derived by writing balance equations on the mass and on the kinetic momentum, considering that the system is subject to the gravity force and hydrostatic pressure. In the case of channels of constant section the state variables are the height of the fluid $h(\zeta,t)$ and its velocity $v(\zeta,t)$. Balance equations lead to

$$\frac{\partial h}{\partial t}(\zeta,t) = -\frac{\partial}{\partial \zeta}\left(h(\zeta,t)v(\zeta,t)\right), \tag{1.11}$$

$$\frac{\partial v}{\partial t}(\zeta,t) = -\frac{\partial}{\partial \zeta}\left(\frac{1}{2}v^2(\zeta,t) + gh(\zeta,t)\right). \tag{1.12}$$

Considering control perspectives, the system system is actuated with the help of two gates at the boundary of the spatial domain. Control action is insured by a boundary feedback of the form

$$h(0,t)v(0,t) = \alpha u_1(t)\sqrt{h_u(t) - h(0,t)} \tag{1.13}$$

$$h(L,t)v(L,t) = \alpha u_2(t)\sqrt{h(L,t) - h_d(t)} \tag{1.14}$$

aiming at controlling the fluid level and the velocity profile within the channel. In (1.13), (1.14) $h_u(t)$ and $h_d(t)$ are the upstream and downstream fluid height, $u_1(t)$ and $u_2(t)$ are the control variables associated to the opening rate of the gates. □

We close this session with the first PDE ever derived, namely the wave equation.

Example 1.2.4 *Wave equation.* Consider a flexible string attached to a wall, as shown in Figure 1.5. The movements of the vibrations in the flexible

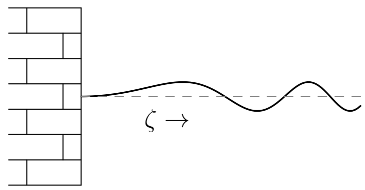

Figure 1.5: Vibrating string.

string are given by

$$\rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial w}{\partial \zeta}\right)(\zeta,t), \tag{1.15}$$

where ρ is the mass density, and T is the elasticity modulus. Since the string is attached to the wall, we have the boundary condition

$$w(0,t) = 0, \tag{1.16}$$

and since there is no force at the right-end,

$$T(L)\frac{\partial w}{\partial \zeta}(L,t) = 0. \tag{1.17}$$

As for the heat equation, this partial differential equation is normally written as

$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = c^2\frac{\partial^2 w}{\partial \zeta^2}(\zeta,t), \tag{1.18}$$

assuming the physical parameters are constant, with $c^2 = T/\rho$. □

Chapter 2
Solutions of PDE's, case studies

In the previous chapter we have seen some models described by partial differential equations (PDE's). In this chapter we consider the problem of finding solutions. This we do in two ways. We begin by showing that explicite expressions for the solution can be found for some simple examples. However, in general this is not possible, and so to obtain an idea of the shape of the solution, numerical techniques have to be used.

In the following chapter we focus on techniques proving the existence of the solutions. This may seem strange, since until now you will have encountered only differential equations that have a unique solution. For partial differential equations existence is not obvious. In fact for non-linear partial differential equations this is a long standing (still) open problem. For linear partial differential equations, the situation is much better, as we will see in the sequel, but by imposing the "wrong" boundary conditions existence of solutions may be lost, see Exercise 2.6.

2.1 Finding solutions

Example 2.1.1 Consider the simple partial differential equation

$$\frac{\partial x}{\partial t}(\zeta, t) = c\frac{\partial x}{\partial \zeta}(\zeta, t), \qquad t \geq 0, \zeta \in \mathbb{R}, \tag{2.1}$$

where c is a constant not equal to zero. This partial differential equation asks you to find a function x depending on two variables, t and ζ, such that

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the derivative with respect to t is c times its derivative with respect to ζ. It is not hard to see that

$$x(\zeta, t) = f(ct + \zeta)$$

is a solution of (2.1) provided f is a differentiable function.

If we add to the partial differential equation (2.1) the *initial condition*

$$x(\zeta, 0) = x_0(\zeta), \qquad \zeta \in \mathbb{R}, \tag{2.2}$$

then it is clear that f must equal the initial condition. Thus the solution becomes

$$x(\zeta, t) = x_0(\zeta + ct). \tag{2.3}$$

Without proof we state that this is the unique solution of (2.1) with initial condition (2.2). □

Example 2.1.2 We consider a similar PDE as in the previous example, but now on the bounded spatial interval $[0, 1]$ and c is assumed to be positive.

$$\frac{\partial x}{\partial t}(\zeta, t) = c\frac{\partial x}{\partial \zeta}(\zeta, t), \qquad t \geq 0, \zeta \in [0, 1], \tag{2.4}$$

with initial condition

$$x(\zeta, 0) = x_0(\zeta), \qquad \zeta \in [0, 1].$$

Inspired by the previous example we get the following solution

$$x(\zeta, t) = \begin{cases} x_0(\zeta + ct) & \zeta \in [0, 1],\ \zeta + ct < 1 \\ x_0(1) & \zeta \in [0, 1],\ \zeta + ct > 1 \end{cases} \tag{2.5}$$

provided x_0 is differentiable with derivative zero at $\zeta = 1$. However, this is not the only solution of (2.4). Another solution is given by (Please check it for yourself).

$$x(\zeta, t) = \begin{cases} x_0(\zeta + ct) & \zeta \in [0, 1],\ \zeta + ct < 1 \\ g(\zeta + ct) & \zeta \in [0, 1],\ \zeta + ct > 1, \end{cases} \tag{2.6}$$

where g is an arbitrary continuous differentiable function on $(1, \infty)$ satisfying $g(1) = x_0(1)$, and $\dot{g}(1) = \dot{x}_0(1)$.

Hence we have that the PDE (2.4) does not possess a unique solution. The reason for this is that we did not impose a *boundary condition*. If we impose the boundary condition $x(1,t) = 0$ for all t, then the unique solution is given by, see also (2.5),

$$x(\zeta,t) = \begin{cases} x_0(\zeta + ct) & \zeta \in [0,1],\ \zeta + ct < 1, \\ 0 & \zeta \in [0,1],\ \zeta + ct > 1. \end{cases} \tag{2.7}$$

Note that we assumed that x_0 satisfies the boundary condition as well, i.e., $x_0(1) = 0$. The rule of thumb is that for a first order PDE we need one boundary condition. However, this is only a rule of thumb. If we impose to the PDE (2.4) the boundary condition $x(0,t) = 0$, then this PDE does not possess any solution if $x_0 \neq 0$, see Exercise 2.6. $\square$

As stated in the previous example the order of the PDE indicates the number of boundary conditions. By the order of the PDE we mean the highest spatial derivative, i.e., the derivative with respect to ζ. Similar to ordinary differential equations the highest derivative with respect to time determines the number of initial conditions.

In the two previous examples we "saw" the solution. In the following example we illustrate another technique for finding solution. This technique is known as *separation of variables*.

Example 2.1.3 Consider the *heat equation* of Example 1.2.2 on the spatial domain $[0,1]$,

$$\frac{\partial x}{\partial t}(\zeta,t) = c\frac{\partial^2 x}{\partial \zeta^2}(\zeta,t), \qquad t \geq 0, \tag{2.8}$$

where $c > 0$. We assume no heat flux at the boundary, i.e., the boundary conditions are

$$\frac{\partial x}{\partial \zeta}(0,t) = 0 = \frac{\partial x}{\partial \zeta}(1,t), \qquad t \geq 0. \tag{2.9}$$

The initial condition is given as

$$x(\zeta,0) = x_0(\zeta), \quad 0 \leq \zeta \leq 1. \tag{2.10}$$

To solve this partial differential equation, we first ignore the initial condition. That is we want to find a function which satisfies (2.8) and (2.9), but not necessarily (2.10).

We begin by trying to find a solution of (2.8) and (2.9) which is of the form

$$x(\zeta,t) = f(\zeta)g(t). \tag{2.11}$$

Since the solution is splitted into two functions both only depending on one of the variables, this technique is known as *separation of variables*. Since the candidate solution (2.11) has to satisfy (2.8), we find for $\zeta \in [0,1]$, $t > 0$ that

$$f(\zeta)\frac{dg}{dt}(t) = \frac{\partial x}{\partial t}(\zeta,t) = c\frac{\partial^2 x}{\partial \zeta^2}(\zeta,t) = c\frac{d^2 f}{d\zeta^2}(\zeta)g(t). \tag{2.12}$$

Assume now that f and g are nowhere zero, then we may divide the above equation by $f(\zeta)g(t)$. This gives the equation

$$\frac{\frac{dg}{dt}(t)}{g(t)} = c\frac{\frac{d^2 f}{d\zeta^2}(\zeta)}{f(\zeta)}. \tag{2.13}$$

Now on the left hand-side we have a function only depending on t, whereas on the right hand-side we have a function depending only on ζ. Since t and ζ are independent, we must have that they are both equal to a (the same) constant. Thus from (2.13) we conclude that

$$\frac{\frac{dg}{dt}(t)}{g(t)} = \lambda \quad \Leftrightarrow \quad \frac{dg}{dt}(t) = \lambda g(t) \tag{2.14}$$

$$c\frac{\frac{d^2 f}{d\zeta^2}(\zeta)}{f(\zeta)} = \lambda \quad \Leftrightarrow \quad c\frac{d^2 f}{d\zeta^2}(\zeta) = \lambda f(\zeta). \tag{2.15}$$

The first equation is easy to solve and gives

$$g(t) = g(0)e^{\lambda t}. \tag{2.16}$$

For the solution of the second differential equation, we also use the boundary conditions (2.9). These boundary conditions imply that not all λ's are possible, if we want to have that f is not identically zero. We find that (see Exercise 2.2)

$$f(\zeta) = \alpha\cos(n\pi\zeta), \quad \lambda = -n^2\pi^2 c, \quad n \in \mathbb{N} = \{0,1,2,\cdots\}. \tag{2.17}$$

Note that the constant function is also included by $n = 0$. The solutions in (2.17) imply that the lambda's in (2.16) are restricted as well. So solutions of the form (2.11) exists, and they are given by

$$x_n(\zeta,t) = \alpha g(0)e^{-n^2\pi^2 ct}\cos(n\pi\zeta), \qquad n \in \mathbb{N}. \tag{2.18}$$

So the positive outcome of our endeavour is that we have not found one solution of the form (2.11), but infinitely many. The drawback is that we cannot allow for arbitrary initial conditions. Namely for the solutions (2.18) there holds

$$x_n(\zeta,0) = \alpha g(0)\cos(n\pi\zeta) = \alpha_n\cos(n\pi\zeta).$$

Hence only if the initial condition is a special sinusoid, then we have obtained a solution of (2.8)–(2.10).

To find more solutions, we use the linearity of the PDE (2.8) with boundary conditions (2.9). This means that if $z(\zeta,t)$ and $y(\zeta,t)$ satisfy (2.8) and the boundary conditions (2.9), then for any $\alpha,\beta \in \mathbb{C}$ the function $\alpha z(\zeta,t)+\beta y(\zeta,t)$ satisfies (2.8) and (2.9). (The reader should check this for her/himself.) Thus

$$\alpha_1 e^{-\pi^2 ct}\cos(\pi\zeta)+\alpha_2 e^{-4\pi^2 ct}\cos(2\pi\zeta)$$

is a solution, for any α_1 and α_2. By repeating the argument, we find for any $N>0$ and for any choice of $\alpha_0,\alpha_1,\dots,\alpha_N$ that

$$S_N(\zeta,t)=\sum_{n=0}^{N}\alpha_n e^{-n^2\pi^2 ct}\cos(n\pi\zeta) \tag{2.19}$$

satisfies (2.8) and (2.9). Now simply evaluating this solution for $t=0$, we see that the associated initial condition equals

$$S_N(\zeta,0)=\sum_{n=0}^{N}\alpha_n\cos(n\pi\zeta). \tag{2.20}$$

So we can now allow for any initial condition which is a (finite) linear combination of our special sinusoids. On the left hand-side we have a function defined on the spatial interval $[0,1]$, whereas on the right hand-side we have a sum of sinusoids. This is very similar to the Fourier cosine series. Roughly speaking, the Fourier series tells us that a periodic function q of period T which is even, i.e., $q(-\zeta)=q(\zeta)$ can be decomposed as

$$q(\zeta)=\sum_{n=0}^{\infty}b_n\cos(n\omega_0\zeta), \tag{2.21}$$

where $\omega_0=2\pi/T$ and

$$b_n=\begin{cases}\frac{2}{T}\int_{-T/2}^{T/2}q(\zeta)\cos(n\omega_0\zeta)d\zeta & n\geq 1,\\ \frac{1}{T}\int_{-T/2}^{T/2}q(\zeta)d\zeta & n=0.\end{cases} \tag{2.22}$$

If we now see the initial condition $x_0(\zeta)$ as the restriction of an even and periodic function (with period 2) to the interval $[0,1]$, then we have that $\omega_0=\pi$,

$$x_0(\zeta)=\sum_{n=0}^{\infty}b_n\cos(n\pi\zeta), \tag{2.23}$$

with

$$b_n = 2\int_0^1 x_0(\zeta)\cos(n\pi\zeta)d\zeta, \quad n \geq 1, \text{ and } b_0 = \int_0^1 x_0(\zeta)d\zeta. \tag{2.24}$$

Hence based on (2.19), (2.20) and (2.23) we expect that the solution of (2.8)–(2.10) is given by

$$x(\zeta,t) = \sum_{n=0}^{\infty} b_n e^{-n^2\pi^2 ct}\cos(n\pi\zeta), \tag{2.25}$$

with b_n is given by (2.24).

Already when Fourier introduced his series expansion, now known as the Fourier series, there was a debate on how to interpret this infinite sum, (2.21) and (2.23). Very soon it became clear that when q (or x_0) is piecewise smooth, then (2.21) (or (2.23)) holds pointwise, except for the points where there is a jump. A hundred years later it was shown that for functions which are square integrable, i.e., $\int_0^T |q(\zeta)|^2 d\zeta < \infty$, (2.21) also holds but in a weaker sense. Namely the integral over the (squared) error goes to zero when we take more and more terms. Thus

$$\lim_{N\to\infty}\int_0^T \left| q(\zeta) - \sum_{n=0}^{N} b_n\cos(n\pi\zeta)\right|^2 d\zeta = 0.$$

Fifties years later it was shown that for initial conditions x_0 that are square integrable, the expression (2.25) is the solution of (2.8)–(2.10), but also in a weaker sense. We shall say some more about this in the following chapter.□

As we have seen in the modelling chapter, the wave equation models a completely different behaviour than the heat equation, but as will be shown next, obtaining its solution goes very similar.

Example 2.1.4 Consider the homogeneous wave equation with no force at the boundary. For simplicity, we assume that the length is one. Thus the model is given by, see also Example 1.2.4

$$\frac{\partial^2 x}{\partial t^2}(\zeta,t) = c^2\frac{\partial^2 x}{\partial \zeta^2}(\zeta,t), \qquad t \geq 0, \zeta \in [0,1], \tag{2.26}$$

with no force at the boundary, i.e., with the boundary condition

$$c^2\frac{\partial x}{\partial \zeta}(0,t) = 0 = c^2\frac{\partial x}{\partial \zeta}(1,t), \qquad t \geq 0 \tag{2.27}$$

and initial condition

$$x(\zeta,0) = x_0(\zeta), \quad \frac{\partial x}{\partial t}(\zeta,t) = x_1(\zeta) \quad 0 \le \zeta \le 1. \tag{2.28}$$

Thus we prescribe the initial position and the initial velocity.

To solve this partial differential equation, we first ignore the initial condition. That is, we want to find a function which satisfies (2.26) and (2.27). We proceed as in Example 2.1.3. Thus we try to find a solution of the form

$$x(\zeta,t) = f(\zeta)g(t),$$

where we assume that f and g are nowhere zero. As in the previous example this leads to the following ordinary differential equations

$$\frac{d^2g}{dt^2}(t) = \lambda g(t) \tag{2.29}$$

and

$$\frac{d^2f}{d\zeta^2}(\zeta) = \frac{\lambda}{c^2} f(\zeta), \qquad c^2\frac{df}{d\zeta}(0) = 0 = c^2\frac{df}{d\zeta}(1). \tag{2.30}$$

Since this last equation is the same as equation (2.15) we already know that

$$f(\zeta) = \alpha\cos(n\pi\zeta), \quad \lambda = -n^2\pi^2c^2, \quad n \in \mathbb{N}. \tag{2.31}$$

For these λ's we can solve (2.29) and find for $n \ge 1$

$$g(t) = \beta\cos(n\pi ct) + \gamma\sin(n\pi ct) \tag{2.32}$$

and for $n = 0$

$$g(t) = \beta + \gamma t. \tag{2.33}$$

So for $n = 0, 1, \cdots$, we have found the following solution of (2.26)–(2.27).

$$x_n(\zeta,t) = \begin{cases} a_n\cos(n\pi\zeta)\cos(n\pi ct) + b_n\cos(n\pi\zeta)\sin(n\pi ct) & n \ge 1, \\ a_0 + b_0 t & n = 0. \end{cases}$$

It is easy to show (please check) that an arbitrary sum of these solutions, i.e.,

$$S_N(\zeta,t) = a_0 + b_0 t + \sum_{n=1}^{N} a_n\cos(n\pi\zeta)\cos(n\pi ct) + b_n\cos(n\pi\zeta)\sin(n\pi ct) \tag{2.34}$$

is again a solution of (2.26)–(2.27). The initial condition associated to this solution is

$$S_N(\zeta, 0) = a_0 + \sum_{n=1}^{N} a_n \cos(n\pi\zeta). \tag{2.35}$$

Hence again we recognise on the right hand-side a Fourier cosine series. We did not only have the initial position as an initial condition, but also the initial velocity, see (2.28). From (3.34) we that

$$\frac{\partial S_N}{\partial t}(\zeta, 0) = b_0 + \sum_{n=1}^{N} b_n n\pi c \cos(n\pi\zeta). \tag{2.36}$$

Again we recognise a Fourier cosine series, but the coefficients in front of the cosine terms are not directly the Fourier coefficients.

Based on the above the assert the following solution of (2.26)–(2.28)

$$x(\zeta, t) = a_0 + b_0 t + \sum_{n=1}^{\infty} a_n \cos(n\pi\zeta)\cos(n\pi ct) + b_n \cos(n\pi\zeta)\sin(n\pi ct) \tag{2.37}$$

with

$$a_n = 2\int_0^1 x_0(\zeta)\cos(n\pi\zeta)d\zeta, \quad n \geq 1, \text{ and } a_0 = \int_0^1 x_0(\zeta)d\zeta \tag{2.38}$$

and

$$b_n = \frac{2}{n\pi c}\int_0^1 x_1(\zeta)\cos(n\pi\zeta)d\zeta, \quad n \geq 1, \text{ and } b_0 = \int_0^1 x_1(\zeta)d\zeta. \tag{2.39}$$

$\square$

In the above examples we have constructed solutions of PDE's. However, it is good to remark that there are only a handful of PDE's which you can solve by hand. We decided to present some of these, because it gives an intuition for the solutions, and they serve as examples for more complicated examples. Hence in general the emphasis will not lie on the exact expression of the solution, but on knowing that the PDE with appropriate boundary and initial conditions possesses a unique solution, and to know properties of these. Finding this solution of the PDE is typically done via numerical techniques.

2.2 Exercises

2.1. Check that the function given in (2.5) is a solution of the PDE (2.4).

2.2. Check that the function defined in equation (2.19) is a solution of the PDE (2.8) and satisfies the boundary conditions (2.9).

2.3. In the solutions of Examples 2.1.3 and 2.1.4 we have seen that the functions are expressed using sinusoids and exponential function. These functions can only be evaluated at a dimension-less argument. On the other hand, our PDE's expresses physical processes, and so they contain units. To see that these two can meet, we study the heat equation of Example 2.1.3 once more, but pay extra attention to the units. So we consider

$$\frac{\partial x}{\partial t}(\zeta,t) = c\frac{\partial^2 x}{\partial \zeta^2}(\zeta,t), \qquad \zeta \in [0,L], \quad t \geq 0,$$

with boundary conditions

$$\frac{\partial x}{\partial \zeta}(0,t) = 0 = \frac{\partial x}{\partial \zeta}(L,t), \qquad t \geq 0.$$

We take as units for t, ζ and x, seconds, meter, and Kelvin, respectively.

(a) Determine the dimension of c.

(b) Find a non-zero solution of the PDE satisfying the boundary conditions.

(c) Check that the arguments of the functions, like cos and the exponential, found in the previous part are dimension-less.

2.4. Consider a bar of length one whose temperature is zero at both ends. The model is given as

$$\begin{aligned} &\frac{\partial w}{\partial t}(\zeta,t) = \alpha\frac{\partial^2 w}{\partial \zeta^2}(\zeta,t) \qquad w(\zeta,0) = w_0(\zeta), \\ &w(0,t) = 0, \quad w(1,t) = 0. \end{aligned} \tag{2.40}$$

$w(\zeta,t)$ represents the temperature at position $\zeta \in [0,1]$ at time $t \geq 0$ and $w_0(\zeta)$ the initial temperature profile. Furthermore, α is a positive constant.

(a) Use separation of variable to determine infinitely many non-zero solutions of (2.40).

(b) Show that a finite sum of solutions found in part (a) still satisfies (2.40).

(c) For which initial conditions w_0 have you found the solution? What would be your guess for the solution of (2.40) for arbitrary initial conditions?

2.5. Consider the vibrating string attached to a wall at both ends, described by the PDE

$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = c^2 \frac{\partial^2 w}{\partial \zeta^2}(\zeta,t), \qquad 0 \le \zeta \le L, t \ge 0 \tag{2.41}$$

with boundary condition

$$w(0,t) = 0 = w(L,t), \qquad t \ge 0 \tag{2.42}$$

(a) If w, ζ and t have units meter, meter, and seconds, respectively, determine the unit of c.

(b) Use separation of variable to determine a non-zero solution of (2.41)–(2.42).

(c) Show that a finite sum of solutions found in part b, still satisfy (2.41)–(2.42).

2.6. In this exercise you will prove that the partial differential equation

$$\frac{\partial x}{\partial t}(\zeta,t) = \frac{\partial x}{\partial \zeta}(\zeta,t), \qquad \zeta \in [0,1], \quad t \ge 0,$$

with non-zero initial condition $x(\zeta,0) = x_0(\zeta)$ and boundary condition

$$x(0,t) = 0$$

does not possess a solution.

(a) Assume that the PDE possesses a solution, show that for any continuously differentiable f satisfying $f(\eta) = 0$ for $\eta \ge 1$ the function $q(t) = \int_0^1 f(\zeta + t)x(\zeta,t)d\zeta$ has derivative zero for $t \ge 0$.

(b) For the function defined in the previous item, show that $q(1) = 0$, independently of the value of f in the interval $[0,1)$.

(c) Conclude from the previous two items that $\int_0^1 f(\zeta)x_0(\zeta)d\zeta$ is zero for all continuously differentiable functions f.

(d) Prove that for any non-zero initial condition the PDE with the chosen boundary condition does not possess a solution in positive time.

Chapter 3

Solutions of linear PDE's, weak, strong, and general

In the previous chapter we have seen that for some standard PDE's a solution could be found. For instance, for the transmission line, see Example 2.1.1,

$$\frac{\partial x}{\partial t}(\zeta,t) = c\frac{\partial x}{\partial \zeta}(\zeta,t), \qquad t \geq 0, \zeta \in \mathbb{R}, \tag{3.1}$$

$$x(\zeta,0) = x_0(\zeta) \tag{3.2}$$

the solution was given by

$$x(\zeta,t) = x_0(ct+\zeta). \tag{3.3}$$

If x_0 is smooth, then it is easy to check that this is a solution. However, the expression (3.3) is also well defined for $x_0(\zeta) = |\cos(\zeta)|$, or for x_0 being the step function, i.e.,

$$x_0(\zeta) = \begin{cases} 1 & \zeta \geq 0 \\ 0 & \zeta < 0. \end{cases}$$

We can clearly imagine that this step-function is moving with speed c over the real line, and in that sense it should be a solution of the transmission line equation (3.1), although it is not differentiable everywhere. To include this as a solution, the concept of weak solution will be introduced.

[0]March 21, 2024

3.1 Weak and strong solutions

Consider a PDE with initial and boundary conditions. We say that $w(\zeta, t)$ is a *strong solution* or *classical solution* when

1. w is sufficiently times differentiable with respect to ζ and t,
2. w satisfies the PDE, and
3. w satisfies the initial and boundary conditions.

If we return to our example in the beginning of this chapter, i.e., (3.1)–(3.3), then we see that for $x_0(\zeta) = \cos(\zeta)$, the function $x(\zeta, t) := \cos(t + c\zeta)$ is a classical/strong solution. Similarly, the function

$$w(\zeta) := \sum_{n=0}^{N} \alpha_n e^{-n^2\pi^2 t} \cos(n\pi\zeta) \tag{3.4}$$

is a classical/strong solution of the heat equation (2.8) with boundary condition (2.9) with initial condition $x_0(\zeta) = \sum_{n=0}^{N} \alpha_n \cos(n\pi\zeta)$. Please check this yourself.

As mention in the introduction of this chapter and in the previous chapter, it seems that classical solutions are not the whole story. For instance, we would like to take $N = \infty$ in (3.4) or to allow for a step function in (3.2). We see that one difficulty lies in the condition that have to differentiate the solution. This we can bypass, by integrating the equation.

Consider again the PDE (3.1). We multiply it with a smooth function of ζ which is zero at $\zeta = \infty$ and $\zeta = -\infty$ and integrate this with respect to ζ

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(\zeta) \frac{\partial x}{\partial t}(\zeta, t) d\zeta &= \int_{-\infty}^{\infty} \phi(\zeta) c \frac{\partial x}{\partial \zeta}(\zeta, t) d\zeta \\ &= -\int_{-\infty}^{\infty} \frac{d\phi}{d\zeta}(\zeta) c x(\zeta, t) d\zeta,\end{aligned}$$

where we have applied integration by parts with respect to the ζ-variable and used that ϕ is zero at plus and minus infinity. Integrating the above equality with respect to t, we find

$$\begin{aligned}-\int_0^{t_f} \int_{-\infty}^{\infty} \frac{d\phi}{d\zeta}(\zeta) c x(\zeta, t) d\zeta dt &= \int_0^{t_f} \int_{-\infty}^{\infty} \phi(\zeta) \frac{\partial x}{\partial t}(\zeta, t) d\zeta dt \\ &= \int_{-\infty}^{\infty} \phi(\zeta) x(\zeta, t_f) d\zeta - \int_{-\infty}^{\infty} \phi(\zeta) x_0(\zeta) d\zeta,\end{aligned}$$

where we have interchanged the order of integrating, and used the initial condition. So we have reformulated the PDE (3.1) with initial condition (3.2) to the integral equation

$$\int_{-\infty}^{\infty} \phi(\zeta)x(\zeta,t_f)d\zeta - \int_{-\infty}^{\infty} \phi(\zeta)x_0(\zeta)d\zeta = \\ -\int_0^{t_f}\int_{-\infty}^{\infty} \frac{d\phi}{d\zeta}(\zeta)cx(\zeta,t)d\zeta dt \qquad (3.5)$$

which has to hold for all $t_f > 0$ and all smooth ϕ which are zero at $\pm\infty$. Since we don't need to differentiate x anymore, we see that (3.5) has a meaning for a larger class solutions. These solutions we call weak, i.e., a *weak solution* is defined by the following two properties;

1. It satisfied the integrated form of the PDE;

2. It satisfies the initial condition.

We illustrate this by showing that (3.3) is a weak solution of (3.1)–(3.2)

Example 3.1.1 Let x_0 be a piecewise continuous function, and let x be given by (3.3). To see that this is a weak solution of weak solution of (3.1)–(3.2) we must check (3.5). After inserting the candidate weak solution, we do the substitution $\eta = ct + \zeta$ in the ζ-integral

$$\begin{aligned}
\int_0^{t_f}\int_{-\infty}^{\infty} \frac{d\phi}{d\zeta}(\zeta)cx(\zeta,t)d\zeta dt &= \int_0^{t_f}\int_{-\infty}^{\infty} \frac{d\phi}{d\zeta}(\zeta)cx_0(ct+\zeta)d\zeta dt \\
&= \int_0^{t_f}\int_{-\infty}^{\infty} \frac{d\phi}{d\eta}(\eta - ct)cx_0(\eta)d\eta dt \\
&= \int_0^{t_f}\int_{-\infty}^{\infty} (-1)\frac{d\phi}{dt}(\eta - ct)x_0(\eta)d\eta dt \\
&= \int_{-\infty}^{\infty}\int_0^{t_f} (-1)\frac{d\phi}{dt}(\eta - ct)x_0(\eta)dt d\eta \\
&= \int_{-\infty}^{\infty} (-1)c\phi(\eta - ct_f)x_0(\eta)d\eta - \int_{-\infty}^{\infty} (-1)\phi(\eta)x_0(\eta)d\eta \\
&= -\int_{-\infty}^{\infty} \phi(\zeta)x(\zeta,t_f)d\zeta + \int_{-\infty}^{\infty} \phi(\zeta)x_0(\zeta)d\zeta,
\end{aligned}$$

where in the last step with did the integral substitutions $\eta = ct_f + \zeta$ and $\eta = \zeta$, respectively. So we see that in the above we only used that x_0 is integrable and so for many initial conditions we have shown that (3.3) is a weak solution. $\square$

In the previous example we did not need to consider boundary conditions. To see how boundary conditions pop up, we consider the heat equation of Example 1.2.2 and 2.1.3.

Example 3.1.2 Consider the heat equation with boundary and initial conditions

$$\frac{\partial x}{\partial t}(\zeta,t) = c\frac{\partial^2 x}{\partial \zeta^2}(\zeta,t), \qquad t \geq 0, \zeta \in [0,1], \tag{3.6}$$

$$\frac{\partial x}{\partial \zeta}(0,t) = 0 = \frac{\partial x}{\partial \zeta}(1,t), \qquad t \geq 0, \tag{3.7}$$

$$x(\zeta,0) = x_0(\zeta), \qquad \zeta \in [0,1]. \tag{3.8}$$

As before, we multiply the PDE with the smooth function ϕ and integrate it with respect to ζ. Performing now integration by parts twice, we obtain

$$\begin{aligned}\int_0^1 \phi(\zeta)\frac{\partial x}{\partial t}(\zeta,t)d\zeta &= \int_0^1 \phi(\zeta)c\frac{\partial^2 x}{\partial \zeta^2}(\zeta,t)d\zeta \\ &= \left[\phi(\zeta)c\frac{\partial x}{\partial \zeta}(\zeta,t)\right]_0^1 - \int_0^1 \frac{d\phi}{d\zeta}(\zeta)c\frac{\partial x}{\partial \zeta}(\zeta,t)d\zeta \\ &= 0 - \int_0^1 \frac{d\phi}{d\zeta}(\zeta)c\frac{\partial x}{\partial \zeta}(\zeta,t)d\zeta \\ &= -\left[\frac{d\phi}{d\zeta}(\zeta)cx(\zeta,t)\right]_0^1 + \int_0^1 \frac{d^2\phi}{d\zeta^2}(\zeta)cx(\zeta,t)d\zeta,\end{aligned}$$

where we used the boundary conditions. Before we integrate this expresion with respect to t, we pay some more attention to the boundary term $\left[\frac{d\phi}{d\zeta}(\zeta)cx(\zeta,t)\right]_0^1 = c\frac{d\phi}{d\zeta}(1)x(1,t) - c\frac{d\phi}{d\zeta}(0)cx(0,t)$. If we would have that x is a continuous function of ζ, evaluating it in $\zeta = 0$ or $\zeta = 1$ is no problem, but as we have seen in the transmission line, we could allow for functions with jumps, and then it becomes harder. Moreover, looking that the (candidate) solution (2.23), we are forced that the series $\sum_{n=0}^{\infty} b_n$ $(= x_0(0))$ has a meaning. Since the aim is to allow for as many initial conditions as possible, we choose another way out. We just assume that $\frac{d\phi}{d\zeta}(\zeta)$ is zero at $\zeta = 0$ and at $\zeta = 1$. Since ϕ is chosen by ourselves, this is allowed. So for these ϕ's there holds

$$\int_0^1 \phi(\zeta)\frac{\partial x}{\partial t}(\zeta,t)d\zeta = \int_0^1 \frac{d^2\phi}{d\zeta^2}(\zeta)cx(\zeta,t)d\zeta$$

which leads to the (time) integrated form;

$$\int_0^1 \phi(\zeta)x(\zeta,t_f)d\zeta - \int_0^1 \phi(\zeta)x_0(\zeta)d\zeta = \int_0^{t_f}\int_0^1 \frac{d^2\phi}{d\zeta^2}(\zeta)cx(\zeta,t)d\zeta dt, \tag{3.9}$$

which has to be satisfied for all $t_f > 0$ and all smooth ϕ satisfying $\frac{d\phi}{d\zeta}(0) = \frac{d\phi}{d\zeta}(1) = 0$. $\square$

In the previous examples we saw how we can extend the concept of a solution by no longer requiring that the function is differentiable. However, for the integrated form like (3.9) it is not immediately clear that there will exist a solution. To formulate existence theory, we have the introduce some notation and concepts.

3.2 Weighted L^2-spaces

In all our examples we have seen functions. In this section we introduce some function spaces, i.e., sets whose elements are functions. They will appear as our state spaces.

Definition 3.2.1. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ with $a < b$ be given. Let $w : (a, b) \mapsto \mathbb{R}$ be a (strictly) positive function[1]. We say that $f : (a, b) \mapsto \mathbb{R}$ is in the weighted L^2-space $L^2_w(a, b)$ whenever

$$\int_a^b f(\zeta)^2 w(\zeta) d\zeta < \infty. \tag{3.10}$$

When $w(\zeta) = 1$ for all $\zeta \in (a, b)$, then we simply write $L^2(a, b)$.

When f may take complex values, (3.10) is replaced by

$$\int_a^b |f(\zeta)|^2 w(\zeta) d\zeta < \infty. \tag{3.11}$$

and the space is denoted by $L^2_w((a, b); \mathbb{C})$. $\square$

Elements from $L^2_w((a, b); \mathbb{C})$ and $L^2_w(a, b)$ possess nice properties which we will use a lot. We list them first for $L^2_w(a, b)$ and later extend them to vector valued functions.

1. If $f, g \in L^2_w(a, b)$, then $\alpha f + \beta g \in L^2_w(a, b)$ for all $\alpha, \beta \in \mathbb{R}$.

2. For $f, g \in L^2_w(a, b)$ we define their *inner product* as

$$\langle f, g\rangle_w = \int_a^b f(\zeta) g(\zeta) w(\zeta) d\zeta. \tag{3.12}$$

[1] Thus $\inf_{\zeta\in(a,b)} w(\zeta) > 0$ or equivalently there exists an $m > 0$ such that $w(\zeta) \geq m$ for all $\zeta \in (a, b)$.

3. The L^2_w-*norm* of f is defined as

$$\|f\|_w = \sqrt{\int_a^b f(\zeta)^2 w(\zeta) d\zeta}. \tag{3.13}$$

4. For the norm and inner product the following properties hold;
   (a) $\langle \alpha f + \beta g, h\rangle_w = \alpha\langle f, h\rangle_w + \beta\langle g, h\rangle_w$; $\alpha, \beta \in \mathbb{R}$
   (b) $\|f\|_w^2 = \langle f, f\rangle_w$
   (c) $|\langle f, h\rangle_w| \leq \|f\|_w \|g\|_w$ (*Cauchy-Schwarz inequality*)

5. Let $\{f_n, n \in \mathbb{N}\}$ be a sequence in $L^2_w(a, b)$ satisfying that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|f_n - f_m\|_w < \varepsilon$ for all $n, m \geq N$. Then there exists an $f \in L^2_w(a, b)$ such that $f_n \to f$, i.e., $\|f_n - f\|_w \to 0$ as $n \to \infty$.

For $f, g, h \in L^2((a, b); \mathbb{C})$ the inner product and norm must be adjusted to

$$\langle f, g\rangle_w = \int_a^b f(\zeta)\overline{g(\zeta)} w(\zeta) d\zeta,$$

$$\|f\|_w = \sqrt{\int_a^b |f(\zeta)|^2 w(\zeta) d\zeta}.$$

The properties stay the same, but in item 1. and 4.(a), α and β may be chosen to be complex. Property 5. is known as the *completeness property*. Note that $\mathbb{R}$ has a similar property.

It is easy to see that the zero function has norm zero. However, so has the function which is zero except in one point. To overcome this we say that f is the zero function (in $L^2(a, b)$) when it is zero almost everywhere. Similarly, we say that $f = g$ (in $L^2(a, b)$) when they are equal almost everywhere.

From courses like linear algebra, we know that for vectors in $\mathbb{R}^n$ we have the Euclidian inner product

$$\langle v, w\rangle = v^T w, \qquad v, w \in \mathbb{R}^n$$

which has similar properties as those stated above. So we have extended this (Euclidian) inner product between vectors to an inner product between functions. We can combine the two inner products to build an inner product on vector valued functions. We recall that the $n \times n$ matrix Q is *strictly positive* when it is symmetric, $Q^T = Q$, and $v^T Q v > 0$ for all non-zero $v \in \mathbb{R}^n$.

Definition 3.2.2. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ with $a < b$ be given. Let $W : (a, b) \mapsto \mathbb{R}^{n \times n}$ be a strictly positive matrix valued function. We say that $f : (a, b) \mapsto \mathbb{R}^n$ is in the weighted L^2-space $L^2_W((a, b); \mathbb{R}^n)$ whenever

$$\int_a^b f(\zeta)^T W(\zeta) f(\zeta) d\zeta < \infty. \tag{3.14}$$

When $w(\zeta) = I$ for all $\zeta \in (a, b)$, then we simply write $L^2((a, b); \mathbb{R}^n)$. $\square$

The inner product and norm on $L^2_W((a, b); \mathbb{R}^n)$ are given by

$$\langle f, g \rangle_W = \int_a^b f(\zeta)^T W(\zeta) g(\zeta) d\zeta, \quad \|f\|_W = \sqrt{\int_a^b f(\zeta)^T W(\zeta) f(\zeta) d\zeta}.$$

They too possess the same properties as stated for the scalar case. If we don't specify which weighted L^2-space we take we will denote it by X, and refer to it a *Hilbert space* or our *state space*.

3.3 Setting up a state space theory for PDE's.

Consider the ordinary differential equation

$$\ddot{y}(t) + 4\dot{y}(t) - 5y(t) = 0, \quad y(0) = y_0, \dot{y}(0) = y_1. \tag{3.15}$$

By introducing the state vector

$$x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix},$$

we can write the ordinary differential equation (3.15) as the first order vector-valued differential equation, or state-differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \tag{3.16}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix}.$$

and the state space equals $\mathbb{R}^2$. We would like to do a similar thing for our PDE's, i.e., to find the state space and the A.

For that we return to Example 2.1.1. Hence we consider the partial differential equation

$$\frac{\partial w}{\partial t}(\zeta, t) = c \frac{\partial w}{\partial \zeta}(\zeta, t), \qquad \zeta \in \mathbb{R}, \quad t \geq 0, \tag{3.17}$$

with c a positive constant. As initial condition we take $w_0(\zeta)$.

For the state-differential equation (3.16) we know that the state $x(t)$ lies in the state space. In particular, the initial state lies in the state space. Applying this reasoning to our PDE we see that the state space must be some function space. We choose it to be $L^2(-\infty, \infty)$, where the whole real axis is chosen since it is the spatial domain of our PDE.

So we have our state space, but how to obtain A and how to interpret $x(t)$? We start with the latter.

We see from the ordinary differential equation, that if we freeze time, then we are in the state space, i.e., $x(t)$ must be in the state space for all t.

In Section 3.1 we showed that the solution is given by

$$w(\zeta, t) = w_0(\zeta + ct). \tag{3.18}$$

Combining this with the condition that $x(t) \in L^2(-\infty, \infty)$, we see that

$$\left(x(t)\right)(\zeta) = w(\zeta, t). \tag{3.19}$$

So what this choice for the state does, is seeing the function w first as a function of time t, and next as a function of the spatial coordinate ζ, instead of seeing it as a function of the pair (ζ, t). A subtle, but important difference.

Now we have settled our choice for the state and state space, it is time to move to the next question *what do we take as A?*

To answer this question, it is good to remember what is the use of A. For ordinary differential equations the A is used to map the state at time t to the time derivative of the state, $\dot{x}(t)$, see (3.16).

Recall that our state $x(t)$ is still a function of time, and so by writing $\frac{dx}{dt}(t)$, we mean (see (3.19)) that we differentiate the function w with respect to time. Thus

$$\left(\frac{dx}{dt}(t)\right)(\zeta) = \frac{\partial w}{\partial t}(\zeta, t). \tag{3.20}$$

By the PDE (3.17) we know that the right hand-side equals $c\frac{\partial w}{\partial \zeta}(\zeta, t)$. Using (3.19) once more we find

$$\left(\frac{dx}{dt}(t)\right)(\zeta) = \frac{\partial w}{\partial t}(\zeta, t) = c\frac{\partial w}{\partial \zeta}(\zeta, t) = \left(Ax(t)\right)(\zeta), \tag{3.21}$$

with

$$\left(Af\right)(\zeta) = c\frac{df}{d\zeta}(\zeta). \tag{3.22}$$

So A just becomes the right hand-side of the PDE. However, from the matrix case, we know that A maps any vector from the state space to a vector in the state space. For our general case, we see that here is a small problem. For instance, if we take f to be a function with a jump, for instance

$$f(\zeta) = \begin{cases} 0 & \zeta < 0 \\ e^{-\zeta} & \zeta \geq 0, \end{cases}$$

then this function lies in $X = L^2(-\infty, \infty)$ (see Definition 3.2.1 or equation (3.10)), but Af (the derivative of f) does not exist. Or if you know about the delta function, it will have a delta-function at zero and thus the answer is not an element of X. So for PDE's Af will not always be defined for all $f \in X$. Since we want to have that A maps into X, we restrict the set on which A may work. Thus Af is only done for f's in the following set

$$\{f \in X \mid \frac{df}{d\zeta} \in X\}$$

This set is called the *domain of* A. Notation: $D(A)$.

So the domain of A will be those functions f (of ζ) such that Af makes sense, i.e., lies in X.

Concluding, we are able to rewrite the PDE (3.17) into the abstract form

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

for a suitable state and on a suitable state space X.

There is problem which remains open. Namely, what to do with boundary conditions?

3.3.1 Boundary conditions.

In order to obtain an intuition on where the boundary conditions have to go in the abstract set-up, we return to Example 2.1.2. Thus the PDE is given by

$$\frac{\partial w}{\partial t}(\zeta, t) = c\frac{\partial w}{\partial \zeta}(\zeta, t), \qquad t \geq 0, \zeta \in [0, 1], \tag{3.23}$$

with initial condition

$$w(\zeta, 0) = x_0(\zeta), \qquad \zeta \in [0, 1]$$

and boundary condition

$$w(1, t) = 0.$$

Since the PDE is very similar to the one presented above, see equation (3.17), we choose a similar state, state space, and A. That is

$$(x(t))(\zeta) = w(\zeta, t), \qquad X = L^2(0,1),$$

and

$$(Af)(\zeta) = c\frac{df}{d\zeta}(\zeta).$$

Again the question is on which functions we want A to work. Since we want that Af lies in the state space, functions with a jump inside the spatial domain $[0,1]$ are not allowed, but on the other hand, we want that our state satisfy the boundary conditions. Now we could add the boundary conditions into the state space X, but in X two functions can be the same whereas they are not pointwise the same. For example, in X the following two functions are the same, see Section 3.2

$$f_1(\zeta) = 1, \zeta \in [0,1] \quad \text{and} \quad f_2(\zeta) = \begin{cases} 1 & \zeta \in [0,1) \\ 0 & \zeta = 1 \end{cases}.$$

Since the first does not satisfy the boundary conditions, we see that adding the boundary conditions to the state space is not possible. However, for the derivative to exist we need that the function is at least continuous, and so we see that we can impose point values for functions in the domain of A. Thus we take as domain of A the following set

$$D(A) = \{f \in X \mid \frac{df}{d\zeta} \in X \text{ and } f(1) = 0\}.$$

Concluding, we see that the domain will also contain the boundary conditions.

3.4 Existence of solutions.

In the previous section we have introduced our class of state spaces and shown how to rewrite PDE's into abstract differential equations on such a state space. These facts enable us to formulate existence theorem. We will not do this in its full generality, but only for solution which don't grow in norm. In Chapter 4 we will see that the norm is often directly related to the energy in the system, and so to demand of a solution that is does not increase in energy is natural.

We consider a linear time-invariant PDE rewritten as the abstract differential equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \tag{3.24}$$

with $x_0 \in X$ (a weighted L^2-space) with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. The (linear) operator A has the domain $D(A)$ and maps into X.

Theorem 3.4.1. *Consider the abstract differential equation (3.24) on the state space X. This differential equation has for every initial condition $x_0 \in X$ a unique weak solution $x(t)$ (in X) satisfying $\|x(t)\| \leq \|x_0\|$ if and only if the following conditions hold*

1. *For all $x_0 \in D(A)$ we have that $\langle Ax_0, x_0 \rangle + \langle x_0, Ax_0 \rangle \leq 0$;*
2. *For all $z \in X$ there exists a $x_0 \in D(A)$ such that $(I - A)x_0 = z$.*

For solutions that do not loose norm/energy, we have a similar theorem.

Theorem 3.4.2. *Consider the abstract differential equation (3.24) on the state space X. This differential equation has for every initial condition $x_0 \in X$ a unique weak solution $x(t)$ (in X) satisfying $\|x(t)\| = \|x_0\|$ if and only if*

1. *for all $x_0 \in D(A)$ we have that $\langle Ax_0, x_0 \rangle + \langle x_0, Ax_0 \rangle = 0$;*
2. *For all $z \in X$ there exists a $x_0 \in D(A)$ such that $(I - A)x_0 = z$.*

Note that the inner product and norm in the theorems is chosen belonging to X. So if we take a different weight function in L^2_w, then the inner product and norm has to change accordingly.

Example 3.4.3 Based on the example in Section 3.3, we define A as

$$Ax = c\frac{dx}{d\zeta} \tag{3.25}$$

with domain

$$D(A) = \{x \in L^2(\mathbb{R}) \mid \frac{dx}{d\zeta} \in L^2(\mathbb{R})\}. \tag{3.26}$$

We will check the conditions of Theorem 3.4.2. Let $x_0 \in D(A)$, then

$$\begin{aligned}
\langle Ax_0, x_0 \rangle + \langle x_0, Ax_0 \rangle &= \int_{-\infty}^{\infty} c\frac{dx_0}{d\zeta}(\zeta)x_0(\zeta) + x_0(\zeta)c\frac{dx_0}{d\zeta}(\zeta)d\zeta \\
&= \int_{-\infty}^{\infty} c\frac{dx_0^2}{d\zeta}(\zeta)d\zeta \\
&= cx_0^2(\infty) - cx_0^2(-\infty) = 0,
\end{aligned}$$

where we have used that for a function in $D(A)$ the values in $\pm\infty$ are zero. Thus the first condition of Theorem 3.4.2 is satisfied. To check the second condition, we choose a $z \in X = L^2(\mathbb{R})$, and we have to construct a $x_0 \in D(A)$ such that $x_0 - Ax_0 = z$, or equivalently

$$x_0(\zeta) - c\frac{dx_0}{d\zeta}(\zeta) = z(\zeta), \qquad \zeta \in \mathbb{R}.$$

It is not hard to see that for $c > 0$ the solution (in X) of this differential equation is given by

$$x_0(\zeta) = \frac{1}{c}\int_\zeta^\infty e^{\frac{\zeta-\tau}{c}} z(\tau)d\tau.$$

For $c < 0$ the solution (in X) of this differential equation is given by

$$x_0(\zeta) = \frac{-1}{c}\int_{-\infty}^\zeta e^{\frac{\zeta-\tau}{c}} z(\tau)d\tau.$$

In both cases this is an element of $D(A)$, and so the second condition in Theorem 3.4.2 is also satisfies. Hence we conclude that the PDE associate to A in (3.25) possesses unique weak solution which stay norm-constant, i.e., $\int_{-\infty}^\infty x(\zeta,t)^2 d\zeta = \int_{-\infty}^\infty x_0(\zeta)^2 d\zeta$ for all t. $\square$

Example 3.4.4 Consider the PDE of Example 2.1.2;

$$\frac{\partial x}{\partial t}(\zeta,t) = c\frac{\partial x}{\partial \zeta}(\zeta,t), \qquad t \geq 0, \zeta \in [0,1], \tag{3.27}$$

with boundary condition $x(1,t) = 0$. Based on our previous observation, see Section 3.3, we define A (corresponding to (3.27)) as

$$Ax = c\frac{dx}{d\zeta} \tag{3.28}$$

with domain

$$D(A) = \{x \in L^2(0,1) \mid \frac{dx}{d\zeta} \in L^2(0,1) \text{ and } x(1) = 0\}. \tag{3.29}$$

We will check the conditions of Theorem 3.4.2. Let $x_0 \in D(A)$, then

$$\begin{aligned}\langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle &= \int_0^1 c\frac{dx_0}{d\zeta}(\zeta)x_0(\zeta) + x_0(\zeta)c\frac{dx_0}{d\zeta}(\zeta)d\zeta \\ &= \int_0^1 c\frac{dx_0^2}{d\zeta}(\zeta)d\zeta \\ &= cx_0^2(1) - cx_0^2(0) = 0 - cx_0^2(0) \leq 0,\end{aligned}$$

where we have used the boundary condition and the fact that $c > 0$. Thus the first condition is satisfied. To check the second condition, we choose a $z \in X = L^2(0,1)$, and we have to construct a $x \in D(A)$ such that $x_0 - Ax_0 = z$, or equivalently

$$x_0(\zeta) - c\frac{dx_0}{d\zeta}(\zeta) = z(\zeta), \qquad \zeta \in (0,1) \quad \text{and} \quad x_0(1) = 0.$$

It is not hard to see that the solution (in X) of this differential equation is given by

$$x_0(\zeta) = \frac{1}{c}\int_\zeta^1 e^{\frac{\zeta-\tau}{c}} z(\tau) d\tau, \qquad \zeta \in (0,1).$$

This is an element of $D(A)$, and so the second condition in Theorem 3.4.2 is also satisfies. Thus the PDE (3.27) possesses for every initial condition in $L^2(0,1)$ a (unique) weak solution whose norm will never be larger than the initial norm, i.e., $\int_0^1 x(\zeta,t)^2 d\zeta \leq \int_0^1 x_0(\zeta)^2 d\zeta$ for all $t \geq 0$. □

Looking at the above examples we see that the domain of A is a subset of L^2 for which the derivative(s) up to a certain order are still in the same L^2 space. These spaces are called Sobolev spaces.

Let $-\infty \leq a < b \leq \infty$, then $H^k(a,b)$ is the set of all functions for which the ℓ'th derivative lies in $L^2(a,b)$ for every $\ell \in \{1,\cdots,k\}$. This is known as the k'th *Sobolev space*. So we had the Sobolev space $H^1(-\infty,\infty) = H^1(\mathbb{R})$ in Example 3.4.3, and for the diffusion equation of the following example we use $H^2(0,1)$.

Example 3.4.5 In this example we consider the heat equation with no heat flux at the boundary of Example 2.1.3. The candidate A with its domain is given as

$$Ax = c\frac{d^2x}{d\zeta^2} \tag{3.30}$$

with domain

$$D(A) = \{x \in L^2(0,1) \mid x \in H^2(0,1) \text{ and } \frac{dx}{d\zeta}(1) = 0 = \frac{dx}{d\zeta}(0)\}, \tag{3.31}$$

where we have chosen as state space $X = L^2(0,1)$. Note that the domain involves the Sobolev space $H^2(0,1)$ as now we need functions in the domain to be twice differentiable.

For $x_0 \in D(A)$ we find

$$\begin{aligned}\langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle &= \int_0^1 c\frac{d^2x_0}{d\zeta^2}(\zeta)x_0(\zeta) + x_0(\zeta)c\frac{d^2x_0}{d\zeta^2}(\zeta)d\zeta \\ &= 2\left[c\frac{dx_0}{d\zeta}(\zeta)x_0(\zeta)\right]_0^1 - 2c\int_0^1 \left[\frac{dx_0}{d\zeta}(\zeta)\right]^2 d\zeta \leq 0,\end{aligned}$$

where we used integration by parts, the boundary conditions and the positivity of c.

Next we solve $(I-A)x = z$ for an arbitrary $z \in X$ and $x \in D(A)$. Hence we must solve

$$x(\zeta) - c\frac{d^2x}{d\zeta^2}(\zeta) = z(\zeta), \quad \zeta \in (0,1) \quad \text{and} \quad \frac{dx}{d\zeta}(1) = 0 = \frac{dx}{d\zeta}(0).$$

We can write this as a system of differential equations, like

$$\frac{d}{d\zeta}\begin{bmatrix} x \\ \frac{dx}{d\zeta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{c} & 0 \end{bmatrix}\begin{bmatrix} x \\ \frac{dx}{d\zeta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-1}{c} \end{bmatrix} z.$$

The solution of this is[2]

$$\begin{aligned}\begin{bmatrix} x(\zeta) \\ \frac{dx}{d\zeta}(\zeta) \end{bmatrix} &= \begin{bmatrix} \cosh(\gamma\zeta) & \frac{1}{\gamma}\sinh(\gamma\zeta) \\ \gamma\sinh(\gamma\zeta) & \cosh(\gamma\zeta) \end{bmatrix}\begin{bmatrix} x(0) \\ \frac{dx}{d\zeta}(0) \end{bmatrix} + \\ &\qquad \int_0^\zeta \begin{bmatrix} \cosh(\gamma(\zeta-\tau)) & \frac{1}{\gamma}\sinh(\gamma(\zeta-\tau)) \\ \gamma\sinh(\gamma(\zeta-\tau)) & \cosh(\gamma(\zeta-\tau)) \end{bmatrix}\begin{bmatrix} 0 \\ \frac{-1}{c} \end{bmatrix} z(\tau)d\tau \\ &= \begin{bmatrix} \cosh(\gamma\zeta) \\ \gamma\sinh(\gamma\zeta) \end{bmatrix} x(0) - \frac{1}{c}\int_0^\zeta \begin{bmatrix} \frac{1}{\gamma}\sinh(\gamma(\zeta-\tau)) \\ \cosh(\gamma(\zeta-\tau)) \end{bmatrix} z(\tau)d\tau,\end{aligned}$$

where $\gamma^2 = \frac{1}{c}$ and we have used the boundary condition at zero. Using the other boundary condition gives

$$x(\zeta) = \cosh(\gamma\zeta)\left[\frac{\int_0^1 \cosh(\gamma(1-\tau))z(\tau)d\tau}{c\gamma\sinh(\gamma)}\right] - \frac{1}{c}\int_0^\zeta \frac{1}{\gamma}\sinh(\gamma(\zeta-\tau))z(\tau)d\tau.$$

Thus the heat equation with no heat flux at the boundary possesses for every initial condition in $L^2(0,1)$ a unique (weak) solution, whose L^2-norm will not increase. We know from Example 2.1.3 that this solution is given by (2.25) and (2.24). □

[2] $\sinh(\zeta) = \frac{e^\zeta - e^{-\zeta}}{2}$, $\cosh(\zeta) = \frac{e^\zeta + e^{-\zeta}}{2}$

3.4.1 Strong and weak solutions, revisited

In Section 3.1 we have introduced the concept of weak and strong solutions, and in Theorems 3.4.1 and 3.4.2 we have given conditions under which weak solutions (bounded by the initial state norm) exist. Now it is a natural question when these weak solutions are classical/strong ones.

It can be shown that if the initial condition lies in the domain of A, $D(A)$, then $x(t)$ stays in the domain of A. Furthermore, the function $t \mapsto x(t)$ is differentiable and its derivative equals $Ax(t)$. Thus when $x_0 \in D(A)$, then the weak solution satisfies the abstract differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0.$$

Since this abstract differential equation is just a short-hand notation for the PDE, we see that this solution satisfies the PDE, and is thus a classical solution. It is good to investigate this on the example of the transport equation as given in equations (3.1)–(3.2), see also Example 2.1.1.

In Section 3.3 we choose as our state space $X = L^2(-\infty, \infty)$. So even if for $x_0(\zeta) = \zeta^2$ the solution as given in (3.3) clearly satisfies the PDE, we don't consider it, because $x_0 \notin L^2(-\infty, \infty)$.

Since $x_0(\zeta) := \frac{|\sin(\zeta)|}{\zeta^2+1}$ is an element of X and since

$$\frac{d}{d\zeta}\left[\frac{|\sin(\zeta)|}{\zeta^2+1}\right] = \begin{cases} \frac{\cos(\zeta)(\zeta^2+1)-2\zeta\sin(\zeta)}{(\zeta^2+1)^2} & \zeta \in (2n\pi, (2n+1)\pi) \\ \frac{-\cos(\zeta)(\zeta^2+1)+2\zeta\sin(\zeta)}{(\zeta^2+1)^2} & \zeta \in ((2n+1)\pi, 2n\pi) \end{cases}$$

is also an element of X, we see that $x_0 \in D(A)$, and thus by the above $x(\zeta, t) = x_0(ct + \zeta)$ is the classical solution of (3.1)–(3.2), even though it is not differentiable in every point. These are subtle differences with what you might expect, and they are caused by our choice of the state space.

3.5 Linearity at all places[3]

In this and the previous chapter we only considered linear PDE's. So it good to have a more detailed look to them, and to the consequences which that choice had on the mathematical objects we work with. First of all, how do you recognise that a PDE is linear?

A PDE is *linear* when any sum and scalar multiple of solutions is again a solution, or said differently *superposition* is allowed. So you assume that

[3] This section is not part of the exam material

$f(\zeta,t)$ and $g(\zeta,t)$ are solutions of the given PDE, and you check whether that implies that $h(\zeta,t) = \alpha f(\zeta,t) + \beta g(\zeta,t)$ is a solution as well. Here α and β are (arbitrary) real constants. Let us check this on two examples of Chapter 1.

Example 3.5.1 We consider the Euler-Bernoulli beam model of Example 1.2.1 with no bending moments applied to the tip. So the movements of the beam are modelled via the PDE, see also (1.1),

$$\rho \frac{\partial^2 w}{\partial t^2}(\zeta,t) = EI \frac{\partial^4 w}{\partial \zeta^4}(\zeta,t), \tag{3.32}$$

and the boundary conditions are given by (1.2), (1.3), and

$$\rho \frac{\partial^2 w}{\partial \zeta^2}(L,t) = -J \frac{\partial^3 w}{\partial t^2 \partial \zeta}(L,t).$$

Now since

$$\frac{\partial^2}{\partial t^2}\left[\alpha f(\zeta,t) + \beta g(\zeta,t)\right] = \alpha \frac{\partial^2 f}{\partial t^2}(\zeta,t) + \beta \frac{\partial^2 g}{\partial t^2}(\zeta,t)$$

and since a similar relation holds for $\frac{\partial^4}{\partial \zeta^4}$, we see that if f and g satisfy (3.32) so will $h = \alpha f + \beta g$. A similar argument gives that h will also satisfy the boundary conditions, because f and g do. Thus this PDE with its boundary conditions in linear. $\square$

Next we look at our model of the shallow water equations of Example 1.2.3.

Example 3.5.2 The PDE describing the water height and its velocity are given by

$$\frac{\partial h}{\partial t}(\zeta,t) = -\frac{\partial}{\partial \zeta}\left(h(\zeta,t)v(\zeta,t)\right), \tag{3.33}$$

$$\frac{\partial v}{\partial t}(\zeta,t) = -\frac{\partial}{\partial \zeta}\left(\frac{1}{2}v^2(\zeta,t) + gh(\zeta,t)\right). \tag{3.34}$$

We assume that there is no control, and so the boundary conditions are

$$h(0,t)v(0,t) = 0 = h(L,t)v(L,t)$$

It is easy to see that h and v identically zero form a solution of (3.33) and (3.34) and that they satisfy the boundary conditions.

Let $h(\zeta,t), v(\zeta,t)$ be another solution, then if the PDE would be linear, then the pair $(2h(\zeta,t), 2v(\zeta,t)) = 2(h(\zeta,t), v(\zeta,t)) + (0,0)$ would be a solution as well. Using equation (3.33) this implies that

$$2\frac{\partial h}{\partial t}(\zeta,t) = \frac{\partial 2h}{\partial t}(\zeta,t) = -\frac{\partial}{\partial \zeta}\left(2h(\zeta,t)2v(\zeta,t)\right) = -4\frac{\partial}{\partial \zeta}\left(h(\zeta,t)v(\zeta,t)\right),$$

or equivalently

$$\frac{\partial h}{\partial t}(\zeta,t) = -2\frac{\partial}{\partial \zeta}\left(h(\zeta,t)v(\zeta,t)\right).$$

However, $\frac{\partial h}{\partial t}$ satisfies (3.33), and so

$$-\frac{\partial}{\partial \zeta}\left(h(\zeta,t)v(\zeta,t)\right) = \frac{\partial h}{\partial t}(\zeta,t) = -2\frac{\partial}{\partial \zeta}\left(h(\zeta,t)v(\zeta,t)\right).$$

This implies that $\frac{\partial h}{\partial t} = 0$, or $h(\zeta,t)$ does not depend on time, i.e. $h(\zeta,t) = h(0,t) = h_0(\zeta)$. Since we took h as an arbitrary solution, this already indicates that linearity is lost.

We continue by noticing that the above equality also implies $h(\zeta,t)$ times $v(\zeta,t)$ cannot depend on ζ either. Hence $h(\zeta,t)v(\zeta,t) = h(0,t)v(0,t) = 0$, by the first boundary condition. Thus by choosing an initial condition h_0, v_0 for which this does not hold, we have a contradiction. Concluding, we have that the model of the shallow water is non-linear.

The rule of thumb is that whenever there is a product of signals in your equation it is non-linear. In our equations we have two places which point to non-linearity. Namely, hv and v^2 in (3.33) and (3.34), respectively. $\square$

When a PDE with its boundary conditions is linear, then this linearity carries over to the abstract notions we have introduced.

MORE TO BE ADDED

3.6 Exercises

3.1. Consider the PDE

$$\frac{\partial x}{\partial t}(\zeta,t) = \alpha\frac{\partial x}{\partial \zeta}(\zeta,t), \qquad \zeta \in [0,1], \quad t \geq 0, \tag{3.35}$$

with $\alpha > 0$. Show that on the state space $X = L^2(0,1)$ this PDE has for any initial condition a unique weak solution, when the boundary conditions are given as;

(a) The state at the right-hand side is set to zero, i.e., $x(1,t) = 0$.

(b) The state at both ends are equal, i.e., $x(1,t) = x(0,t)$.

(c) Determine for $\alpha = 1$, all the boundary conditions such that the PDE (3.35) possesses unique weak solutions constant in norm.

3.2. Consider the PDE

$$\frac{\partial x}{\partial t}(\zeta,t) = -\frac{\partial x}{\partial \zeta}(\zeta,t), \qquad \zeta \in [0,1], \quad t \geq 0, \tag{3.36}$$

Show that on the state space $X = L^2(0,1)$ this PDE has for any initial condition a unique weak solution, when the boundary conditions are given as;

(a) The state at the left-hand side is set to zero, i.e., $x(0,t) = 0$.

(b) The states at both ends are equal, i.e., $x(1,t) = x(0,t)$.

(c) Determine all the boundary conditions such that the PDE (3.36) possesses unique weak solutions constant in norm.

3.3. Consider a bar of length one whose temperature is zero at both ends. The model is given as

$$\begin{aligned} &\frac{\partial w}{\partial t}(\zeta,t) = \alpha\frac{\partial^2 w}{\partial \zeta^2}(\zeta,t) \qquad w(\zeta,0) = w_0(\zeta), \\ &w(0,t) = 0, \quad w(1,t) = 0. \end{aligned} \tag{3.37}$$

$w(\zeta,t)$ represents the temperature at position $\zeta \in [0,1]$ at time $t \geq 0$ and $w_0(\zeta)$ the initial temperature profile. Furthermore, α is a positive constant.

(a) Write the PDE (3.37) as an abstract differential equation on the state space $X = L^2(0,1)$.

(b) Show that for every $w_0 \in X$ the PDE possesses a weak solution.

3.4. To illustrate/explain the two conditions in Theorem 3.4.1, we consider the differential operator of Example 3.4.4, but with different boundary conditions. The state space remains $X = L^2(0,1)$.

(a) Show that with the boundary conditions $x(1) = 0 = x(0)$, part 1. of Theorem 3.4.1 is satisfied, but part 2. not.

(b) Show that if we impose no boundary conditions, part 2. of Theorem 3.4.1 is satisfied, but part 1. not

Hence together the conditions make sure that you impose just the right amount of boundary conditions.

Chapter 4
Port-Hamiltonian PDE's

4.1 Class of port-Hamiltonian systems

In Example 1.2.4 we introduced a PDE model describing the vibrations in a flexible string. This model is given as:

$$\rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial w}{\partial \zeta}\right)(\zeta,t), \tag{4.1}$$

where ρ is the mass density, and T is the elasticity modulus. There is a natural energy associated to this model which is given by

$$\frac{1}{2}\int_0^L \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t)^2 + T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)^2 d\zeta. \tag{4.2}$$

In the first term we recognise the kinetic energy "half times mass times velocity squared". The second term represents the potential energy. We use these "energy variables" to rewrite the PDE (4.1) into the abstract form $\dot{x}(t) = Ax(t)$.

Choose the state

$$(x_1(t))(\zeta) = \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t), \text{ and } (x_2(t))(\zeta) = \frac{\partial w}{\partial \zeta}(\zeta,t). \tag{4.3}$$

[0]March 21, 2024

Then it is easy to see that for $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ we can write the PDE as

$$\begin{aligned}
\left(\frac{dx(t)}{dt}\right)(\zeta) = \frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial}{\partial t}\begin{pmatrix} \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t) \\ \frac{\partial w}{\partial \zeta}(\zeta,t) \end{pmatrix} \\
&= \begin{pmatrix} \rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) \\ \frac{\partial^2 w}{\partial t \partial \zeta}(\zeta,t) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial w}{\partial \zeta}\right)(\zeta,t) \\ \frac{\partial^2 w}{\partial \zeta \partial t}(\zeta,t) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial^2 w}{\partial \zeta \partial t}(\zeta,t) \\ \frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial w}{\partial \zeta}\right)(\zeta,t) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial \zeta}\begin{pmatrix} \frac{\partial w}{\partial t}(\zeta,t) \\ \left(T(\zeta)\frac{\partial w}{\partial \zeta}\right)(\zeta,t) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial \zeta}\left[\begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix}\begin{pmatrix} \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t) \\ \frac{\partial w}{\partial \zeta}(\zeta,t) \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial \zeta}\left[\begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix} x(\zeta,t)\right].
\end{aligned} \tag{4.4}$$

We see that in the new formulation of the undamped vibrating string (4.1) the use of the energy variables as state variables leads to a first order in space and in time partial differential equation. We shall see that this is the case of many physical systems. They all fit in the general first order formulation

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1\frac{\partial}{\partial \zeta}\left[\mathcal{H}(\zeta)x(\zeta,t)\right] + P_0\left[\mathcal{H}(\zeta)x(\zeta,t)\right], \quad \zeta \in (a,b), t \geq 0. \tag{4.5}$$

Where P_1 is symmetric, i.e., $P_1^T = P_1$, P_0 is *anti-symmetric*, i.e., $P_0^T = -P_0$. Furthermore, they are both constant. Finally, $\mathcal{H}$ is a (strictly) positive symmetric matrix, independent of t, but may depend on ζ.

Remark 4.1.1. In the case of the vibrating string

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; P_0 = 0, \text{ and } \mathcal{H}(\zeta) = \begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix}.$$

□

The energy or Hamiltonian can be expressed by using x and $\mathcal{H}$. That is

$$E(t) = \frac{1}{2}\int_a^b x(\zeta,t)^T \mathcal{H}(\zeta)x(\zeta,t)d\zeta. \tag{4.6}$$

In the following theorem we show that for any system which is of the form (4.5) with Hamiltonian/energy (4.6), the change of energy (power) is only possible via the boundary of its spatial domain.

Theorem 4.1.2. *Consider the partial differential equation* (4.5) *in which* P_0, P_1 *are constant matrices satisfying* $P_1^T = P_1$ *and* $P_0^T = -P_0$. *Furthermore,* $\mathcal{H}$ *is independent on* t *and is symmetric, i.e. for all* ζ*'s we have that* $\mathcal{H}(\zeta)^T = \mathcal{H}(\zeta)$. *For the Hamiltonian/energy given by* (4.6) *the following balance equation holds for all classical solutions of* (4.5)

$$\frac{dE}{dt}(t) = \frac{1}{2}\left[(\mathcal{H}x)^T(\zeta,t)P_1(\mathcal{H}x)(\zeta,t)\right]_a^b. \tag{4.7}$$

Proof: By using the partial differential equation, we find that

$$\begin{aligned}\frac{dE}{dt}(t) &= \frac{1}{2}\int_a^b \frac{\partial x}{\partial t}(\zeta,t)^T\mathcal{H}(\zeta)x(\zeta,t)d\zeta + \frac{1}{2}\int_a^b x(\zeta,t)^T\mathcal{H}(\zeta)\frac{\partial x}{\partial t}(\zeta,t)d\zeta \\ &= \frac{1}{2}\int_a^b \left[P_1\frac{\partial}{\partial\zeta}(\mathcal{H}x)(\zeta,t) + P_0(\mathcal{H}x)(\zeta,t)\right]^T \mathcal{H}(\zeta)x(\zeta,t)d\zeta + \\ &\quad \frac{1}{2}\int_a^b x(\zeta,t)^T\mathcal{H}(\zeta,t)\left[P_1\frac{\partial}{\partial\zeta}(\mathcal{H}x)(\zeta,t) + P_0(\mathcal{H}x)(\zeta,t)\right]d\zeta.\end{aligned}$$

Using now the fact that $P_1, \mathcal{H}(\zeta)$ are symmetric, and P_0 is anti-symmetric, we write the last expression as

$$\begin{aligned}&\frac{1}{2}\int_a^b \left[\frac{\partial}{\partial\zeta}(\mathcal{H}x)(\zeta,t)\right]^T P_1\mathcal{H}(\zeta)x(\zeta,t) + \\ &\qquad [\mathcal{H}(\zeta)x(\zeta,t)]^T\left[P_1\frac{\partial}{\partial\zeta}(\mathcal{H}x)(\zeta,t)\right]d\zeta + \\ &\frac{1}{2}\int_a^b -[\mathcal{H}(\zeta)x(\zeta,t)]^T P_0\mathcal{H}(\zeta)x(\zeta,t) + \\ &\qquad [\mathcal{H}(\zeta)x(\zeta,t)]^T[P_0\mathcal{H}(\zeta)x(\zeta,t)]\,d\zeta \\ &= \frac{1}{2}\int_a^b \frac{\partial}{\partial\zeta}\left[(\mathcal{H}x)^T(\zeta,t)P_1(\mathcal{H}x)(\zeta,t)\right]d\zeta \\ &= \frac{1}{2}\left[(\mathcal{H}x)^T(\zeta,t)P_1(\mathcal{H}x)(\zeta,t)\right]_a^b.\end{aligned}$$

Hence we have proved the theorem. □

The balance equation (4.7) will turn out to be very important, and will guide us in many problems. For instance, it will help us to show existence of solutions. As we have seen existence and uniqueness of solutions depends on the correct formulation of the boundary conditions. These boundary conditions will expressed in (some linear combination of) $x(a,t)$ and $x(b,t)$. Because of the fact that both the partial differential equation and the energy balance equation involve the combination $\mathcal{H}(\zeta)x(\zeta,t)$, it is more natural to formulate the boundary conditions in this combination as well: some linear combination of $\mathcal{H}(b)x(b,t)$ and $\mathcal{H}(a)x(a,t)$ is zero. More precisely, for some $n \times 2n$ matrix W_B we have

$$W_B \begin{pmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{pmatrix} = 0. \tag{4.8}$$

A PDE of the form (4.5) with P_1, P_0, $\mathcal{H}$ satisfying the conditions as stated below (4.5), and boundary conditions (4.8) will called a *port-Hamiltonian PDE*, or *port-Hamiltonian system*.

In what follows we revisit some classical examples and show that they all lie in the class of systems (4.5).

Example 4.1.3 (Undamped Timoshenko beam)
In recent years the boundary control of flexible structures has attracted much attention with the increase of high technology applications such as space science and robotics. In these applications the control of vibrations is crucial. These vibrations can be modeled by beam equations. Among them the *Timoshenko beam* equation incorporates shear and rotational inertia effects, which makes it a more precise model than *Euler Bernoulli* model or *Rayleigh* models. The Timoshenko beam model is well known in the following form

$$\begin{aligned} \rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) &= \frac{\partial}{\partial \zeta}\left[K(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta,t) - \phi(\zeta,t)\right)\right], \quad \zeta \in (a,b),\ t \geq 0, \\ I_\rho(\zeta)\frac{\partial^2 \phi}{\partial t^2}(\zeta,t) &= \frac{\partial}{\partial \zeta}\left(EI(\zeta)\frac{\partial \phi}{\partial \zeta}(\zeta,t)\right) + K(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta,t) - \phi(\zeta,t)\right), \end{aligned} \tag{4.9}$$

where $w(\zeta,t)$ is the transverse displacement of the beam and $\phi(\zeta,t)$ is the rotation angle of a filament of the beam. The coefficients $\rho(\zeta)$, $I_\rho(\zeta)$, $E(\zeta)$, $I(\zeta)$, and $K(\zeta)$ are the mass per unit length, the rotary moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus respectively. In order to describe this model from balance equations we have to define the extensive variables that are used as state variables:

$$\begin{aligned}
x_1(\zeta,t) &= \frac{\partial w}{\partial \zeta}(\zeta,t) - \phi(\zeta,t) && \text{shear displacement}\\
x_2(\zeta,t) &= \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t) && \text{momentum}\\
x_3(\zeta,t) &= \frac{\partial \phi}{\partial \zeta}(\zeta,t) && \text{angular displacement}\\
x_4(\zeta,t) &= I_\rho(\zeta)\frac{\partial \phi}{\partial t}(\zeta,t) && \text{angular momentum}
\end{aligned}$$

Calculating the time derivative of the variables $x_1, \dots, x_4$, we find by using (4.9)

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1(\zeta,t)\\ x_2(\zeta,t)\\ x_3(\zeta,t)\\ x_4(\zeta,t)\end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial \zeta}\left(\frac{x_2(\zeta,t)}{\rho(\zeta)}\right) - \frac{x_4(\zeta,t)}{I_\rho(\zeta)}\\ \frac{\partial}{\partial \zeta}\left(K(\zeta)x_1(\zeta,t)\right)\\ \frac{\partial}{\partial \zeta}\left(\frac{x_4(\zeta,t)}{I_\rho(\zeta)}\right)\\ \frac{\partial}{\partial \zeta}\left(E(\zeta)I(\zeta)x_3(\zeta,t)\right) + K(\zeta)x_1(\zeta,t)\end{pmatrix}. \tag{4.10}$$

We can write this in a form similar to those presented in (4.4). However, as we shall see, we also need a "constant" term. Since this is a long expression, we will not write down the coordinates ζ and t.

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0\end{pmatrix}\frac{\partial}{\partial \zeta}\left[\begin{pmatrix} K&0&0&0\\ 0&\frac{1}{\rho}&0&0\\ 0&0&EI&0\\ 0&0&0&\frac{1}{I_\rho}\end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}\right] +$$
$$\begin{pmatrix} 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\end{pmatrix}\begin{pmatrix} K&0&0&0\\ 0&\frac{1}{\rho}&0&0\\ 0&0&EI&0\\ 0&0&0&\frac{1}{I_\rho}\end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}. \tag{4.11}$$

Again this system can be expressed on the general form (4.5) with

$$P_1 = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0\end{pmatrix}, \; P_0 = \begin{pmatrix} 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\end{pmatrix},$$

and

$$\mathcal{H}(\zeta) = \begin{pmatrix} K(\zeta)&0&0&0\\ 0&\frac{1}{\rho(\zeta)}&0&0\\ 0&0&EI(\zeta)&0\\ 0&0&0&\frac{1}{I_{\rho(\zeta)}}\end{pmatrix}.$$

Formulating the energy/Hamiltonian in the variables $x_1, \ldots, x_4$ is easier,

$$\begin{aligned}E(t) =& \frac{1}{2}\int_a^b \left(Kx_1^2 + \frac{1}{\rho}x_2^2 + EIx_3^2 + \frac{1}{I_\rho}x_4^2\right) d\zeta \\ =& \frac{1}{2}\int_a^b \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^T \begin{pmatrix} K & 0 & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & EI & 0 \\ 0 & 0 & 0 & \frac{1}{I_\rho} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right] d\zeta. \end{aligned} \tag{4.12}$$

and leads from (4.7) to the balance equation (now with the spatial and temporal variables in)

$$\begin{aligned}\frac{dE}{dt}(t) = & K(b)x_1(b)\frac{x_2(b,t)}{\rho(b)} + E(b)I(b)x_3(b,t)\frac{x_4(b)}{I_\rho(b)} \\ & - K(a)x_1(a,t)\frac{x_2(a,t)}{\rho(a)} - E(a)I(a)x_3(a,t)\frac{x_4(a,t)}{I_\rho(a)}. \end{aligned}$$ □

4.2 Solutions to the port-Hamiltonian system

In this section, we apply the general result presented in the previous chapter to PDE's in the port-Hamiltonian formulation, i.e., we consider

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta}\left[\mathcal{H}(\zeta)x(\zeta,t)\right] + P_0\left[\mathcal{H}(\zeta)x(\zeta,t)\right], \tag{4.13}$$

where $x(\zeta, t)$ is in $\mathbb{R}^n$ for all ζ and t. Recall that $P_1 = P_1^T$ and $P_0 = -P_0^T$ are $n \times n$ matrices and $\mathcal{H}(\zeta)$ has the property that there are positive numbers m and M such that $m \cdot I_n \leq \mathcal{H}(\zeta) \leq M \cdot I_n$. We also have the energy introduced in the previous section, and the energy balance equation describing it's derivative

$$\frac{dE}{dt}(t) = \frac{1}{2}\left[(\mathcal{H}(\zeta)x(\zeta,t))^T P_1 \mathcal{H}(\zeta)x(\zeta,t)\right]_{\zeta=a}^{\zeta=b}.$$

Together with the partial differential equation we have boundary conditions, formulated as

$$W_B \begin{pmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{pmatrix} = 0. \tag{4.14}$$

In order to apply the theory of the previous chapter, we do not regard $x(\cdot, \cdot)$ as a function of place and time, but as a function of time, which takes values in a function space, i.e., we see $x(\zeta, t)$ as the function $x(\cdot, t)$ evaluated at ζ, see Section 3.3. With a little bit of misuse of notation, we write $x(\cdot, t) = (x(t))(\cdot)$. Hence we "forget" the spatial dependence, and we write the PDE as the (abstract) ordinary differential equation

$$\frac{dx}{dt}(t) = P_1 \frac{\partial}{\partial \zeta} [\mathcal{H}x(t)] + P_0 [\mathcal{H}x(t)]. \tag{4.15}$$

Hence we consider the operator

$$Ax := P_1 \frac{d}{d\zeta} [\mathcal{H}x] + P_0 [\mathcal{H}x] \tag{4.16}$$

on a domain which includes the boundary conditions. The domain should be a part of the *state space* X, which we identify next. For our class of PDE's we have a natural energy function, see (4.6). Hence it is quite natural to consider only states which have a finite energy. That is we take as our state space all functions for which $\int_a^b x(\zeta, t)^T \mathcal{H}(\zeta) x(\zeta, t) d\zeta$ is finite. Since $\mathcal{H}(\zeta)$ is strictly positive definite, i.e., $mI \leq \mathcal{H}(\zeta)$, we see that this condition is the same as saying that for every $t \geq 0$, $x(t)$ is in the weighted L_2 space, $L^2_{\mathcal{H}}((a, b); \mathbb{R}^n)$, see Definition 3.2.2.

So we take as our state space

$$X = L^2_{\mathcal{H}}((a, b); \mathbb{R}^n) \tag{4.17}$$

with inner product

$$\langle f, g \rangle_{\mathcal{H}} = \frac{1}{2} \int_a^b f(\zeta)^T \mathcal{H}(\zeta) g(\zeta) d\zeta. \tag{4.18}$$

This implies that the squared norm of a state x equals the energy of this state.

Thus we have formulated our state space X, see (4.17) and (4.18), and our operator, A, see (4.16). The domain of this operator involves the boundary conditions, and is given by, see (4.8),

$$\begin{aligned} D(A) = \{x \in L^2_{\mathcal{H}}((a, b); \mathbb{R}^n) \mid & \mathcal{H}x \in H^1((a, b); \mathbb{R}^n), \\ & W_B \left(\begin{smallmatrix} \mathcal{H}(b)x(b) \\ \mathcal{H}(a)x(a) \end{smallmatrix} \right) = 0\}. \end{aligned} \tag{4.19}$$

Here $H^1((a, b); \mathbb{R}^n)$ are all functions from (a, b) to $\mathbb{R}^n$ which are square integrable and have a derivative which is again square integrable, i.e., a Sobolev space, see also page 29.

The following theorem shows that the PDE (4.13) with boundary conditions (4.14) has for every initial condition (with finite energy) a solution which does not grow in energy precisely when the power is negative, see (4.7).

Theorem 4.2.1. *Consider the operator A defined in (4.16) and (4.19), where we assume the following*

- *P_1 is an invertible, symmetric real $n \times n$ matrix;*
- *P_0 is an anti-symmetric real $n \times n$ matrix;*
- *For all $\zeta \in [a, b]$ the $n \times n$ matrix $\mathcal{H}(\zeta)$ is real, symmetric, and $mI \leq \mathcal{H}(\zeta) \leq MI$, for some $M, m > 0$ independent of ζ;*
- *W_B is a full rank real matrix of size $n \times 2n$.*

Then for every $x_0 \in X = L^2_{\mathcal{H}}((a, b); \mathbb{R}^n)$ there exists a unique (weak) solution of $\dot{x}(t) = Ax(t), x(0) = x_0$ satisfying $\|x(t)\|_{\mathcal{H}} \leq \|x_0\|_{\mathcal{H}}$ if and only if

$$\langle Ax_1, x_1 \rangle_{\mathcal{H}} + \langle x_1, Ax_1 \rangle_{\mathcal{H}} \leq 0, \qquad \forall x_1 \in D(A).$$

Furthermore, for every $x_0 \in X$ there exists a unique (weak) solution of $\dot{x}(t) = Ax(t), x(0) = x_0$ satisfying $\|x(t)\|_{\mathcal{H}} = \|x_0\|_{\mathcal{H}}$ if and only if

$$\langle Ax_1, x_1 \rangle_{\mathcal{H}} + \langle x_1, Ax_1 \rangle_{\mathcal{H}} = 0, \qquad \forall x_1 \in D(A).$$

We see a similarity with Theorems 3.4.1 and 3.4.2. However, the second condition on $(I - A)x_0 = z$ seems not to be there. This was often the hardest condition to be checked. In turns out that for this class of systems this condition has been taken over by the (much simpler) condition that W_B must have full rank.

We apply this theorem to our standard example from Chapter 2, see also Example 2.1.2 and 3.4.3.

Example 4.2.2 Consider the homogeneous PDE on the spatial interval $[0, 1]$.

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial x}{\partial \zeta}(\zeta, t), && \zeta \in [0, 1],\ t \geq 0 \\ x(\zeta, 0) &= x_0(\zeta), && \zeta \in [0, 1] \\ 0 &= x(1, t), && t \geq 0. \end{aligned}$$

We see that the first equation can be written in the from (4.13) by choosing $P_1 = 1$, $\mathcal{H} = 1$ and $P_0 = 0$. We have that $n = 1$ and the boundary condition can be written

$$0 = x(1,t) = \mathcal{H}(1)x(1,t) \Longleftrightarrow 0 = W_B \begin{bmatrix} \mathcal{H}(1)x(1,t) \\ \mathcal{H}(0)x(0,t) \end{bmatrix}, \tag{4.20}$$

with $W_B = \begin{bmatrix} 1 & 0 \end{bmatrix}$. This has clearly full rank.

Since

$$\begin{aligned} \langle Ax, x\rangle_{\mathcal{H}} + \langle x, Ax\rangle_X &= \frac{1}{2}\int_0^1 \frac{dx}{d\zeta}(\zeta)x(\zeta) + x(\zeta)\frac{dx}{d\zeta}(\zeta)x(\zeta)d\zeta \\ &= \frac{1}{4}\left[x^2(\zeta)\right]_0^1 = \frac{1}{4}[0 - x^2(0)] \le 0, \end{aligned}$$

we conclude by Theorem 4.2.1 that the PDE with the boundary condition possesses a unique weak solution in the state space $L^2(0,1)$, and $\|x(t)\|^2 \le \|x_0\|^2$. The expression for this weak solution is given in Example 2.1.2, equation (2.7). □

Next we consider the vibrating string of Example 1.2.4.

Example 4.2.3 Consider the vibrating string of the beginning of this chapter. We assume that its spatial interval is $[0, L]$, and that it can move freely at $\zeta = L$, but is fixed at $\zeta = 0$, see also Figure 1.5. The rewriting into a port-Hamiltonian model is done in the beginning of this chapter, and we have taken the states, see (4.3)

$$x_1 = \rho\frac{\partial w}{\partial t} \text{ and } x_2 = \frac{\partial w}{\partial \zeta}.$$

We have to write the boundary conditions in the form of (4.14). We see that $\mathcal{H}x$ consists of the variables $\frac{\partial w}{\partial t}$ and $T\frac{\partial w}{\partial \zeta}$. Because of the configuration of the string, we find

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} T(L)\frac{\partial w}{\partial \zeta}(L,t) \\ \frac{\partial w}{\partial t}(0,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial w}{\partial t}(L,t) \\ T(L)\frac{\partial w}{\partial \zeta}(L,t) \\ \frac{\partial w}{\partial t}(0,t) \\ T(0)\frac{\partial w}{\partial \zeta}(0,t) \end{pmatrix}.$$

So $W_B = \left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{smallmatrix}\right)$ which has clearly rank 2, and thus of full rank.

Now for $x_0 = \left(\begin{smallmatrix} x_{10} \\ x_{20} \end{smallmatrix}\right) \in D(A)$ we find that, see Exercise 4.1,

$$\begin{aligned}\langle Ax_0, x_0\rangle_{\mathcal{H}} + &\langle x_0, Ax_0\rangle_{\mathcal{H}} \\ &= x_{10}(L)\frac{T(L)}{\rho(L)}x_{20}(L) - x_{10}(0)\frac{T(0)}{\rho(0)}x_{20}(0) = 0.\end{aligned}$$

Theorem 4.2.1 now gives that for every initial condition in the state space, there exists a unique weak solution. This solution will at every time has the same energy as the initial state. □

4.3 Exercises

4.1. In this exercise we show that the energy balance as found in equation (4.7) of Theorem 4.1.2 has a direct relation with the expression

$$\langle Ax_0, x_0\rangle_{\mathcal{H}} + \langle x_0, Ax_0\rangle_{\mathcal{H}}.$$

Consider the port-Hamiltonian PDE (4.5) and its abstract formulation (4.16).

(a) Let x_0 be an element of $D(A)$. Prove that

$$\begin{aligned}\langle Ax_0, x_0\rangle_{\mathcal{H}} + &\langle Ax_0, x_0\rangle_{\mathcal{H}} \\ &= \frac{1}{2}\left[(\mathcal{H}(b)x_0(b))^T P_1 \mathcal{H}(b)x_0(b) - \right. \\ &\qquad \left.(\mathcal{H}(a)x_0(a))^T P_1 \mathcal{H}(a)x_0(a)\right].\end{aligned} \tag{4.21}$$

(b) Interpret the conditions of Theorem 4.2.1 on the expression

$$\langle Ax_0, x_0\rangle_{\mathcal{H}} + \langle x_0, Ax_0\rangle_{\mathcal{H}}$$

in terms of the power balance. Explain why these conditions are logical.

(c) Write the expression of (4.21) out for the wave equation of Example 4.2.3.

4.2. Consider the vibrating string of Example 1.2.4 on the spatial interval $[0, L]$, but now freely moving at both sides

(a) Show that by imposing the no-force boundary conditions, i.e., $T(L)\frac{\partial w}{\partial \zeta}(L, t) = T(0)\frac{\partial w}{\partial \zeta}(0, t) = 0$ as shown Figure 4.1, then weak solutions exists. What can you say about the energy of these solutions?

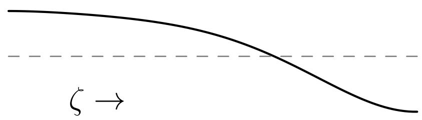

Figure 4.1: Freely vibrating string.

(b) If the endpoints of string are hold at a constant position, i.e., $w(0,t) = w(L,t) = p$, $p \in \mathbb{R}$ independent of time, do solutions exist? If so, what can you say about the energy of these solutions?

4.3. Consider the Timoshenko beam from Example 4.1.3. Show that under boundary conditions as stated in (a) or in (b) weak solutions exists. What can you say about the energy of these solutions?

(a) $\frac{\partial w}{\partial t}(a,t) = \frac{\partial w}{\partial t}(b,t) = 0$, and $\frac{\partial \phi}{\partial t}(a,t) = \frac{\partial \phi}{\partial t}(b,t) = 0$.

(b) $\frac{\partial w}{\partial t}(a,t) = 0$, $\frac{\partial \phi}{\partial t}(a,t) = 0$, $\frac{\partial w}{\partial t}(b,t) = -\frac{\partial w}{\partial \zeta}(b,t) + \phi(b,t)$, and $\frac{\partial \phi}{\partial \zeta}(b,t) = -Q\frac{\partial \phi}{\partial t}(b,t)$, $Q \geq 0$.

4.4. In the theory developed in this chapter, we considered the PDE's of the spatial interval $[a,b]$. However, the theory is independent of this spatial interval. In this exercise, we show that if we have proved a theorem for the spatial interval $[a,b]$, then we can easily formulate the result for the general interval $[0,L]$.

(a) Assume that the spatial coordinate ζ lies in the interval $[a,b]$. Introduce the new spatial coordinate η as

$$\eta = \frac{L(\zeta - a)}{b-a}.$$

Reformulate the PDE (4.13) in the new spatial coordinate.

(b) What are the new $\mathcal{H}$, P_0, when P_1 remains the same?

(c) How do the boundary conditions (4.8) change when using the new spatial variable?

4.5. Consider the transmission line of as shown in Figure 4.2. The model is given by

$$\begin{aligned} \frac{\partial Q}{\partial t}(\zeta,t) &= -\frac{\partial}{\partial \zeta}\frac{\phi(\zeta,t)}{L(\zeta)} \\ \frac{\partial \phi}{\partial t}(\zeta,t) &= -\frac{\partial}{\partial \zeta}\frac{Q(\zeta,t)}{C(\zeta)}, \end{aligned} \tag{4.22}$$

where $Q(\zeta,t), \phi(\zeta,t)$ are the charge and magnetic flux, respectively, at postion ζ and time t. C denotes the (distributed) capacity and L is the distributed inductance.

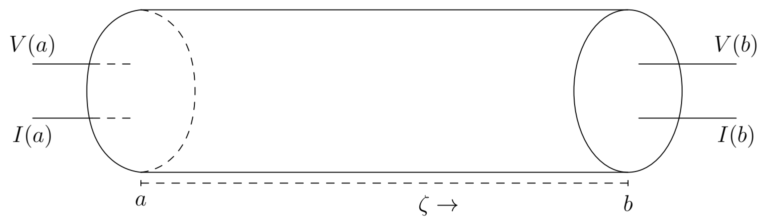

Figure 4.2: Transmission line.

The energy is given by

$$E(t) = \frac{1}{2}\int_a^b \frac{\phi(\zeta,t)^2}{L(\zeta)} + \frac{Q(\zeta,t)^2}{C(\zeta)} d\zeta.$$

(a) Show that the change of energy can be expressed in the voltage $V = Q/C$ and the current $I = \phi/L$ at the boundary.

(b) Show that when the voltages at the boundary are set to zero, then the system is a port-Hamiltonian system, whose solutions stay constant in energy.

(c) Now we put a resistor at the right-end, i.e, $V(b,t) = RI(b,t)$ with $R > 0$, whereas at the other end we put the current equal to zero. Show that the weak solutions exist and are non-increasing in energy.

4.6. Consider coupled vibrating strings as given in the figure below. We

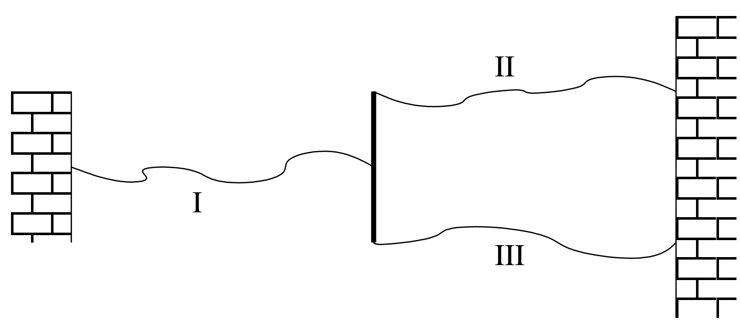

Figure 4.3: Coupled vibrating strings

assume that all the length of the strings are equal. The model for every vibrating string is given by (4.1) with physical parameters, $\rho_{\mathrm{I}}, T_{\mathrm{I}}, \rho_{\mathrm{II}}$, etc. Furthermore, we assume that the three strings are connected via a (massless) bar, as shown in Figure 4.3. This bar can only move in the vertical direction. This implies that the velocity of string I at its right-hand side equals those of the other two strings at their left-hand side. Furthermore, the force of string I at its right-end side equals the sum of the forces of the other two at their left-hand side, i.e.,

$$T_{\mathrm{I}}(b)\frac{\partial w_{\mathrm{I}}}{\partial \zeta}(b) = T_{\mathrm{II}}(a)\frac{\partial w_{\mathrm{II}}}{\partial \zeta}(a) + T_{\mathrm{III}}(a)\frac{\partial w_{\mathrm{III}}}{\partial \zeta}(a).$$

As depictured, the strings are attached to a wall.

(a) Identify the boundary conditions for the system given in Figure 4.3.

(b) Formulate the coupled strings as depictured in Figure 4.3 as a Port-Hamiltonian system (4.13) and (4.8). Furthermore, determine the energy space X.

(c) Show that PDE associated to these vibrating strings possesses for every initial condition in the energy space X, a unique weak solution.

Chapter 5
Inputs and Outputs

In this chapter we study the formulation and existence of solutions for partial differential equations with a control and observation term. Before we do so, we first reconsider the finite-dimensional case. Thus let the following ordinary differential equation be given

$$\ddot{y}(t) + 4\dot{y}(t) + 8y(t) = -3u(t), \tag{5.1}$$

where u is the input, and y is the output of this system. With the state $x(t) = \left(\begin{smallmatrix} y(t) \\ \dot{y}(t) \end{smallmatrix}\right)$ this ODE can be written in the state space form

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -8 & -4 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ -3 \end{pmatrix} u(t) \tag{5.2}$$

$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t), \tag{5.3}$$

or with symbols

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5.4}$$

$$y(t) = Cx(t). \tag{5.5}$$

It is well-known that this inhomogeneous state-space equation possesses a unique solution. We even know the expression for this solution, namely,

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{5.6}$$

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau, \tag{5.7}$$

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where x_0 is the initial condition.

In this chapter we show that PDE's with control and observation within the spatial domain and/or control and observation on the boundary of this spatial domain, an (abstract) equation similar to (5.4) can be derived, and shown to have a unique solution. We will not derive an expression, similar to (5.6) for it. Such an equation does exists, but since we will not use it and since it needs more background we don't treat it in this course.

5.1 Inputs

Similar as writing a homogeneous PDE into an abstract differential equation, we can write an inhomogeneous PDE into an abstract differential equation with an input term. We show this in an example first.

Example 5.1.1 Consider the following controlled partial differential equation on the spatial interval $[0,1]$ in which $c > 0$

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= c\frac{\partial x}{\partial \zeta}(\zeta,t) + u(t) \\ x(1,t) &= 0. \end{aligned} \tag{5.8}$$

For $u = 0$, we have seen in Section 3.3 that the PDE can be written on the state space $X = L^2(0,1)$ as

$$\dot{x}(t) = Ax(t),$$

with

$$\begin{aligned} Ax &= c\frac{dx}{d\zeta}, \\ D(A) &= \left\{x \in L^2(0,1) \mid x \in H^1(0,1) \text{ and } x(1) = 0\right\}. \end{aligned}$$

As for the transformation from ordinary differential equations to state space equations, the input is added to the right hand-side, and so if we define Bu as

$$(Bu)(\zeta) = \mathbb{1}(\zeta) \cdot u, \tag{5.9}$$

where $\mathbb{1}(\zeta)$ is the function identically equal to one. So B is the mapping, which maps the scalar u to the function $\mathbb{1}(\zeta) \cdot u$ (u times the constant-one function). Since the constant functions are elements of the state space X, we see that B maps the scalar $u \in \mathbb{R} = U$ (the input values) into the state space. With the choice of A and B defined in (5.9) our PDE (5.8) can be written as.

$$\dot{x}(t) = Ax(t) + Bu(t). \qquad \square$$

Before we study existence of weak and classical solutions for the inhomogeneous equation, we treat another example first.

Example 5.1.2 Consider the heat equation on the spatial domain $[0,1]$ with boundary and initial conditions like in Example 3.1.2, but now with two heaters

$$\frac{\partial x}{\partial t}(\zeta,t) = c\frac{\partial^2 x}{\partial \zeta^2}(\zeta,t) + \mathbb{1}_{[\frac{1}{8},\frac{3}{8}]}(\zeta)u_1(t) + \mathbb{1}_{[\frac{4}{7},\frac{6}{7}]}(\zeta)u_2(t), \tag{5.10}$$

$$\frac{\partial x}{\partial \zeta}(0,t) = 0 = \frac{\partial x}{\partial \zeta}(1,t), \qquad t \geq 0, \tag{5.11}$$

$$x(\zeta,0) = x_0(\zeta). \tag{5.12}$$

Here by $\mathbb{1}_{[a,b]}$ we mean the function which is identically one when $\zeta \in [a,b]$ and zero elsewhere. So we can put heat into the bar at two places. Namely, uniformly in the interval $[\frac{1}{8},\frac{3}{8}]$ and uniformly in the interval $[\frac{4}{7},\frac{6}{7}]$. This can be done independently of each other.

In Example 3.4.5 we have seen that the homogenous equation can be written as $\dot{x}(t) = Ax(t)$ on the state space $X = L^2(0,1)$, with A and its domain given by, see (3.30) and (3.31),

$$\begin{aligned} Ax &= c\frac{d^2x}{d\zeta^2} \\ D(A) &= \{x \in L^2(0,1) \mid x \in H^2(0,1) \text{ and } \frac{dx}{d\zeta}(1) = 0 = \frac{dx}{d\zeta}(0)\}. \end{aligned}$$

Now we have two inputs, and so the values of the inputs will come from $\mathbb{R}^2$. Keeping in mind that the term Bu must represent the inhomogeneous term, we choose

$$(Bu)(\zeta) = b_1(\zeta)u_1 + b_2(\zeta)u_2, \tag{5.13}$$

where $u = \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right)$, and $b_1(\zeta) = \mathbb{1}_{[\frac{1}{8},\frac{3}{8}]}(\zeta)$, $b_2(\zeta) = \mathbb{1}_{[\frac{4}{7},\frac{6}{7}]}(\zeta)$. Since b_1, b_2 are elements of the state space X, we see that B maps the input space $U = \mathbb{R}^2$ into the state space. With the choice (5.13) and our expression of A, we see that the PDE (5.10)–(5.12) can be written as

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0$$

on the state space $X = L^2(0,1)$. □

The following theorem shows that if we have existence and uniqueness of solutions when $u = 0$, i.e, the homogeneous situation, then that implies existence and uniqueness for a very large class of inputs.

By $L^1_{\text{loc}}([0,\infty);\mathbb{R}^m)$ we denote the set of all functions from $[0,\infty)$ to $\mathbb{R}^m$ which satisfy $\int_0^{t_1} \|u(t)\| dt < \infty$ for all $t_1 > 0$. Finally, by $C^1([0,\infty);\mathbb{R}^m)$ we denote the set of continuously differentiable functions from $[0,\infty)$ to $\mathbb{R}^m$.

Theorem 5.1.3. *Consider on the state space X the inhomogeneous abstract differential equation*

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0. \tag{5.14}$$

Assume that the following holds;

- *The homogeneous equation $\dot{x}(t) = Ax(t), x(0) = x_0$ has for every $x_0 \in X$ a unique weak solution in X;*
- *$u(t)$ takes values in $\mathbb{R}^m$, and B can be written as*

$$Bu = \sum_{j=1}^{m} b_j u_j, \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$$

  with $b_j \in X$, $j = 1, \cdots, m$.

Under these conditions the inhomogeneous equation(5.14) has for every $x_0 \in X$ and every $u \in L^1_{\text{loc}}([0,\infty);\mathbb{R}^m)$ a unique weak solution.

Furthermore, when $u \in C^1([0,\infty);\mathbb{R}^m)$ and $x_0 \in D(A)$, then this weak solution is the unique classical solution of (5.14).

Having the general existence of the solution, we return to the PDE of Example 5.1.1.

Example 5.1.4 Consider the controlled partial differential equation of Example 5.1.1. Thus on the spatial interval $[0,1]$ we have

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= c\frac{\partial x}{\partial \zeta}(\zeta,t) + u(t) \\ x(1,t) &= 0. \end{aligned} \tag{5.15}$$

To show that this system possesses a unique weak solution, we check the conditions of the above theorem.

In Example 5.1.1 this PDE has been rewritten in the abstract form

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with state space $X = L^2(0,1)$, input space $U = \mathbb{R}$,

$$\begin{aligned} Ax &= c\frac{dx}{d\zeta}, \\ D(A) &= \left\{x \in L^2(0,1) \mid x \in H^1(0,1) \text{ and } x(1) = 0\right\}, \\ Bu &= \mathbb{1} \cdot u. \end{aligned}$$

In Example 3.4.4 we have shown that the PDE related to $\dot{x}(t) = Ax(t)$ possesses for every $x_0 \in X$ a unique weak solution. Since $U = \mathbb{R}$ and $Bu = b \cdot u$, with $b(\zeta) = \mathbb{1}(\zeta)$, which is an element of X, we see that both conditions in Theorem 5.1.3 are satisfied, and so the PDE (5.15) possesses a unique weak solution for every initial condition and every input in $L^1_{\text{loc}}((0,\infty);\mathbb{R})$.

In this simple case this solution can be calculated, and is given by

$$x(\zeta,t) = x_0(\zeta + ct)\mathbb{1}_{[0,1]}(\zeta + ct) + \int_0^t \mathbb{1}_{[0,1]}(\zeta + ct - cs)u(s)\,ds. \qquad \square$$

In the examples in this section we applied a control within the spatial domain. However, we could have applied a control at the boundary. When doing so, we cannot rewrite this system in our standard form (5.14). This is general the case when controlling a PDE via its boundary. Thus systems with control at the boundary form a new class of systems, and are introduced in Section 5.3. We first add outputs to the input-state equation treated in this section.

5.2 Outputs

In the previous section we have added an input function to our system. In this section additionally an output is added. As often, we begin with an example. Therefor we take the heat equation of Example 5.1.2 and add a measurement.

Example 5.2.1 Consider the heat equation on the spatial domain $[0,1]$ with boundary, initial conditions, and two inputs as in Example 5.1.2, but with a measurement

$$\frac{\partial x}{\partial t}(\zeta,t) = c\frac{\partial^2 x}{\partial \zeta^2}(\zeta,t) + \mathbb{1}_{[\frac{1}{8},\frac{3}{8}]}(\zeta)u_1(t) + \mathbb{1}_{[\frac{4}{7},\frac{6}{7}]}(\zeta)u_2(t), \tag{5.16}$$

$$\frac{\partial x}{\partial \zeta}(0,t) = 0 = \frac{\partial x}{\partial \zeta}(1,t), \qquad t \geq 0, \tag{5.17}$$

$$x(\zeta,0) = x_0(\zeta), \tag{5.18}$$

$$y(t) = \int_{\frac{1}{3}}^{\frac{2}{3}} x(\zeta,t)d\zeta. \tag{5.19}$$

So we can put heat into the bar at two places and we measure the average temperature on the interval $[\frac{1}{3},\frac{2}{3}]$. This means that the output is a scalar-valued function, and thus the output space Y equals $\mathbb{R}$.

Our state space X equals $L^2(0,1)$ which has the inner product $\langle f,g\rangle = \int_0^1 f(\zeta)g(\zeta)d\zeta$. Knowing this, we see that we can write the output equation as

$$y(t) = \int_{\frac{1}{3}}^{\frac{2}{3}} x(\zeta,t)d\zeta = \int_0^1 \mathbb{1}_{[\frac{1}{3},\frac{2}{3}]}(\zeta)x(\zeta,t)d\zeta = \langle \mathbb{1}_{[\frac{1}{3},\frac{2}{3}]}, x(t)\rangle,$$

with $\mathbb{1}_{[\frac{1}{3},\frac{2}{3}]} \in L^2(0,1)$. If the weak solution exists of the state-differential equation $\dot{x}(t) = Ax(t)+Bu(t)$, then $x(t) \in X$ for every $t \geq 0$. Now since the inner product exists for any two elements in X, the output is well-defined.$\square$

The observation made in the above example is summarised in the following theorem.

Theorem 5.2.2. *Consider on the state space X, input space U and output space Y the abstract system*

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \tag{5.20}$$

$$y(t) = Cx(t) + Du(t). \tag{5.21}$$

Assume that the following holds;

- *The equation $\dot{x}(t) = Ax(t)+Bu(t), x(0) = x_0$ has for the given $x_0 \in X$ and input function $u(t) \in L^1_{\text{loc}}((0,\infty);\mathbb{R}^m)$ a unique weak solution in X;*
- *$y(t)$ takes values in $\mathbb{R}^k$, and Cx can be written as*

$$Cx = \begin{pmatrix} \langle c_1, x\rangle \\ \vdots \\ \langle c_k, x\rangle \end{pmatrix}$$

with $c_j \in X$, $j = 1, \cdots, k$;

- *D is a $k \times m$ matrix.*

Under these conditions the output equation(5.21) is well-defined and is a continuous function.

It may seem obvious that $Cx(t)$ is well-defined, but for observations at the boundary of the spatial domain it is not, as we will see the the following section.

5.3 Boundary control systems

We first explain the idea by means of the controlled transport equation (2.4), see also Example 5.1.1. Consider the following system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= c\frac{\partial x}{\partial \zeta}(\zeta,t), && \zeta \in [0,1],\ t \geq 0 \\ x(\zeta,0) &= x_0(\zeta), && \zeta \in [0,1] \\ x(1,t) &= u(t), && t \geq 0. \end{aligned} \tag{5.22}$$

for an input $u \in L^1_{\text{loc}}(0,\infty)$ and $c > 0$.

Boundary control problems like the one above occur frequently in our applications, but unfortunately they do not fit into our standard formulation (5.14). However, for sufficiently smooth inputs it is possible to reformulate such problems so that they do lead to an associated system in the standard form (5.14). In order to find the associated system for the controlled transport equation we use the following trick. Assume that x is a classical solution of the PDE (5.22) and that u is continuously differentiable. Defining

$$v(\zeta,t) = x(\zeta,t) - u(t),$$

we obtain the following partial differential equation for v

$$\begin{aligned} \frac{\partial v}{\partial t}(\zeta,t) &= c\frac{\partial v}{\partial \zeta}(\zeta,t) - \dot{u}(t), \qquad \zeta \in [0,1],\ t \geq 0 \\ v(1,t) &= 0, \qquad t \geq 0. \end{aligned}$$

As we have seen in Section 5.1, this partial differential equation for v can be written in the standard form as

$$\dot{v}(t) = Av(t) + B\tilde{u}(t)$$

for $\tilde{u} = \dot{u}$. Hence by applying a simple trick, we can reformulate a PDE with boundary control into a PDE with internal control. The price we have to pay is that u has to be smooth. So in particular, not an arbitrary function in L^1, but a more smooth function. Namely, it should have its derivative in L^1.

As in Sections 3.3 and 5.1 we formulate the boundary control abstractly. In the example at the start of this section, we clearly recognise a PDE and a controlled boundary condition. As in Section 3.3 the state $x(t)$ will take values in the state space, which is here $L^2(0,1)$. If we now introduce the operator on $X = L^2(0,1)$

$$\mathfrak{A}f = c\frac{df}{d\zeta},$$

then we can write the PDE as

$$\dot{x}(t) = \mathfrak{A}x(t).$$

To account for the boundary control, we define

$$\mathfrak{B}f = f(1).$$

The boundary control now can be written as

$$\mathfrak{B}x(t) = u(t).$$

Note that as in the previous chapters the ζ-dependence of $x(t)$ is not written down explicitly.

As in Section 3.3 we face the problem that the derivative of a function in $L^2(0,1)$ need not to have a derivative in $L^2(0,1)$ (if it exists). Furthermore, taking the value at $\zeta = 1$ can be troublesome for an arbitrary function in the state space X. So we give both operators a domain.

$$\begin{aligned} \mathfrak{A}f &= c\frac{df}{d\zeta}, \qquad D(\mathfrak{A}) = \left\{f \in L^2(0,1) \mid f \in H^1(0,1)\right\}, \\ \mathfrak{B}f &= f(1), \qquad D(\mathfrak{B}) = D(\mathfrak{A}). \end{aligned}$$

So our system in (5.22) is now formulated in the following way

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0, \\ \mathfrak{B}x(t) &= u(t). \end{aligned} \tag{5.23}$$

We study existence of weak solutions for system written in this abstract form.

Definition 5.3.1. The control system (5.23) is a *boundary control system* if $\mathfrak{A} : D(\mathfrak{A}) \subset X \mapsto X$, $u(t) \in \mathbb{R}^m$, and the *boundary operator* $\mathfrak{B} : D(\mathfrak{A}) \subset X \mapsto \mathbb{R}^m$. Moreover, the following holds:

a. The abstract differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

has for all $x_0 \in X$ a unique weak solution in X. Here A is defined as the operator $A : D(A) \mapsto X$ with $D(A) = D(\mathfrak{A}) \cap \ker(\mathfrak{B})$

$$Ax = \mathfrak{A}x \qquad \text{for } x \in D(A). \tag{5.24}$$

b. There exist functions $b_1, \cdots, b_m \in D(\mathfrak{A})$ such that for all $u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in U = \mathbb{R}^m$

$$\mathfrak{B}\left[\sum_{j=1}^{m} b_j u_j\right] = u, \qquad u \in \mathbb{R}^m. \tag{5.25}$$

□

Part b. of the definition is equivalent to the fact that the range of the operator $\mathfrak{B}$ equals U. So it allows us to choose every value in U for $u(t)$. In other words, the values of inputs are not restricted, which is a logical condition for inputs.

Part a. of the definition guarantees that the system possesses a unique solution when the input term is absent, i.e., when the input is identically zero. In other words, the homogeneous equation is well-posed. This is also a logical condition, since we would like that the trivial input ($u = 0$) is possible.

We say that the function $x(t)$ is a *classical solution* of the boundary control system of Definition 5.3.1 if $x(t)$ is a continuously differentiable function, $x(t) \in D(\mathfrak{A})$ for all t, and $x(t)$ satisfies (5.23) for all t.

For a boundary control system, we can apply a similar trick as the one applied in the beginning of this section. This is the subject of the following theorem.

We show that the solutions of (5.23) are strongly related to solutions of the abstract differential equation

$$\begin{aligned} \dot{v}(t) &= Av(t) - B\dot{u}(t) + \mathfrak{A}Bu(t), \\ v(0) &= v_0, \end{aligned} \tag{5.26}$$

with $Bu = B\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = \sum_{j=1}^m b_j u_j$. Since, by assumption, we have that $\dot{v}(t) = Av(t), v(0) = v_0$ has for every v_0 a unique weak solution, and since $B\dot{u}$ and $\mathfrak{A}Bu$ are of the form $\sum_j p_j u_j$ with $p_j \in X$, we have from Theorem 5.1.3 that equation (5.26) has a unique classical solution for $v_0 \in D(A)$. Furthermore, we prove the relation $v(t) = x(t) - Bu(t)$ between the (classical) solutions of (5.23) and (5.26). Since the proof is quite insightful, we include it.

Theorem 5.3.2. *Consider the boundary control system* (5.23)*, satisfying the conditions of Definition 5.3.1 and the abstract Cauchy equation* (5.26)*. Assume that $u \in C^2([0,\infty); U)$. Then, if $v_0 = x_0 - Bu(0) \in D(A)$, the classical solutions of* (5.23) *and* (5.26) *are related by*

$$v(t) = x(t) - Bu(t). \tag{5.27}$$

Furthermore, the classical solution of (5.23) *is unique.*

PROOF: Suppose that $v(t)$ is a classical solution of (5.26). Then $v(t) \in D(A) \subset D(\mathfrak{A})$, $Bu(t) \in D(\mathfrak{B})$, and so

$$\mathfrak{B}x(t) = \mathfrak{B}[v(t) + Bu(t)] = \mathfrak{B}v(t) + \mathfrak{B}Bu(t) = u(t),$$

where we have used that $v(t) \in D(A) \subset \ker \mathfrak{B}$ and equation (5.25). Furthermore, from (5.27) we have

$$\begin{aligned} \dot{x}(t) &= \dot{v}(t) + B\dot{u}(t) \\ &= Av(t) - B\dot{u}(t) + \mathfrak{A}Bu(t) + B\dot{u}(t) && \text{by (5.26)} \\ &= Av(t) + \mathfrak{A}Bu(t) \\ &= \mathfrak{A}(v(t) + Bu(t)) && \text{by (5.24)} \\ &= \mathfrak{A}x(t) && \text{by (5.27).} \end{aligned}$$

Thus, if $v(t)$ is a classical solution of (5.26), then $x(t)$ defined by (5.27) is a classical solution of (5.23).

The other implication is proved similarly. The uniqueness of the classical solutions of (5.23) follows from the uniqueness of the classical solutions of (5.26). □

As an example we study once more the controlled transport equation.

Example 5.3.3 We consider the following system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= c\frac{\partial x}{\partial \zeta}(\zeta,t), && \zeta\in[0,1],\ t\geq 0\\ x(\zeta,0) &= x_0(\zeta), && \zeta\in[0,1]\\ x(1,t) &= u(t), && t\geq 0 \end{aligned}$$

with $c>0$. We have seen that this system can be written in the form (5.23) by using $X=L^2(0,1)$ and

$$\begin{aligned} \mathfrak{A}f &= c\frac{df}{d\zeta}, && D(\mathfrak{A}) = \left\{f\in L^2(0,1)\mid f\in H^1(0,1)\right\},\\ \mathfrak{B}f &= f(1), && f\in D(\mathfrak{A}). \end{aligned}$$

To show that these two operators satisfy the assumptions of a boundary control system, we first look at the domain of A,

$$D(A) := D(\mathfrak{A})\cap\ker(\mathfrak{B}) = \left\{f\in L^2(0,1)\mid f\in H^1(0,1), f(1)=0\right\}.$$

By Example 3.4.4 we know that $\dot{x}(t)=Ax(t), x(0)=x_0$ with $A=c\frac{d}{d\zeta}$ and this domain, possesses for every initial condition in $X=L^2(0,1)$ a unique weak solution, and so item a. of Definition 5.3.1 is satisfied.

We see that $\mathfrak{B}$ maps into $\mathbb{R}$, and so we choose $U=\mathbb{R}$. Furthermore, when we choose the function b_1 as $b_1(\zeta)=1$ for all ζ, then $b_1\in D(\mathfrak{A})$ and $\mathfrak{B}b_1u=u$ for $u\in\mathbb{R}$. This shows that the second item of Definition 5.3.1 is satisfied as well, and so our boundary controlled transport equation is a boundary control system, and so for sufficiently smooth input and initial condition it possesses a unique classical solution. It is not hard to check that this solution is given by, for $\zeta\in[0,1]$,

$$x(\zeta,t) = \begin{cases} x_0(\zeta+ct) & \zeta+ct\leq 1\\ u(\frac{\zeta-1}{c}+t) & \zeta+ct>1. \end{cases} \tag{5.28}$$

It is easy to see that when $u(0)\neq x_0(1)$, then this solution contains a jump for $\zeta\in[0,1]$ with $\zeta+ct=1$. Thus in that case it is not a classical solution, no matter how smooth u and x_0 are. It is good to realise that the condition $x_0-b_1u(0)\in D(A)$ (see Theorem 5.3.2) implies (among others) that this jump is not present.

Looking at the solution (5.28), we see that for every $t>0$ it is well-defined (as a function in $L^2(0,1)$) for any $u\in L^2$. So in this case the classical solution can be extended to a wider class of inputs, but this falls outside the scope of these lecture notes. $\square$

The approach which we have followed in the section follows the theory as developed by Fattorini in [10]. However, the technique is much older, and is sometimes referred to as "lifting". Namely, you lift the control from the boundary to the interior of the spatial domain. For more on the history, we refer to [30].

As in Section 5.2 it is logical to add an output equation to our boundary control system.

5.3.1 Boundary output

Consider the boundary controlled system of Definition 5.3.1

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0, \\ \mathfrak{B}x(t) &= u(t). \end{aligned}$$

To this equation we add an output. We distinguish two types. Namely, within the spatial domain

$$y(t) = Cx(t) \tag{5.29}$$

and at the boundary

$$y(t) = \mathfrak{C}x(t). \tag{5.30}$$

In both cases we assume that the output space is $Y = \mathbb{R}^k$. For the C in equation (5.29) we assume that it can be written as, see also Theorem 5.2.2,

$$Cx = \begin{pmatrix} \langle c_1, x\rangle \\ \vdots \\ \langle c_k, x\rangle \end{pmatrix}, \quad \text{with } c_j \in X, j = 1, \cdots, k. \tag{5.31}$$

For the $\mathfrak{C}$ in (5.30) we assume that it is a (linear) map from $D(\mathfrak{A})$ to Y. Using Theorem 5.3.2 the following is immediately clear.

Theorem 5.3.4. *Consider the boundary control system* (5.23)*, satisfying the conditions of Definition 5.3.1. Let the output y be given by*

a. *equation (5.29) with C satisfying (5.31), or by*

b. *equation (5.30), with $\mathfrak{C} : D(\mathfrak{A}) \mapsto Y = \mathbb{R}^k$.*

Then for all $u \in C^2([0,\infty); U)$ and $x_0 \in D(\mathfrak{A})$ satisfying $x_0 - Bu(0) \in D(A)$, we have that the output $y(t)$ is well-defined.

PROOF: From the assumptions in the Theorem, we see by Theorem 5.3.2 that there exists a unique classical solution of (5.23). In particular, we have that $x(t) \in D(\mathfrak{A})$ for all $t \geq 0$, and so the output $y(t)$ is well-defined. □

We close this section by adding an output to the controlled transport equation of Example 5.3.3

Example 5.3.5 We consider the system, see also (5.22),

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= c\frac{\partial x}{\partial \zeta}(\zeta, t), \qquad && \zeta \in [0,1],\ t \geq 0 \\ x(\zeta, 0) &= x_0(\zeta), && \zeta \in [0,1] \\ u(t) &= x(1,t), && t \geq 0, \end{aligned} \tag{5.32}$$

with $c > 0$. Now we add the output equation

$$y(t) = x(0,t). \tag{5.33}$$

We can write this in the form (5.30) by defining

$$\mathfrak{C}f = f(0).$$

Since this is well-defined (and linear) on $D(\mathfrak{A}) = \{f \in L^2(0,1) \mid f \in H^1(0,1)\}$, the results from Example 5.3.3 and Theorem 5.3.4 give that the above system has a well-defined output for every smooth input $u(t)$ and initial condition $x_0 \in D(\mathfrak{A})$ satisfying $x_0(1) = u(0)$.

Using (5.28) we find that the output is given by

$$y(t) = \begin{cases} x_0(ct) & ct \leq 1 \\ u(t - \frac{1}{c}) & ct > 1. \end{cases}$$

□

The uncontrolled transport equation is a simple example of our general class of port-Hamiltonian systems. We showed that the controlled version can be written as a boundary control system with an output. In the following section, we do this for general port-Hamiltonian systems.

5.4 Port-Hamiltonian systems as boundary control systems

In this section we add a boundary control and observation to our Hamiltonian system and we show that the assumptions of a boundary control system are satisfied. The port-Hamiltonian system with control and observation is given by

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= P_1\frac{\partial}{\partial \zeta}[\mathcal{H}x(\zeta,t)] + P_0[\mathcal{H}x(\zeta,t)] && (5.34)\\ u(t) &= W_{B,1}\begin{pmatrix}\mathcal{H}x(b,t)\\ \mathcal{H}x(a,t)\end{pmatrix} && (5.35)\\ 0 &= W_{B,2}\begin{pmatrix}\mathcal{H}x(b,t)\\ \mathcal{H}x(a,t)\end{pmatrix} && (5.36)\\ y(t) &= W_C\begin{pmatrix}\mathcal{H}x(b,t)\\ \mathcal{H}x(a,t)\end{pmatrix}. && (5.37)\end{aligned}$$

We first recall the defining operators of the port-Hamiltonian system. As in Section 4.2 we assume that

- P_1 is an invertible, symmetric real $n \times n$-matrix;
- P_0 is an anti-symmetric real $n \times n$-matrix;
- $\mathcal{H}(\zeta)$ is a symmetric, invertible $n \times n$-matrix for every $\zeta \in [a,b]$ and $mI \le \mathcal{H}(\zeta) \le MI$ for some $m, M > 0$ independent of ζ;
- $W_B := \left(\begin{smallmatrix} W_{B,1} \\ W_{B,2}\end{smallmatrix}\right)$ is a full rank real matrix of size $n \times 2n$.

As in Section 4.2 we choose the weighted L^2-space $X = L^2_{\mathcal{H}}((a,b);\mathbb{R}^n)$ equipped with the inner product

$$\langle f,g\rangle_{\mathcal{H}} := \frac{1}{2}\int_a^b f(\zeta)^T\mathcal{H}(\zeta)g(\zeta)\,d\zeta$$

as our state space. The input space U equals $\mathbb{R}^m$, where m is the number of rows of $W_{B,1}$. The output space Y equals $\mathbb{R}^k$, where k is the number of rows of W_C.

We are now in the position to show that this controlled port-Hamiltonian system is indeed a boundary control system.

Therefor we write the controlled port-Hamiltonian system in the abstract form

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0,\\ \mathfrak{B}x(t) &= u(t),\\ y(t) &= \mathfrak{C}x(t),\end{aligned}$$

with

$$\begin{aligned} \mathfrak{A}x &= P_1 \frac{\partial}{\partial \zeta}[\mathcal{H}x] + P_0[\mathcal{H}x], && (5.38)\\ D(\mathfrak{A}) &= \Big\{x \in L^2((a,b);\mathbb{R}^n) \mid \mathcal{H}x \in H^1((a,b);\mathbb{R}^n), && (5.39)\\ &\qquad\qquad W_{B,2}\begin{pmatrix}\mathcal{H}x(b)\\ \mathcal{H}x(a)\end{pmatrix} = 0\Big\}, \\ \mathfrak{B}x &= W_{B,1}\begin{pmatrix}\mathcal{H}x(b)\\ \mathcal{H}x(a)\end{pmatrix}, && (5.40)\\ \mathfrak{C}x &= W_C\begin{pmatrix}\mathcal{H}x(b)\\ \mathcal{H}x(a)\end{pmatrix}. && (5.41) \end{aligned}$$

So we have written the controlled port-Hamiltonian system in the language of a boundary control system. It remains to show that the conditions of Definition 5.3.1 are satisfied. We will only do that for the case that the homogeneous PDE possesses solution non-increasing in the energy norm.

Theorem 5.4.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be given as in* (5.38)–(5.40). *If*

$$\langle \mathfrak{A}x, x\rangle + \langle x, \mathfrak{A}x\rangle \leq 0 \quad \text{for all } x \in D(\mathfrak{A}) \cap \ker \mathfrak{B}, \tag{5.42}$$

then the system (5.34)–(5.36) is a boundary control system on X. Furthermore, for the input u identically zero, the energy of the solution, i.e. $\|x(t)\|^2_{\mathcal{H}}$, will not increase.

Furthermore, for classical solutions of the boundary control problem, the output $y(t)$ is well-defined.

Proof: We begin with the simple observation that

$$\begin{pmatrix}\mathcal{H}x(b)\\ \mathcal{H}x(a)\end{pmatrix} \in \ker \begin{pmatrix}W_{B,1}\\ W_{B,2}\end{pmatrix}$$

is the same as

$$W_{B,1}\begin{pmatrix}\mathcal{H}x(b)\\ \mathcal{H}x(a)\end{pmatrix} = 0 \text{ and } W_{B,2}\begin{pmatrix}\mathcal{H}x(b)\\ \mathcal{H}x(a)\end{pmatrix} = 0.$$

Thus $D(A) := D(\mathfrak{A}) \cap \ker \mathfrak{B}$ equals

$$D(A) = \left\{\mathcal{H}x \in H^1((a,b);\mathbb{R}^n) \mid \begin{pmatrix}\mathcal{H}x(b)\\ \mathcal{H}x(a)\end{pmatrix} \in \ker \begin{pmatrix}W_{B,1}\\ W_{B,2}\end{pmatrix}\right\}. \tag{5.43}$$

Since A is given by, see Definition 5.3.1,

$$Ax = \mathfrak{A}x = P_1 \frac{\partial}{\partial \zeta}[\mathcal{H}x] + P_0[\mathcal{H}x] \tag{5.44}$$

we obtain by our assumptions on the matrices, equation (5.42), and Theorem 4.2.1 that $\dot{x}(t) = Ax(t)$, $x(0) = x_0$ has for every $x_0 \in X$ a unique weak solution in X satisfying $\|x(t)\|_{\mathcal{H}} \leq \|x_0\|_{\mathcal{H}}$. Hence part a. of Definition 5.3.1 is satisfied.

Thus it remains to show that part b. is satisfied as well. The $n \times 2n$-matrix W_B is of full rank n. Thus there exists a $2n \times n$-matrix S such that

$$W_B S = \begin{pmatrix} W_{B,1} \\ W_{B,2} \end{pmatrix} S = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.45}$$

where I_m is the identity matrix on $U = \mathbb{R}^m$. The zeros denote zero block matrices of appropriate sizes, such that the matrix on the right is $n \times n$. Note that (5.45) can be equivalently written as

$$W_{B,1} S = \begin{pmatrix} I_m & 0 \end{pmatrix} \text{ and } W_{B,2} S = \begin{pmatrix} 0 & 0 \end{pmatrix}. \tag{5.46}$$

We write $S = \left(\begin{smallmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{smallmatrix}\right)$ with S_{11} and S_{21} $n \times m$ matrices and we define the for $u \in \mathbb{R}^m$ the function $Bu \in X$ by

$$(Bu)(\zeta) := \mathcal{H}(\zeta)^{-1} \left(S_{11} \frac{\zeta - a}{b - a} + S_{21} \frac{b - \zeta}{b - a} \right) u. \tag{5.47}$$

The definition of Bu implies that for every $u \in \mathbb{R}^m$, Bu is a square integrable function and that $\mathcal{H}Bu \in H^1((a,b);\mathbb{R}^n)$. Next we determine the boundary values which B satisfies.

$$W_{B,2} \begin{pmatrix} (\mathcal{H}Bu)(b) \\ (\mathcal{H}Bu)(a) \end{pmatrix} = W_{B,2} \begin{pmatrix} S_{11}u \\ S_{21}u \end{pmatrix} = W_{B,2} S \begin{pmatrix} u \\ 0 \end{pmatrix} = 0,$$

where we have used (5.46). Using (5.46) once more we see that

$$\mathfrak{B}Bu = W_{B,1} \begin{pmatrix} (\mathcal{H}Bu)(b) \\ (\mathcal{H}Bu)(a) \end{pmatrix} = W_{B,1} \begin{pmatrix} S_{11}u \\ S_{21}u \end{pmatrix} = W_{B,1} S \begin{pmatrix} u \\ 0 \end{pmatrix} = u.$$

Thus the port-Hamiltonian system is indeed a boundary control system.

Since it is easy to see that $\mathfrak{C}$ is well-defined on $D(\mathfrak{A})$, we conclude by Theorem 5.3.4 that the output is well-defined for classical solutions. $\square$

So for smooth controls and initial conditions, satisfying the boundary conditions, we know that solutions of the port-Hamiltonian system exist. Since the energy/Hamiltonian plays an important within this class of systems, it is useful to have a relation between the change of energy (power) and the external signals input and output. In many examples there exists such a relation. When we have n inputs and n outputs, a general formula can be derived expressing this relation.

Note that if we have n input, then $W_B = W_{B,1}$ or equivalently $W_{B,2} = 0$. Furthermore, we assume that we have n outputs, i.e., W_C is a matrix of size $n \times 2n$. Since we do not want to measure quantities that we already have chosen as inputs, see (5.41), we assume that $\left(\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right)$ is of full rank, or equivalently this matrix is invertible.

With this inverse we define

$$P_{W_B,W_C} = \begin{pmatrix} W_B^T & W_C^T \end{pmatrix}^{-1} \begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix} \begin{pmatrix} W_B \\ W_C \end{pmatrix}^{-1}. \tag{5.48}$$

Theorem 5.4.2. *Consider the system (5.34)–(5.37) with W_B and W_C full rank $n \times 2n$ matrices such that $\left(\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right)$ is invertible.*

If the $n \times n$ right-lower submatrix of P_{W_B,W_C} is non-positive, then for every $u \in C^2((0,\infty);\mathbb{R}^n)$, $\mathcal{H}x(0) \in H^1((a,b);\mathbb{R}^n)$, and $u(0) = W_B \left(\begin{smallmatrix} \mathcal{H}x(b,0) \\ \mathcal{H}x(a,0) \end{smallmatrix}\right)$, the system (5.34)–(5.37) has a unique (classical) solution, with $\mathcal{H}x(t) \in H^1((a,b);\mathbb{R}^n)$. The output $y(\cdot)$ is continuous, and the following balance equation is satisfied:

$$\frac{d}{dt}\|x(t)\|_{\mathcal{H}}^2 = \frac{1}{2} \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_{W_B,W_C} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}. \tag{5.49}$$

PROOF: See Exercise 5.1. □

As an example we once more study the controlled transport equation.

Example 5.4.3 We consider the system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial x}{\partial \zeta}(\zeta,t), \qquad \zeta \in [0,1],\ t \geq 0 \\ x(\zeta,0) &= x_0(\zeta), \qquad\quad\ \zeta \in [0,1]. \end{aligned} \tag{5.50}$$

This system can be written in form (5.34) by choosing $n = 1$, $P_0 = 0$, $P_1 = 1$ and $\mathcal{H} = 1$.

Since $n = 1$, we can either apply one control or no control at all. The control free case has been treated in Chapter 4, and so we choose one control. By using the boundary variables, the control is written as, see (5.35)

$$u(t) = \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} x(1,t) \\ x(0,t) \end{pmatrix}. \tag{5.51}$$

Note that $W_B = (\alpha, \beta)$ has full rank if and only if $\alpha^2 + \beta^2 \neq 0$.

Theorem 5.4.1 gives that the PDE (5.50) together with (5.51) is a boundary control system if $\alpha^2 + \beta^2 \neq 0$ and $\alpha^2 \geq \beta^2$, see (5.42). Thus possible boundary controls are for example

$$\begin{aligned} u(t) &= x(1,t), & (\beta = 0), \\ u(t) &= 3x(1,t) - x(0,t), & (\alpha = 3, \beta = -1). \end{aligned}$$

For the control $u(t) = -x(1,t) + 3x(0,t)$ we don't know the answer.

Now we add the output equation

$$y(t) = \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} x(1,t) \\ x(0,t) \end{pmatrix}. \tag{5.52}$$

Since $W_C = (c\ d)$ must have full rank, we find that $c^2+d^2 \neq 0$. Furthermore, since $\begin{pmatrix} W_B \\ W_C \end{pmatrix}$ must be invertible, we find that $\alpha d - \beta c \neq 0$.

The matrix P_{W_B,W_C} of (5.48) is given by

$$\begin{aligned} P_{W_B,W_C} &= \frac{1}{(\alpha d - \beta c)^2} \begin{pmatrix} d & -c \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} d & -\beta \\ -c & \alpha \end{pmatrix} \\ &= \frac{1}{(\alpha d - \beta c)^2} \begin{pmatrix} d^2 - c^2 & -d\beta + c\alpha \\ -d\beta + c\alpha & \beta^2 - \alpha^2 \end{pmatrix}. \end{aligned}$$

For the particular choice $\alpha = 1, \beta = 0$ i.e. $u(t) = x(1,t)$ and $c = 0, d = 1$, that is $y(t) = x(0,t)$, we find $P_{W_B,W_C} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, or equivalently

$$\frac{d}{dt}\|x(t)\|^2_{\mathcal{H}} = \frac{1}{2}\left(u(t)^2 - y(t)^2\right). \qquad \square$$

5.5 Exercises

5.1. Prove Theorem 5.4.2.

Hint: Calculate $\langle \mathfrak{A}x, x\rangle + \langle x, \mathfrak{A}x\rangle$.

5.2. Consider the vibrating string of Example 1.2.4 with the boundary conditions

$$\frac{\partial x}{\partial \zeta}(0,t) = 0 \quad \text{and} \quad T(L)\frac{\partial x}{\partial \zeta}(L,t) = u(t) \quad t \geq 0.$$

Reformulate this system as a boundary control system.

Hint: see Exercise 4.2

5.3. Consider the Timonshenko beam from Example 4.1.3 for any of the following boundary condition:

(a) $\frac{\partial w}{\partial t}(a,t) = u_1(t)$, $\frac{\partial w}{\partial t}(b,t) = u_2(t)$, and $\frac{\partial \phi}{\partial t}(a,t) = \frac{\partial \phi}{\partial t}(b,t) = 0$.

(b) $\frac{\partial w}{\partial t}(a,t) = u(t)$, $\frac{\partial \phi}{\partial t}(a,t) = 0$, $\frac{\partial w}{\partial t}(b,t) = -\frac{\partial \phi}{\partial \zeta}(b,t) + \phi(b,t)$, and $\frac{\partial \phi}{\partial \zeta}(b,t) = -Q\frac{\partial \phi}{\partial t}(b,t)$, $Q \geq 0$.

Reformulate these systems as boundary control systems.

5.4. We consider again the vibrating string of Exercise 5.2 with the given boundary conditions. Define an output to the system such that

$$\frac{d}{dt}\|x(t)\|^2_{\mathcal{H}} = u(t)^T y(t).$$

5.5. In the formulation of port-Hamiltonian systems as boundary control systems with an output, see Section 5.4 we did not impose conditions on W_C, whereas the matrix W_B had to be of rank n.

Sometimes one finds for the $k \times 2n$ matrix W_C the additional condition

$$\text{rank} \begin{pmatrix} W_B \\ W_C \end{pmatrix} = n + k.$$

Explain why the above rank condition is logical.

5.6. Consider the vibrating string of Exercise 5.2 with the boundary conditions

$$\frac{\partial x}{\partial \zeta}(0,t) = 0 \quad \text{and} \quad T(L)\frac{\partial x}{\partial \zeta}(L,t) = u(t) \quad t \geq 0.$$

and we measure the velocity at $\zeta = 0$.

(a) Prove that this is a well-defined input-output system.

(b) Can the change of energy be expressed in the input and output, i.e. does an expression similar to (5.49) hold in this case?

(c) Repeat the above two questions for the measurement $y(t) = \frac{\partial w}{\partial t}(1,t)$.

5.7. Consider the coupled strings of Exercise 4.6. Now we apply a force $u(t)$, to the bar in the middle, see Figure 5.1. This implies that the

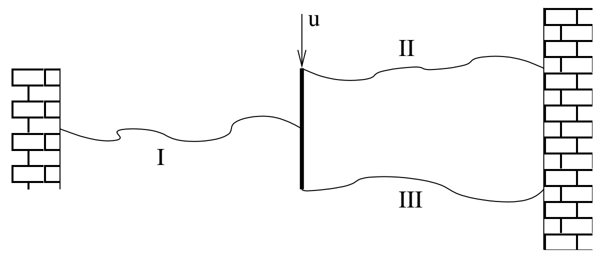

Figure 5.1: Coupled vibrating strings with external force

force balance in the middle becomes

$$T_{\mathrm{I}}(b)\frac{\partial w_{\mathrm{I}}}{\partial \zeta}(b) = T_{\mathrm{II}}(a)\frac{\partial w_{\mathrm{II}}}{\partial \zeta}(a) + T_{\mathrm{III}}(a)\frac{\partial w_{\mathrm{III}}}{\partial \zeta}(a) + u(t).$$

(a) Formulate the coupled vibrating strings with external force as a boundary control system.

(b) Additionally, we measure the velocity of the bar in the middle. Reformulate the system with this output as (5.34)–(5.37).

(c) For the input and output defined above, determine the power balance in terms of the input and output, see (5.49).

Chapter 6
Transfer Functions

In this chapter we discuss the concept of transfer functions. Let us first recapitulate the concept for finite-dimensional systems. Consider the ordinary differential equation

$$\ddot{y}(t) + 3\dot{y}(t) - 7y(t) = -\dot{u}(t) + 2u(t), \tag{6.1}$$

where the dot denotes the derivative with respect to time. In many textbooks one derives the transfer function by taking the Laplace transform of this differential equation under the assumption that the initial conditions are zero. Since the following rules hold for the Laplace transform

$$\begin{aligned}
&\dot{f}(t) \to F(s) - f(0),\\
&\ddot{f}(t) \to s^2F(s) - sf(0) - \dot{f}(0),
\end{aligned}$$

we have that after Laplace transformation the differential equation becomes the algebraic equation:

$$s^2Y(s) - sy(0) - \dot{y}(0) + 3sY(s) - 3y(0) - 7Y(s) = -sU(s) + 2U(s). \tag{6.2}$$

Assuming now that $y(0) = 0$ and that $\dot{y}(0) = 0$, this implies that

$$Y(s) = \frac{-s+2}{s^2+3s-7}U(s). \tag{6.3}$$

The rational function in front of $U(s)$ is called the transfer function associated with the differential equation (6.1).

This is a standard technique, but there are some difficulties with it if we want to extend it to partial differential equations. One of the difficulties is that one has to assume that u and y are Laplace transformable. Since u is chosen, this is not a strong assumption, but once u is chosen, y is dictated by the differential equation, and it is not known a priory whether it is Laplace transformable. Furthermore, the Laplace transform only exists in some right half-plane of the complex plane[1]. This implies that we have the equality (6.3) for those s in the right-half plane for which the Laplace transform of u and y both exist. The right-half plane in which the Laplace transform exists is named the region of convergence. Even for the simple differential equation (6.1) the Laplace transform does not give you the equality (6.3) everywhere. Taking into account the region of convergence of both u and y, after Laplace transform we find that (6.3) only holds for those s which lies right of the *poles*, i.e., the zeros of $s^2 + 3s - 7$.

To overcome all these difficulties we define the transfer function in a different way. We shall look for solutions of the differential equation which are exponentials. Let us illustrate this for the differential equation of (6.1). Given $s \in \mathbb{C}$, we look for a solution pair of the form $(u(t), y(t)) = (e^{st}, y_s e^{st})$. If for an s such a solution exists, and it is unique, then we call y_s the transfer function of (6.1) in the point s. Substituting this pair into our differential equation, we find

$$s^2 y_s e^{st} + 3s y_s e^{st} - 7 y_s e^{st} = -s e^{st} + 2e^{st}. \tag{6.4}$$

We recognise the common term e^{st} which is never zero, and so we may divide by it. After this division, we obtain

$$s^2 y_s + 3s y_s - 7 y_s = -s + 2. \tag{6.5}$$

This is uniquely solvable for y_s if and only if $s^2 + 3s - 7 \neq 0$, and we find that y_s and thus the newly defined transfer function equals

$$\frac{-s+2}{s^2+3s-7}.$$

We see that we have obtained, without running into mathematical difficulties, the transfer function. Moreover, we have it now for all complex s which are not a pole. Since for PDE's or for abstract differential equations the concept of a solution is well-defined, we may define transfer function via exponential functions.

[1] Let $f(t)$ be a function defined for $t \in [0, \infty)$, then its Laplace transform $F(s)$ is defined as $F(s) = \int_0^\infty f(t) e^{-st} dt$ for all $s \in \mathbb{C}$ for which $\int_0^\infty |f(t)| e^{-Re(s)t} dt < \infty$.

6.1 Basic definition and properties

In this section we start with a very general definition of a transfer function, which even applies to systems not described by a PDE, but via e.g. a difference differential equation or an integral equation. To formulate this definition, we first have to introduce a general system[2]. In a general system, we have a time axis, $\mathbb{T}$, which is assumed to be a subset of $\mathbb{R}$. Furthermore, we distinguish three spaces, U, Y, and R. U and Y are the input- and output space, respectively, whereas R contains the rest of the variables. In our examples, R will become the state space. A *system* $\mathfrak{S}$ is a subset of $(U \times R \times Y)^{\mathbb{T}}$, i.e., a subset of all functions from $\mathbb{T}$ to $U \times R \times Y$.

Definition 6.1.1. Let $\mathfrak{S}$ be a system, let s be an element of $\mathbb{C}$, and let $u_0 \in U$. We say that $(u_0 e^{st}, r(t), y(t))$ is an *exponential solution* in $\mathfrak{S}$ if there exist $r_0 \in R$, $y_0 \in Y$, such that $(u_0 e^{st}, r_0 e^{st}, y_0 e^{st}) = (u_0 e^{st}, r(t), y(t))$, $t \in \mathbb{T}$ is in $\mathfrak{S}$.

If for every $u_0 \in U$ the output trajectory, $y_0 e^{st}$, corresponding to an exponential solution is unique, then we call the mapping $u_0 \mapsto y_0$ the *transfer function at s*. We denote this mapping by $G(s)$. Let $\Omega \subset \mathbb{C}$ be the set consisting of all s for which the transfer function at s exists. The mapping $s \in \Omega \mapsto G(s)$ is defined as the *transfer function* of the system $\mathfrak{S}$. $\square$

As you may have noticed, in this definition the system is not defined via equations, but via trajectories. This may seem very strange, since normally we know the set of equations, but not the set of solutions (trajectories), see our discussion in Chapters 3 and 5. However, here it works, since we impose the scape of the solution. We illustrate this on the state space system of the previous chapter. That is, we consider the abstract system of Theorem 5.2.2

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 & (6.6) \\ y(t) &= Cx(t) + Du(t). & (6.7) \end{aligned}$$

Under the conditions stated in Theorems 5.1.3 and 5.2.2, we know that there exist weak solutions, but we don't have an expression for this solution. Since an exponential solution always contains the exponential e^{st} in the input, the input is smooth. For the same reason the state trajectory will be differentiable with respect to t, and so $\dot{x}(t)$ exists. For these reasons we assert that our exponential solutions will be strong/classical solutions

[2] We follow here the behavioural approach, i.e., follow the ideas of J.C. Willems, see [22]

of the differential equation. Thus for the system (6.6)–(6.7) we define the associated "trajectory" system $\mathfrak{S}$ as

$$\mathfrak{S} = \{(u(t), x(t), y(t)) \mid \text{there exists an } x_0 \text{ such that } x(t) \text{ is a classical solution of (6.6) and } y(t) \text{ satisfies (6.7)}\}. \quad (6.8)$$

In the calculation of the transfer function we often encounter equations like

$$(sI - A)x_0 = g$$

with $s \in \mathbb{C}$, $g \in X$ given and $x_0 \in D(A)$ to be found. If this equation is unique solvable for every $g \in X$, then we say that s is in the *resolvent* of A (denoted by $s \in \rho(A)$), and we write

$$x_0 = (sI - A)^{-1}g.$$

With this piece of notation, we can derive the familiar formula $G(s) = C(sI - A)^{-1}B + D$, for the transfer function of (6.6)–(6.7).

Theorem 6.1.2. *Consider the state linear system (6.6)–(6.7). As solutions of this system we take strong/classical solutions, see (6.8)*

The triple $(u(t), x(t), y(t)) = (u_0 e^{st}, x_0 e^{st}, y_0 e^{st})$ is an exponential solution (6.6)–(6.7) if and only if

$$\begin{cases} sx_0 = Ax_0 + Bu_0 \text{ and} \\ y_0 = Cx_0 + Du_0. \end{cases} \quad (6.9)$$

For $s \in \rho(A)$ we have that the transfer function exists and is given by

$$G(s) = C(sI - A)^{-1}B + D. \quad (6.10)$$

PROOF: Since we assume that our exponential solution is a classical one, we may substitute it in equations (6.6) and (6.7). This gives

$$sx_0 e^{st} = Ax_0 e^{st} + Bu_0 e^{st} \quad (6.11)$$
$$y_0 e^{st} = Cx_0 e^{st} + Du_0 e^{st}. \quad (6.12)$$

Since $e^{st} \neq 0$ for all $s \in \mathbb{C}$ and $t \geq 0$, these equations are the same as (6.9).

The top equation of (6.9) can be equivalently written as

$$(sI - A)x_0 = Bu_0. \quad (6.13)$$

Thus for $s \in \rho(A)$,

$$x_0 = (sI - A)^{-1}Bu_0. \quad (6.14)$$

Using the output equation of (6.9), we have that

$$\begin{aligned} y_0 &= Cx_0 + Du_0 \\ &= C(sI - A)^{-1}Bu_0 + Du_0. \end{aligned}$$

Hence $y_0 = C(sI - A)^{-1}Bu_0 + Du_0$ which proves (6.10). □

From this theorem and its proof, we see an alternative way of determining the transfer function of (6.6)–(6.7).

Corollary 6.1.3. *If for for a given $s \in \mathbb{C}$ and all $u_0 \in U$, the equation*

$$sx_0 = Ax_0 + Bu_0$$

is uniquely solvable for $x_0 \in D(A)$, then the transfer function exists at s, and $G(s)u_0 = y_0$, with

$$y_0 = Cx_0 + Du_0$$ □

In the following example we illustrate this construction of the transfer function.

Example 6.1.4 Consider the following system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial^2 x}{\partial \zeta^2}(\zeta, t) + 2\mathbb{1}_{[0,\frac{1}{2}]}(\zeta)u(t) && \zeta \in [0,1],\ t \geq 0 && (6.15) \\ x(\zeta, 0) &= x_0(\zeta), && \zeta \in [0,1] \\ x(0,t) &= 0 = x(1,t) && t \geq 0 \\ y(t) &= 2\int_{\frac{1}{2}}^{1} x(\zeta,t)d\zeta, && t \geq 0. \end{aligned}$$

We can write this in the form (6.6), (6.7) on the state space $X = L^2(0,1)$ with

$$Af = \frac{d^2 f}{d\zeta^2}, \quad D(A) = \{f \in X \mid f \in H^2(0,1), f(0) = f(1) = 0\}$$

and $(Bu)(\zeta) = b_1(\zeta)u = 2\mathbb{1}_{[0,\frac{1}{2}]}(\zeta)u$, and $Cf = 2\int_{\frac{1}{2}}^{1} f(\zeta)d\zeta = \int_0^1 c(\zeta)f(\zeta)d\zeta$ $= \langle c, f\rangle$, with $c(\zeta) = 2\mathbb{1}_{[\frac{1}{2},1]}(\zeta)$. This system satisfies the conditions of Theorems 5.1.3 and 5.2.2.

Now we use the technique of Corollary 6.1.3 to obtain the transfer function. So given $s \in \mathbb{C}$ we try to find a $x_0 \in D(A)$ satisfying

$$sx_0(\zeta) = \frac{d^2 x_0}{d\zeta^2}(\zeta) + 2 \cdot \mathbb{1}_{[0,\frac{1}{2}]}(\zeta)u_0. \tag{6.16}$$

Furthermore, since $x_0 \in D(A)$ we have that $x_0 \in H^2(0,1)$, and

$$x_0(0) = 0 = x_0(1). \tag{6.17}$$

The differential equation (6.16) can be rewritten as the first-order system

$$\frac{d}{d\zeta}\begin{pmatrix} x_0 \\ \frac{dx_0}{d\zeta} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ \frac{dx_0}{d\zeta} \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix}\mathbb{1}_{[0,\frac{1}{2}]}u_0,$$

which has the usual solution for $s \neq 0$

$$\begin{aligned}\begin{pmatrix} x_0(\zeta) \\ \frac{dx_0}{d\zeta}(\zeta) \end{pmatrix} &= \begin{pmatrix} \cosh(\sqrt{s}\zeta) & \frac{1}{\sqrt{s}}\sinh(\sqrt{s}\zeta) \\ \sqrt{s}\sinh(\sqrt{s}\zeta) & \cosh(\sqrt{s}\zeta) \end{pmatrix}\begin{pmatrix} 0 \\ \frac{dx_0}{d\zeta}(0) \end{pmatrix} \\ &\quad -2\int_0^\zeta \begin{pmatrix} \frac{1}{\sqrt{s}}\sinh(\sqrt{s}(\zeta-\tau)) \\ \cosh(\sqrt{s}(\zeta-\tau)) \end{pmatrix}\mathbb{1}_{[0,\frac{1}{2}]}(\tau)u_0 d\tau,\end{aligned}$$

where we have used that $x_0(0) = 0$. In addition, we have

$$\begin{aligned} 0 &= x_0(1) = \frac{\sinh(\sqrt{s})}{\sqrt{s}}\frac{dx_0}{d\zeta}(0) - 2\int_0^{1/2}\frac{\sinh(\sqrt{s}(1-\tau))}{\sqrt{s}}u_0 d\tau \\ &= \frac{\sinh(\sqrt{s})}{\sqrt{s}}\frac{dx_0}{d\zeta}(0) + 2\left[\frac{1}{s}\cosh(\sqrt{s}(1-\xi))\right]_0^{1/2} u_0 \\ &= \frac{\sinh(\sqrt{s})}{\sqrt{s}}\frac{dx_0}{d\zeta}(0) + \frac{2}{s}\left[\cosh(\sqrt{s}\frac{1}{2}) - \cosh(\sqrt{s})\right]u_0. \end{aligned}$$

Thus for all $s \neq 0$ such that $\sinh(\sqrt{s}) \neq 0$ we obtain

$$\frac{dx_0}{d\zeta}(0) = \frac{2[\cosh(\sqrt{s}\frac{1}{2}) - \cosh(\sqrt{s})]}{\sqrt{s}\sinh(\sqrt{s})}\cdot u_0, \tag{6.18}$$

and we have

$$x_0(\zeta) = \frac{\sinh(\sqrt{s}\zeta)}{\sqrt{s}}\frac{dx_0}{d\zeta}(0) - 2\int_0^\zeta \frac{\sinh(\sqrt{s}(\zeta-\tau))\mathbb{1}_{[0,\frac{1}{2}]}(\tau)}{\sqrt{s}}d\tau u_0. \tag{6.19}$$

Calculating the integral, and pugging in (6.18) gives

$$x_0(\zeta) = \begin{cases} \frac{2}{s}[1-\cosh(\sqrt{s}\zeta)]u_0 + \gamma\sinh(\sqrt{s}\zeta), & \zeta \in [0,\frac{1}{2}] \\ \beta\sinh(\sqrt{s}(1-\zeta)) & \zeta \in [\frac{1}{2},1], \end{cases} \tag{6.20}$$

with

$$\begin{aligned}
\gamma &= \frac{2u_0}{s}\frac{\cosh(\sqrt{s}) - \cosh(\sqrt{s}/2)}{\sinh(\sqrt{s})} \\
\beta &= \frac{2u_0[\cosh(\sqrt{s}/2) - 1]}{s\sinh(\sqrt{s})}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
y_0 &= 2\int_{1/2}^{1} x_0(\zeta)d\zeta = 2\beta\int_{1/2}^{1}\sinh(\sqrt{s}(1-\zeta))d\zeta \\
&= \frac{2}{s\sinh(\sqrt{s})}\frac{2}{\sqrt{s}}[\cosh(\sqrt{s}/2) - 1]^2 u_0. \qquad (6.21)
\end{aligned}$$

Thus we may conclude that the transfer function on $\{s \in \mathbb{C} \mid \sinh(\sqrt{s}) \neq 0\} = \{s \in \mathbb{C} \mid s \neq -r^2\pi^2, r = 0, 1, 2, \cdots\}$ is given by

$$G(s) = 4\frac{(\cosh(\frac{\sqrt{s}}{2}) - 1)^2}{s\sqrt{s}\sinh(\sqrt{s})}.$$

Note that at the beginning of this derivation we took $s \neq 0$. To see if $G(0)$ exists, we try to solve (6.16)–(6.17) directly. So we have to solve

$$\frac{d^2x_0}{d\zeta^2}(\zeta) = -2\mathbb{1}_{[0,\frac{1}{2}]}(\zeta)u_0, \qquad x_0(0) = 0 = x_0(1).$$

Integrating once gives

$$\frac{dx_0}{d\zeta}(\zeta) = \begin{cases} -2\zeta u_0 + c_0 & \zeta \in [0, \frac{1}{2}], \\ -u_0 + c_0 & \zeta \in [\frac{1}{2}, 1]. \end{cases}$$

Integrating once more, and using the first boundary condition gives

$$x_0(\zeta) = \begin{cases} -\zeta^2 u_0 + c_0\zeta & \zeta \in [0, \frac{1}{2}], \\ -\zeta u_0 + c_0\zeta + \frac{1}{4}u_0 & \zeta \in [\frac{1}{2}, 1]. \end{cases}$$

Note that we have used the fact that x_0 should be in H^2 and thus continuous differentiable. Using the second boundary condition gives that $c_0 = \frac{3}{4}u_0$. Finally,

$$y_0 = 2\int_{\frac{1}{2}}^{1} x_0(\zeta)d\zeta = 2\int_{\frac{1}{2}}^{1}\left[-\frac{1}{4}\zeta u_0 + \frac{1}{4}u_0\right]d\zeta = \frac{1}{16}u_0.$$

From this we find that $G(0) = \frac{1}{16}$, and thus $s = 0$ is not a pole. $\square$

The above theorem shows that for state-space systems of the form (6.6)–(6.7) the transfer function exists, and is given by the formula well-known from finite-dimensional system theory. Our main class of systems, the port-Hamiltonian systems have their control and observation at the boundary, and are not of the form (6.6)–(6.7). As proved in Theorem 5.4.1, they form a subclass of the boundary control systems. In the following theorem, we formulate transfer functions for boundary control systems. Again since exponential functions are smooth, we take the classical solutions, see Theorem 5.3.2.

Theorem 6.1.5. *Consider the system*

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0 \\ u(t) &= \mathfrak{B}x(t) \\ y(t) &= \mathfrak{C}x(t) \end{aligned} \tag{6.22}$$

where $(\mathfrak{A}, \mathfrak{B})$ satisfies the conditions for boundary control system, see Definition 5.3.1 and $\mathfrak{C}$ is a well-defined (linear) operator from $D(\mathfrak{A})$ to $Y = \mathbb{R}^k$.

If for a given $s \in \mathbb{C}$ equations (6.23)–(6.24) have a unique solution $x_0 \in D(\mathfrak{A})$ for all $u_0 \in U$, then $G(s)$ can be found via (6.25).

$$s x_0 = \mathfrak{A}x_0 \tag{6.23}$$

$$u_0 = \mathfrak{B}x_0 \tag{6.24}$$

$$G(s)u_0 = \mathfrak{C}x_0. \tag{6.25}$$

Proof: Since $x(t) = x_0 e^{st}$ is a classical solution for $u(t) = u_0 e^{st}$, we can substitute this in the differential equation (6.22). By doing so we find

$$\begin{aligned} s x_0 e^{st} &= \mathfrak{A}x_0 e^{st} \\ u_0 e^{st} &= \mathfrak{B}x_0 e^{st} \\ y_0 e^{st} &= \mathfrak{C}x_0 e^{st} \end{aligned}$$

Removing the exponential term, we find the equations (6.23)–(6.25). $\square$

We close this section by calculating the transfer function for the simple Example 5.3.5 and a boundary heat equation.

Example 6.1.6 Consider for $c > 0$ the system

$$\frac{\partial x}{\partial t}(\zeta, t) = c\frac{\partial x}{\partial \zeta}(\zeta, t), \qquad \zeta \in [0,1],\ t \geq 0 \tag{6.26}$$

$$u(t) = x(1,t), \qquad t \geq 0.$$

$$y(t) = x(0,t), \qquad t \geq 0.$$

If we define $\mathfrak{C}x = x(0)$, then it is easy to see that all assumptions in Theorem 6.1.5 are satisfied, see Theorem 5.4.1. Hence we can calculate the transfer function, we do this via the equation (6.23)–(6.25). For the system (6.26) this becomes

$$\begin{aligned} sx_0(\zeta) &= c\frac{\partial x_0}{\partial \zeta}(\zeta) = c\frac{dx_0}{d\zeta}(\zeta) \\ u_0 &= x_0(1) \\ G(s)u_0 &= x_0(0). \end{aligned}$$

The above differential equation has the solution $x_0(\zeta) = \alpha e^{\frac{s}{c}\zeta}$. Using the other two equations, we see that $G(s) = e^{-\frac{s}{c}}$. □

This is nice transfer function, i.e., bounded on the right half-plane. However, this is not always the case as the following example shows.

Example 6.1.7 Consider the system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial^2 x}{\partial \zeta^2}(\zeta, t), & & \zeta \in [0,1],\ t \geq 0 \\ u(t) &= \frac{\partial x}{\partial \zeta}(0, t), \quad x(0,t) = 0 & & t \geq 0. \\ y(t) &= x(1, t), & & t \geq 0. \end{aligned} \tag{6.27}$$

Using the procedure of Theorem 6.1.5 gives the following set of equations

$$\begin{aligned} sx_0(\zeta) &= \frac{d^2 x_0}{d\zeta^2}(\zeta) \\ x_0(0) &= 0, \quad \frac{dx_0}{d\zeta}(0) = u_0 \\ y_0 &= x_0(1). \end{aligned} \tag{6.28}$$

For $s \neq 0$, the first equation gives that

$$x_0(\zeta) = \alpha e^{\sqrt{s}\zeta} + \beta e^{-\sqrt{s}\zeta}, \qquad \zeta \in [0,1],$$

with α, β to be determined by the other two relations. By $x_0(0) = 0$, we find that $\alpha = -\beta$, and $\frac{dx_0}{d\zeta}(0) = u_0$ gives that

$$x_0(\zeta) = \frac{e^{\sqrt{s}\zeta} - e^{-\sqrt{s}\zeta}}{2\sqrt{s}} u_0$$

Hence the transfer function is given by

$$G(s) = \frac{e^{\sqrt{s}} - e^{-\sqrt{s}}}{2\sqrt{s}} = \frac{\sinh(\sqrt{s})}{\sqrt{s}}.$$

It is clear that this function grows without bound when $s \to \infty$.

Again we may ask the question whether $s = 0$ is a pole of our transfer function. To see this we again look at (6.28), but now for $s = 0$. The differential equation has the solution

$$x_0(\zeta) = \alpha + \beta\zeta.$$

Using the second line, we find that $x_0(\zeta) = u_0\zeta$, and thus $G(0) = 1$. □

6.2 Transfer functions for port-Hamiltonian systems

In this section we apply the results found in the previous section to our class of port-Hamiltonian systems. Since we have obtained Theorem 6.1.5 describing transfer functions for general boundary control system, the application to port-Hamiltonian system is straightforward. We obtain the transfer function for the system defined in (5.34)–(5.37). That is, the system is given by

$$\dot{x}(t) = P_1 \frac{\partial}{\partial \zeta}[\mathcal{H}x(t)] + P_0[\mathcal{H}x(t)] \tag{6.29}$$

$$u(t) = W_{B,1} \begin{pmatrix} \mathcal{H}x(b,t) \\ \mathcal{H}x(a,t) \end{pmatrix} \tag{6.30}$$

$$0 = W_{B,2} \begin{pmatrix} \mathcal{H}x(b,t) \\ \mathcal{H}x(a,t) \end{pmatrix} \tag{6.31}$$

$$y(t) = W_C \begin{pmatrix} \mathcal{H}x(b,t) \\ \mathcal{H}x(a,t) \end{pmatrix}. \tag{6.32}$$

It is assumed that (6.29)–(6.32) satisfy the assumption of a port-Hamiltonian system, see Section 5.4. Further we assume that for $u = 0$ the operator associated to (6.29)-(6.31) satisfies the inequality (5.42). This guarantees that (6.29)–(6.32) is a boundary control system, see Theorem 5.4.1.

Theorem 6.2.1. *Consider the system* (6.29)–(6.32). *This system has the transfer function* $G(s)$, *which is determined by*

$$\begin{aligned} sx_0 &= P_1 \frac{d}{d\zeta}[\mathcal{H}x_0] + P_0[\mathcal{H}x_0] && (6.33) \\ u_0 &= W_{B,1} \begin{pmatrix} (\mathcal{H}x_0)(b) \\ (\mathcal{H}x_0)(a) \end{pmatrix} && (6.34) \\ 0 &= W_{B,2} \begin{pmatrix} (\mathcal{H}x_0)(b) \\ (\mathcal{H}x_0)(a) \end{pmatrix} && (6.35) \\ G(s)u_0 &= W_C \begin{pmatrix} (\mathcal{H}x_0)(b) \\ (\mathcal{H}x_0)(a) \end{pmatrix}. && (6.36) \end{aligned}$$

PROOF: The proof is a direct combination of Theorems 5.4.1 and 6.1.5. By the first theorem, we know that the system (6.29)–(6.32) is a well-defined boundary control system and that the output equation is well-defined in the domain of the system operator $\mathfrak{A}$. Hence all conditions of Theorem 6.1.5 are satisfied, and the defining relation for the transfer function, equation (6.23)–(6.25), becomes (6.33)–(6.36). $\square$

Looking at (6.33)–(6.36) we see that the calculation of the transfer function is equivalent to solving an ordinary differential equation. If $\mathcal{H}$ is constant, i.e., independent of ζ, this is easy. However, in general it can be very hard to solve this ordinary differential equation by hand, see Exercise 6.2.

Even if the transfer function is hard/impossible to calculate by hand, we still would like to know properties of it. For instance, where are its poles located? For a port-Hamiltonian system we have the energy function and an associated balance equation, see (4.7). This relates the internal signal (state) with the boundary, and since the input and output are defined at the boundary, we might expect/hope that the power balance can be written in terms of inputs and outputs. This indeed happens when we have full control and full measurements, see Theorem 5.4.2.

We want to show how this power balance reflects in the transfer function. However, since in the definition of the transfer function we used complex exponentials, we have to say how we define the energy for non-real states. If $x(t)$ is complex-valued, then

$$\|x(t)\|_{\mathcal{H}}^2 = E(t) = \frac{1}{2}\int_a^b x(\zeta,t)^* \mathcal{H}(\zeta) x(\zeta,t) d\zeta, \tag{6.37}$$

where for $x(\zeta,t) \in \mathbb{C}^n$, $x(\zeta,t)^*$ denote the complex conjugate. Note that with this definition the energy remains a non-negative quantity. Similar to

Theorem 4.1.2 we have that

$$\frac{dE}{dt}(t) = \frac{1}{2}\left[(\mathcal{H}x)^*(\zeta,t)P_1(\mathcal{H}x)(\zeta,t)\right]_a^b. \tag{6.38}$$

Now we assume that we have full control, i.e, $W_{B,1} = W_B$ and full measurements and that $\begin{pmatrix} W_B \\ W_C \end{pmatrix}$ is of full rank, or equivalently this matrix is invertible. Then based on (6.38), we have for classical solutions the following "complex" version of (5.49);

$$\frac{d}{dt}\|x(t)\|^2_{\mathcal{H}} = \frac{1}{2}\begin{pmatrix} u^*(t) & y^*(t) \end{pmatrix} P_{W_B,W_C}\begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \tag{6.39}$$

where

$$P_{W_B,W_C} = \begin{pmatrix} W_B^T & W_C^T \end{pmatrix}^{-1}\begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix}\begin{pmatrix} W_B \\ W_C \end{pmatrix}^{-1}. \tag{6.40}$$

Theorem 6.2.2. *Consider the system* (6.29)–(6.32) *without homogeneous boundary conditions and with full observation, i.e.,* $W_B = W_{B,1}$ *and* W_C *is an* $n \times 2n$ *matrix. Assume further that* $\begin{pmatrix} W_B \\ W_C \end{pmatrix}$ *is invertible. Then its transfer function satisfies the following equality*

$$Re\,(s)\|x_0\|^2_{\mathcal{H}} = \frac{1}{4}\begin{pmatrix} u_0^T & u_0^T G(s)^* \end{pmatrix} P_{W_B,W_C}\begin{pmatrix} u_0 \\ G(s)u_0 \end{pmatrix}, \tag{6.41}$$

with P_{W_B,W_C} *given by (6.40).*

Proof: The transfer function is by definition related to the exponential solution $(u_0e^{st}, x_0e^{st}, G(s)u_0e^{st})$. Since by Theorem 5.3.2 any classical solution satisfies (6.39), we find that

$$\frac{d}{dt}\|x_0e^{st}\|^2_{\mathcal{H}} = \frac{1}{2}\begin{pmatrix} u_0^Te^{\overline{s}t} & u_0^TG(s)^*e^{\overline{s}t} \end{pmatrix} P_{W_B,W_C}\begin{pmatrix} u_0e^{st} \\ G(s)u_0e^{st} \end{pmatrix}.$$

Note that we have assumed u_0 to be real. Since $e^{\overline{s}t}\cdot e^{st} = e^{2\mathrm{Re}(s)t}$ and since

$$\|x_0e^{st}\|^2_{\mathcal{H}} = |e^{st}|^2\|x_0\|^2_{\mathcal{H}} = e^{2\mathrm{Re}(s)t}\|x_0\|^2_{\mathcal{H}},$$

the above equation can be written as

$$\frac{d}{dt}e^{2\mathrm{Re}(s)t}\|x_0\|^2_{\mathcal{H}} = \frac{1}{2}\begin{pmatrix} u_0^T & u_0^TG(s)^* \end{pmatrix} P_{W_B,W_C}\begin{pmatrix} u_0 \\ G(s)u_0 \end{pmatrix} e^{2\mathrm{Re}(s)t}.$$

Since the derivative of $e^{2\mathrm{Re}(s)t}$ equals $2\mathrm{Re}(s)e^{2\mathrm{Re}(s)t}$, we see that the above equality is equivalent to (6.41). □

In the above theorem we assumed that we had n controls and n measurements. However, an equation similar to (6.41) may hold when we do not have full control and measurements.

In the following example we calculate and investigate properties of the transfer function associated to the transmission line. We start with full control and measurement, and show that from this the finding of the transfer function from just one input to one output is a simple step.

Example 6.2.3 Consider the transmission line of as shown in Figure 6.1. The model is given by

$$\begin{aligned} \frac{\partial Q}{\partial t}(\zeta,t) &= -\frac{\partial}{\partial \zeta}\frac{\phi(\zeta,t)}{L(\zeta)} \\ \frac{\partial \phi}{\partial t}(\zeta,t) &= -\frac{\partial}{\partial \zeta}\frac{Q(\zeta,t)}{C(\zeta)}, \end{aligned} \tag{6.42}$$

where $Q(\zeta,t), \phi(\zeta,t)$ are the charge and magnetic flux, respectively, at postion ζ and time t. C denotes the (distributed) capacity and L is the distributed inductance.

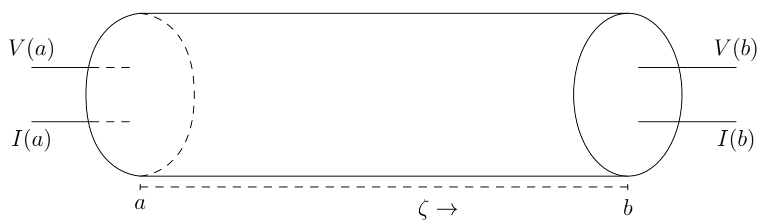

Figure 6.1: Transmission line.

This is a port-Hamiltonian system, see Exercise 4.5, with energy

$$E(t) = \frac{1}{2}\int_a^b \frac{\phi(\zeta,t)^2}{L(\zeta)} + \frac{Q(\zeta,t)^2}{C(\zeta)} d\zeta.$$

We control the voltages $V = Q/C$ at both ends, and measure the currents $I = \phi/L$ at the same points. Furthermore, we assume that the spatial interval $[a,b]$ equals $[0,1]$. Hence the system is the PDE (6.42) together with the control and observation

$$u(t) = \begin{pmatrix} \frac{Q(1,t)}{C(1)} \\ \frac{Q(0,t)}{C(0)} \end{pmatrix} \tag{6.43}$$

$$y(t) = \begin{pmatrix} \frac{\phi(1,t)}{L(1)} \\ \frac{\phi(0,t)}{L(0)} \end{pmatrix}. \tag{6.44}$$

For the transfer function, this PDE is replaced by the ordinary differential equation

$$\begin{aligned} sQ_0(\zeta) &= -\frac{d}{d\zeta}\frac{\phi_0(\zeta)}{L(\zeta)} && (6.45)\\ s\phi_0(\zeta) &= -\frac{d}{d\zeta}\frac{Q_0(\zeta)}{C(\zeta)} \\ u_0 &= \begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix} = \begin{pmatrix} \frac{Q_0(1)}{C(1)} \\ \frac{Q_0(0)}{C(0)} \end{pmatrix} && (6.46)\\ y_0 &= \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \begin{pmatrix} \frac{\phi_0(1)}{L(1)} \\ \frac{\phi_0(0)}{L(0)} \end{pmatrix}. && (6.47) \end{aligned}$$

Since we want to illustrate transfer functions, and their properties, we make the simplifying assumption that $C(\zeta) = L(\zeta) = 1$ for all $\zeta \in [0,1]$. With this assumption, it is easy to see that the solution of (6.45) is given by

$$Q_0(\zeta) = \alpha e^{s\zeta} + \beta e^{-s\zeta}, \qquad \phi_0(\zeta) = -\alpha e^{s\zeta} + \beta e^{-s\zeta}, \tag{6.48}$$

where α, β are (complex) constants. Using (6.46) we can relate these constants to u_0,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{e^s - e^{-s}} \begin{pmatrix} 1 & -e^{-s} \\ -1 & e^s \end{pmatrix} u_0. \tag{6.49}$$

Combining this with (6.47) gives

$$y_0 = \frac{1}{e^s - e^{-s}} \begin{pmatrix} -e^s - e^{-s} & 2 \\ -2 & e^s + e^{-s} \end{pmatrix} u_0.$$

Thus the transfer function is given by

$$G(s) = \begin{pmatrix} -\tanh(s) & -\frac{1}{\sinh(s)} \\ \frac{1}{\sinh(s)} & \tanh(s) \end{pmatrix}. \tag{6.50}$$

Using now the balance equation (6.41), we find

$$\begin{aligned} \mathrm{Re}\,(s)\|x_0\|^2 &= \frac{1}{4} \begin{pmatrix} u_{10} & u_{20} & y_{10}^* & y_{20}^* \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{10} \\ u_{20} \\ y_{10} \\ y_{20} \end{pmatrix} \\ &= \frac{1}{2}\mathrm{Re}\,(u_{20}y_{20}) - \mathrm{Re}\,(u_{10}y_{10}) && (6.51)\\ &= \frac{1}{2}\mathrm{Re}\,(u_{20}G_{21}(s)u_{10} + u_{20}G_{22}(s)u_{20}) - \\ &\quad \frac{1}{2}\mathrm{Re}\,(u_{10}G_{11}(s)u_{10} + u_{10}G_{12}(s)u_{20}). && (6.52) \end{aligned}$$

By taking $u_{10} = 0$, we conclude that the real part of G_{22} is positive for $\mathrm{Re}(s) > 0$. Combined with the fact that G_{22} is *analytic*[3] for $s \in \mathbb{C}_0^+ := \{s \in \mathbb{C} \mid \mathrm{Re}\,(s) > 0\}$, we have that G_{22} is *positive real*. This can also be checked by direct calculation on $G_{22}(s) = \tanh(s)$.

Consider next the system defined by the PDE (6.42) with input $u(t) = Q(1,t)$, output $I(1,t)$ and boundary condition $Q(0,t) = 0$. We can proceed like we did above, but we see that we have already obtained the transfer function by putting $u_{20} = 0$ in (6.46) and only look at y_{10} in (6.47). Hence the transfer function of this single input single output system is $-\tanh(s)$.

The transfer functions (6.50) and $-\tanh(s)$ have their poles on the imaginary axis, and so one cannot draw a *Bode* or *Nyquist plot*. In order to show these concept known from classical control theory can also be used for system described by PDE's we add a damping such that we obtain a system with no poles on the imaginary axis.

We consider the PDE (6.45) with the following conditions

$$\begin{aligned} V(1,t) &= RI(1,t) && (6.53)\\ u(t) &= V(0,t) && (6.54)\\ y(t) &= I(0,t). && (6.55) \end{aligned}$$

Again we take the simplifying assumption that $C(\zeta) = L(\zeta) = 1$, $\zeta \in [0,1]$. Calculation the transfer function leads to the ODE (6.45) with the boundary conditions

$$\begin{aligned} V_0(1) &= RI_0(1) && (6.56)\\ u_0 &= V_0(0) && (6.57)\\ y_0 &= I_0(0). && (6.58) \end{aligned}$$

The ODE has as solution (6.48). The equations (6.56)–(6.58) imply

$$\begin{aligned} \alpha e^s + \beta e^{-s} &= R\left(-\alpha e^s + \beta e^{-s}\right)\\ u_0 &= \alpha + \beta\\ y_0 &= -\alpha + \beta. \end{aligned}$$

Solving this equation gives the following transfer function,

$$G(s) = \frac{\cosh(s) + R\sinh(s)}{\sinh(s) + R\cosh(s)}. \tag{6.59}$$

The Nyquist plot of this is a perfect circle, see Figure 6.2. Again using the

[3] a function from $\Omega \subseteq \mathbb{C}$ to $\mathbb{C}$ is defined to be analytic if its derivative exists for all $s \in \Omega$. The term holomorphic is sometimes used as well.

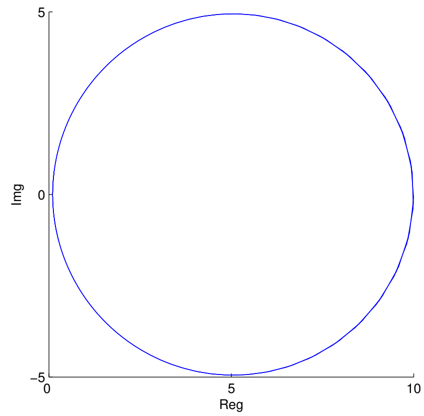

Figure 6.2: Nyquist plot of (6.59) for $R = 10$.

balance equation (6.38), we find that for this system

$$\begin{aligned} 2\mathrm{Re}\,(s)\|x_0\|^2 &= \mathrm{Re}\,(V_0(0)I_0(0)) - \mathrm{Re}\,(V_0(1)I_0(1)) \\ &= \mathrm{Re}\,(u_0 G(s) u_0) - \mathrm{Re}\,(I_0(1) R I_0(1)). \end{aligned}$$

Hence for $\mathrm{Re}(s) > 0$, we have

$$\mathrm{Re}\,(G(s))u_0^2 = 2\mathrm{Re}\,(s)\|x_0\|^2 + R I_0(1)^2 \geq 0, \tag{6.60}$$

and so G is positive real. □

6.3 Exercises

6.1. Determine the transfer function of the system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial \lambda x}{\partial \zeta}(\zeta,t), && \zeta \in [a,b],\ t \geq 0 \\ x(\zeta,0) &= x_0(\zeta), && \zeta \in [0,1] \\ u(t) &= \lambda(b)x(b,t), && t \geq 0. \\ y(t) &= \lambda(a)x(a,t), && t \geq 0, \end{aligned} \tag{6.61}$$

where λ is a (strictly) positive continuous function not depending on t.

6.2. Consider the system of the transmission line given by (6.42)–(6.44).

(a) Show that even when the physical parameter C and L are spatial dependent, the equality (6.52) still holds.

(b) Choose $L(\zeta) = e^{\zeta}$, and $C(\zeta) = 1$, and determine the transfer function.

Hint: You may use a computer package like Maple or Mathematica.

6.3. Our standard port-Hamiltonian system is defined on the spatial interval $[a, b]$. In Exercise 4.4 we have shown that it can easily be transformed to a port-Hamiltonian system on the spatial interval $[0, 1]$. How does the transfer function change?

6.4. Show that the Nyquist plot of transfer function (6.59) is a circle. Furthermore, show that G restricted to the imaginary axis is periodic, and determine the period.

6.5. Consider the vibrating string of Example 1.2.4. We assume that the mass density and Young's modulus are constant. We control this system by controlling the velocity at $\zeta = b$ and the strain at $\zeta = a$, i.e., $u(t) = \begin{pmatrix} \frac{\partial w}{\partial t}(t,b) \\ \frac{\partial w}{\partial \zeta}(t,a) \end{pmatrix}$. We observe the same quantities, but at the opposite ends, i.e., $y(t) = \begin{pmatrix} \frac{\partial w}{\partial t}(t,a) \\ \frac{\partial w}{\partial \zeta}(t,b) \end{pmatrix}$.

Determine the transfer function of this system.

6.6. Since we have defined transfer functions via a different way, it may be good to check some well-known properties of it. Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be two linear and time-invariant systems, with input-output pair u_1, y_1 and u_2, y_2, respectively. Assume that for a given $s \in \mathbb{C}$ both systems have a transfer function.

(a) Show that the series connection, i.e., $u_2 = y_2$ has the transfer function $G(s) = G_2(s)G_1(s)$.

(b) Show that the parallel connection, i.e., $u_1 = u_2 = u$, and $y = y_1 + y_2$ has the transfer function $G_1(s) + G_2(s)$.

(c) Show that the feedback connection, i.e., $u_1 = u - y_2$, $y = y_1$ has the transfer function $G_1(s)\,[I + G_2(s)]^{-1}$ provided $I + G_2(s)$ is invertible.

6.7. Consider the coupled strings of Exercise 5.7. As input we apply a force to the bar in the middle, and as output we measure the velocity of this bar. Assuming that all physical parameters are not depending on ζ, determine the transfer function.

6.4 Notes and references

The ideas for defining the transfer function in the way we did is old, but has hardly been investigated for distributed parameter system. [29] was the first paper where this approach has been used for infinite-dimensional systems. In that paper the concept we named transfer function was called a characteristic function.

One may find the exponential solution in Polderman and Willems [22], where all solutions of this type are called the exponential behaviour.

Chapter 7
Stability and Stabilizability, Time-Domain

7.1 Introduction

In this chapter we study the stability of our systems in time domain. In the following chapter we shall discuss the stability of our system by looking at the frequency domain characterisation.

Here we shall mean by stability, the stability of the state, i.e., we only look at the solutions of the homogeneous differential equation. As for non-linear ordinary differential equations there are two different notions of stability. Namely asymptotic stability and exponential stability, which are defined next.

As before, we write our PDE as the abstract differential equation on the state space X

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0. \tag{7.1}$$

We assume that this equation has a unique weak solution for every initial condition $x_0 \in X$.

We define two concepts of stability.

Definition 7.1.1. The system (7.1) is *asymptotically stable* if for every $x_0 \in X$ the state trajectory $x(t)$ converges to zero for t going to infinity. $\square$

Asymptotic stability is also known under the name *strong stability*. However, we shall use asymptotical stability since that is the standard terminology within systems theory.

Definition 7.1.2. The system (7.1) is *exponentially stable*, if there exists a $M \geq 1$, $\omega < 0$ such that for all $x_0 \in X$ the following holds

$$\|x(t)\| \leq M e^{\omega t} \|x_0\|, \qquad t \geq 0. \tag{7.2}$$

□

Since for $\omega < 0$ there holds $e^{\omega t} \to 0$ as $t \to \infty$, it is easy to see that exponential stability implies asymptotic stability. The converse does not hold, as will be shown in the next example. However, before we show it, it is good to have the following result. If the system is exponentially stable, then at time $t_{\frac{1}{2}} := \log(2M)/(-\omega) > 0$ we have that

$$\|x(t_{\frac{1}{2}})\| \leq M e^{-\log(2M)} \|x_0\| = \frac{M}{2M}\|x_0\| = \frac{1}{2}\|x_0\|. \tag{7.3}$$

So at that time instance every solution has lost at least half of its initial norm (energy). If the system is only asymptotically stable, then this point in time will depend on the initial state. With this observation we study the shift on an infinite long interval.

Example 7.1.3 Consider the partial differential equation of Example 2.1.1, but now on the interval $[0, \infty)$ and with $c = 1$

$$\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial x}{\partial \zeta}(\zeta, t), \quad \zeta, t \geq 0. \tag{7.4}$$

We choose as state space $X = L^2(0, \infty)$. As in Example 3.1.1 it is not hard to show that unique weak solution of (7.4) is given by, see Exercise 7.1,

$$x(\zeta, t) = x_0(t + \zeta), \tag{7.5}$$

where x_0 is the initial condition. First we shall show that this PDE is asymptotically stable. Given the solution (7.5) we have

$$\|x(t)\|^2 = \int_0^\infty x_0(t + \zeta)^2 d\zeta = \int_t^\infty x_0(\eta)^2 d\eta, \tag{7.6}$$

where we made the integral substitution $\eta = t + \zeta$. So we see that

$$\lim_{t\to\infty} \|x(t)\|^2 = \lim_{t\to\infty} \int_t^\infty x_0(\eta)^2 d\eta = 0,$$

since $x_0 \in L^2(0, \infty)$. So this shows asymptotic stability. Next it remains to show that it is not exponentially stable. For this we use the observation made just before this example, see (7.3). If it would be exponentially stable, then should exists a $t_{\frac{1}{2}}$ such that for all $x_0 \in X$, we have $\|x(t_{\frac{1}{2}})\| \leq \frac{1}{2}\|x_0\|$. We construct next an initial condition for which this does not hold. Suppose $t_{\frac{1}{2}}$ exists, then we define

$$\tilde{x}_0(\zeta) = \begin{cases} 1 & \zeta \in [t_{\frac{1}{2}}, t_{\frac{1}{2}} + 1], \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\|\tilde{x}_0\| = 1$. By (7.6) and our choice of $\tilde{x}_0$ we find that

$$\|x(t_{\frac{1}{2}})\|^2 = \int_{t_{\frac{1}{2}}}^{\infty} \tilde{x}_0(\eta)^2 d\eta = \int_{t_{\frac{1}{2}}}^{t_{\frac{1}{2}}+1} 1 d\eta = 1 = \|\tilde{x}_0\|^2.$$

This contradicts (7.3), and so it cannot exist. Concluding we have that this system is asymptotically, but not exponentially stable. □

Hence in contrast to ordinary differential equation with constant coefficients, there is for PDE's (with constant coefficients) a difference between exponential and asymptotic stability. In this chapter we present some techniques which helps you in proving stability. However, it is good to remark that there are many papers on stability, and so very often it is not easy to prove it. In Section 7.2 we show that for port-Hamiltonian systems there is a sufficient condition for exponential stability. However, since port-Hamiltonian systems don't have any internal damping, these conditions often imply that at all but one boundary there is damping.

We end with a small lemma, showing the converse of (7.3).

Lemma 7.1.4. *Consider the (abstract) differential equation $\dot{x}(t) = Ax(t)$, $x(0) = x_0$. If there exists a $t_r > 0$ and a $r \in [0, 1)$ such that for every initial condition $x_0 \in X$, the corresponding solution satisfies*

$$\|x(t_r)\| \leq r\|x_0\|,$$

then the system is exponentially stable.

PROOF: We shall not give the detailed proof but just give the argument on which it is based. Important to realise is that the differential equation is time invariant. This implies that we can see $x(2t_r)$ as the solution of $\dot{x}(t) = Ax(t)$ at time t_r, when we would have taken the initial condition $x(t_r)$. Since the estimate holds for every initial condition, we have that

$\|x(2t_r)\| \le r\|x(t_r)\|$. Combining this with the initial estimate, we obtain $\|x(2t_r)\| \le r\|x(t_r)\| \le r^2\|x_0\|$. Repeating this argument gives

$$\|x(kt_r)\| \le r^k\|x_0\|, \quad k \in \mathbb{N}.$$

We can write

$$r^k = e^{k\log(r)} = e^{\frac{kt_r}{t_r}\log(r)}.$$

Combining these, we find that for $t = kt_r$ there holds

$$\|x(t)\| \le e^{\omega t}\|x_0\|,$$

where $\omega = \log(r)/t_r$, which is negative since $r \in (0,1)$. Some extra work gives that $\|x(t)\| \le Me^{\omega t}\|x_0\|$ for all $t \ge 0$. □

In the following section we consider our class of port-Hamiltonian system, and we show that a simple condition is guaranteeing exponential stability.

7.2 Exponential stability of port-Hamiltonian systems

We return to our homogeneous port-Hamiltonian system of Section 4.2. That is we consider the PDE

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1\frac{\partial}{\partial \zeta}\left[\mathcal{H}(\zeta)x(\zeta,t)\right] + P_0\left[\mathcal{H}(\zeta)x(\zeta,t)\right] \tag{7.7}$$

with the boundary condition

$$W_B\begin{pmatrix} (\mathcal{H}x)(b,t) \\ (\mathcal{H}x)(a,t)\end{pmatrix} = 0, \tag{7.8}$$

As in Theorem 4.2.1 we assume that the following holds

ASSUMPTION 7.2.1:

- P_1 is an invertible, symmetric real $n \times n$ matrix;
- P_0 is an anti-symmetric real $n \times n$ matrix;
- For all $\zeta \in [a,b]$ the $n \times n$ matrix $\mathcal{H}(\zeta)$ is real, symmetric, and $mI \le \mathcal{H}(\zeta) \le MI$, for some $M, m > 0$ independent of ζ;
- $\mathcal{H}$ is continuously differentiable on the interval $[a,b]$;

- W_B be a full rank real matrix of size $n \times 2n$; $\square$

The above, with the exception of the differentiability of $\mathcal{H}$, have been our standard assumptions in many previous chapters. However, we would like to remark that our main Theorem 7.2.3 also holds if P_0 satisfies $P_0 + P_0^T \leq 0$. Under the conditions as listed in Assumption 7.2.1 we know that the abstract differential equation $\dot{x}(t) = Ax(t), x(0) = x_0$ with

$$Ax := P_1 \frac{d}{d\zeta} [\mathcal{H}x] + P_0 [\mathcal{H}x] \tag{7.9}$$

and domain

$$D(A) = \{x \in L^2((a,b);\mathbb{R}^n) \mid \mathcal{H}x \in H^1((a,b);\mathbb{R}^n),\ W_B \left(\begin{smallmatrix} \mathcal{H}(b)x(b) \\ \mathcal{H}(a)x(a) \end{smallmatrix} \right) = 0\} \tag{7.10}$$

possesses for every $x_0 \in X$ a unique weak solution $x(t) \in X$ with $\|x(t)\|_{\mathcal{H}} \leq \|x_0\|_{\mathcal{H}}$, where

$$X = L^2((a,b);\mathbb{R}^n) \tag{7.11}$$

with inner product

$$\langle f, g \rangle_{\mathcal{H}} = \frac{1}{2} \int_a^b f(\zeta)^T \mathcal{H}(\zeta) g(\zeta) d\zeta, \tag{7.12}$$

provided that

$$\langle Ax_0, x_0 \rangle_{\mathcal{H}} + \langle x_0, Ax_0 \rangle_{\mathcal{H}} \leq 0. \tag{7.13}$$

This result can be found in Section 4.2, in particular Theorem 4.2.1.

The following lemma shows that there exists a time instant $\tau > 0$ such that the norm/energy of a state at that time can be bounded by the energy observed at one of the boundaries during the period $0, \tau]$.

Lemma 7.2.2. *Consider the operator A given by (7.9) and (7.10) and assume that it satisfies (7.13). Let $x_0 \in X$ be any initial condition, then for sufficiently large $\tau > 0$ the state trajectory $x(t) := T(t)x_0$ satisfies*

$$\|x(\tau)\|_{\mathcal{H}}^2 \leq c \int_0^\tau \|(\mathcal{H}x)(b,t)\|^2 \, dt \qquad \text{and} \tag{7.14}$$

$$\|x(\tau)\|_{\mathcal{H}}^2 \leq c \int_0^\tau \|(\mathcal{H}x)(a,t)\|^2 \, dt, \tag{7.15}$$

where $c > 0$ is a constant that only depends on τ and not on x_0.

With this technical lemma, the proof of exponential stability is easy.

Theorem 7.2.3. *Consider the operator A defined in (7.9) and (7.10). Furthermore, we assume that the conditions in Assumption 7.2.1 are satisfied. If for some positive constant k one of the following conditions is satisfied for all $x_0 \in D(A)$*

$$\langle Ax_0, x_0\rangle_{\mathcal{H}} + \langle x_0, Ax_0\rangle_{\mathcal{H}} \leq -k\|(\mathcal{H}x_0)(b)\|^2 \tag{7.16}$$

$$\langle Ax_0, x_0\rangle_{\mathcal{H}} + \langle x_0, Ax_0\rangle_{\mathcal{H}} \leq -k\|(\mathcal{H}x_0)(a)\|^2, \tag{7.17}$$

then the system is exponentially stable.

Proof: Without loss of generality we assume that the first inequality (7.16) holds. Since that inequality implies (7.13), we know by Lemma 7.2.2 that (7.14) holds for some $\tau > 0$.

Let $x(t)$ be the solution of $\dot{x}(t) = Ax(t), x(0) = x_0$. If we take $x_0 \in D(A)$, then this is a classical solution, i.e., $x(t) \in D(A)$, $x(t)$ is differentiable and $x(t)$ satisfies the differential equation. Using this last fact, it is easy to see that

$$\frac{d\|x(t)\|^2_{\mathcal{H}}}{dt} = \frac{d\langle x(t), x(t)\rangle_{\mathcal{H}}}{dt} = \langle Ax(t), x(t)\rangle_{\mathcal{H}} + \langle x(t)Ax(t)\rangle_{\mathcal{H}}. \tag{7.18}$$

Using this and equation (7.16), we have that

$$\begin{aligned}
\|x(\tau)\|^2_{\mathcal{H}} - \|x(0)\|^2_{\mathcal{H}} &= \int_0^\tau \frac{d\|x(t)\|^2_{\mathcal{H}}}{dt}(t)\, dt \\
&= \int_0^\tau \langle Ax(t), x(t)\rangle_{\mathcal{H}} + \langle x(t), Ax(t)\rangle_{\mathcal{H}}\, dt \\
&\leq -k\int_0^\tau \|(\mathcal{H}x)(b,t)\|^2 dt.
\end{aligned}$$

Combining this with (7.14), we find that

$$\|x(\tau)\|^2_{\mathcal{H}} - \|x(0)\|^2_{\mathcal{H}} \leq \frac{-k}{c}\|x(\tau)\|^2_{\mathcal{H}}.$$

Thus $\|x(\tau)\|^2_{\mathcal{H}} \leq \frac{c}{c+k}\|x(0)\|^2_{\mathcal{H}}$. Since $c, k > 0$ we have that $0 < \frac{c}{c+k} < 1$, and so the conditions of Lemma 7.1.4 are satisfies. Applying that lemma, we conclude that our system is exponentially stable. □

Estimate (7.16) provides a simple way to prove the exponential stability property. We note that Theorem 4.1.2 implies

$$\langle Ax, x\rangle_{\mathcal{H}} + \langle x, Ax\rangle_{\mathcal{H}} = (\mathcal{H}x)^T(b)P_1(\mathcal{H}x)(b) - (\mathcal{H}x)^T(a)P_1(\mathcal{H}x)(a). \tag{7.19}$$

This equality can be used on a case by case bases to show exponential stability. However, when we have full damping, i.e., no homogeneous boundary conditions, then exponential stability will hold. However, that case is not very interesting, since it hardly happens. We will illustrate it in the following example.

Example 7.2.4 Consider the *transmission line* on the spatial interval $[a, b]$

$$\begin{aligned} \frac{\partial Q}{\partial t}(\zeta, t) &= -\frac{\partial}{\partial \zeta} \frac{\phi(\zeta, t)}{L(\zeta)} \\ \frac{\partial \phi}{\partial t}(\zeta, t) &= -\frac{\partial}{\partial \zeta} \frac{Q(\zeta, t)}{C(\zeta)}. \end{aligned} \tag{7.20}$$

Here $Q(\zeta, t)$ is the charge at position $\zeta \in [a, b]$ and time $t > 0$, and $\phi(\zeta, t)$ is the flux at position ζ and time t. C is the (distributed) capacity and L is the (distributed) inductance. This example we studied in Exercise 4.5 and Example 6.2.3. To the above PDE we add the following input and output

$$u(t) = \begin{pmatrix} \frac{Q(b,t)}{C(b)} \\ \frac{Q(a,t)}{C(a)} \end{pmatrix} = \begin{pmatrix} V(b, t) \\ V(a, t) \end{pmatrix} \tag{7.21}$$

$$y(t) = \begin{pmatrix} \frac{\phi(b,t)}{L(b)} \\ \frac{\phi(a,t)}{L(a)} \end{pmatrix} = \begin{pmatrix} I(b, t) \\ I(a, t) \end{pmatrix}. \tag{7.22}$$

First we want to know whether the homogeneous system, i.e., $u(t) = 0, t \geq 0$, is (exponentially) stable. A simple calculation gives that for $x_0 \in D(A)$

$$\langle Ax_0, x_0 \rangle_{\mathcal{H}} + \langle x_0, Ax_0 \rangle_{\mathcal{H}} = V_0(a)I_0(a) - V_0(b)I_0(b). \tag{7.23}$$

Thus when there is no input, this expression is zero. By Theorem 4.2.1 we conclude that for every initial condition we have that the energy stays constant, i.e., $\|x(t)\|_{\mathcal{H}} = \|x_0\|_{\mathcal{H}}$ and thus this PDE cannot be (asymptotically) stable.

Now we apply an output feedback. If we apply a full output feedback, then it is not hard to show that we have obtained an exponentially stable system, see Exercise ??

We want to consider a more interesting example, in which we only apply a feedback on one of the boundaries. This is, we set the first input to zero, and put a resistor at the other end. This implies that we have the PDE (7.20) with boundary conditions

$$V(a, t) = 0, \qquad V(b, t) = RI(b, t), \tag{7.24}$$

with $R > 0$. Using (7.23) we find for $x_0 \in D(A)$

$$\langle Ax_0, x_0\rangle_{\mathcal{H}} + \langle x_0, Ax_0\rangle_{\mathcal{H}} = -RI_0(b)^2. \tag{7.25}$$

Furthermore, we have that $(\mathcal{H}x)(b) = \begin{pmatrix} V(b) \\ I(b) \end{pmatrix}$. Thus

$$\|(\mathcal{H}x)(b)\|^2 = V(b)^2 + I(b)^2 = (R^2+1)I(b)^2. \tag{7.26}$$

Combining the two previous equations, we find that for $x_0 \in D(A)$

$$\langle Ax_0, x_0\rangle_{\mathcal{H}} + \langle x_0, Ax_0\rangle_{\mathcal{H}} \leq -\frac{R}{1+R^2}\|(\mathcal{H}x_0)(b)\|^2. \tag{7.27}$$

Hence by Theorem 7.2.3 we conclude that putting a resistor at one end of the transmission line stabilises the system exponentially. Note that we have assumed that the conditions as stated in Assumption 7.2.1 hold. This is left as an exercise to the reader, see Exercise 4.5. □

7.3 Lyapunov functions

In the previous section we concentrated on port-Hamiltonian systems. Although they form an important class, we have already seen other PDE's. So we would like to have a technique to study stability of these as well. We do this via Lyapunov functions. As you might know from a course on ODE's, Lyapunov functions can be used to prove stability for a wide class of differential equations, including non-linear ones. In this section we concentrate on linear differential equations.

Definition 7.3.1. Consider the abstract differential equation (7.1) on the state space X. We say that $X \mapsto [0,\infty)$ is a *Lyapunov function* when the following conditions are satisfied

1. $V(0) = 0$, and $V(x_0) > 0$ whenever $x_0 \neq 0$;
2. For all $x_0 \in D(A)$ there holds

$$\dot{V}(x_0) := \frac{dV}{dx}(Ax_0) \leq 0. \tag{7.28}$$

□

The last condition can also be read as

$$\frac{dV(x(t))}{dt} \leq 0 \text{ for } t \geq 0 \tag{7.29}$$

for all classical solutions of (7.1), or equivalently,

$$V(x(t_2)) \leq V(x(t_1)), \text{ whenever } t_2 \geq t_1. \tag{7.30}$$

So a Lyapunov function is a positive function which is non-increasing along the solutions. The following theorem shows how Lyapunov functions can be used to prove exponential stability.

Theorem 7.3.2. *Consider the abstract differential equation $\dot{x}(t) = Ax(t)$, $x(0) = x_0$ on the state space X. We assume that for every $x_0 \in X$ there exists a weak solution of this equation.*

Let V be a Lyapunov function for which there exists a $\alpha, \beta > 0$ and $n \in \mathbb{N}$ such that for all $x_0 \in D(A)$

$$\dot{V}(x_0) \leq -\alpha V(x_0), \tag{7.31}$$

$$\beta \|x_0\|^n \leq V(x_0), \tag{7.32}$$

and

$$\sup_{\{x_0 \in X \mid \|x_0\|=1\}} V(x_0) < \infty, \tag{7.33}$$

then the system is exponentially stable.

Proof: Let $x_0 \in D(A)$ and let $x(t)$ be the (classical) solution of $\dot{x}(t) = Ax(t)$, $x(0) = x_0$. Then by (7.31), see also (7.29), there holds

$$\frac{dV(x(t))}{dt} \leq -\alpha V(x(t)), \quad t \geq 0.$$

Thus

$$V(x(t)) \leq e^{-\alpha t} V(x(0)),$$

and by using (7.32) we find

$$\|x(t)\| \leq \sqrt[n]{\frac{V(x_0)}{\beta}} e^{\omega t}$$

with $\omega = -\alpha/n$. Using now (7.33) we have that by choosing $M = \sqrt[n]{\frac{m}{\beta}}$ with $m = \sup_{\{x_0 \in X \mid \|x_0\|=1\}} V(x_0)$, for all $x_0 \in D(A)$ satisfying $\|x_0\| = 1$ there holds

$$\|x(t)\| \leq M e^{\omega t}.$$

Since the system is linear we find that if z_0 does not have norm one, then $x_0 := \frac{z_0}{\|z_0\|}$ has norm one, and the solution corresponding to z_0 is $\|z_0\|$ times the one corresponding to x_0.

Hence $\|x(t)\| \leq Me^{\omega t}\|x_0\|$ for all initial conditions in the domain of A, and so we conclude the system is exponentially stable for initial conditions in the domain of A. By using the denseness of the domain of A in X, we can get $\|x(t)\| \leq Me^{\omega t}\|x_0\|$ for all $x_0 \in X$. □

Although this theorem gives a nice condition for exponential stability, in practise the condition (7.31) can be quite hard to obtain, and most times clever estimates are needed as is illustrated in the next example. The conditions (7.32) and (7.33) normally always holds and will not be hard to check.

Example 7.3.3 Consider a heated bar of length one, which is hold at constant temperature at its boundary

$$\frac{\partial w}{\partial t}(\zeta, t) = c\frac{\partial^2 w}{\partial \zeta^2}(\zeta, t), \quad \zeta \in (0, 1),$$

where $c > 0$,

$$w(0, t) = 0 = w(1, t).$$

As Lyapunov function we try the (squared) $L^2(0, 1)$-norm, i.e.,

$$V(x) = \frac{1}{2}\int_0^1 x(\zeta)^2 d\zeta,$$

where we have added a half for conveniences. Note that $V(x) = \frac{1}{2}\|x\|^2$, since we work with the state space $L^2(0, 1)$, and so (7.32) and (7.33) clearly hold.

Differentiating V along classical solutions gives

$$\begin{aligned}
\frac{dV}{dt}(x(t)) &= \frac{1}{2}\frac{d}{dt}\left(\int_0^1 w(\zeta, t)^2 d\zeta\right) \\
&= \int_0^1 w(\zeta, t)\frac{\partial w}{\partial t}(\zeta, t) d\zeta \\
&= \int_0^1 w(\zeta, t) c\frac{\partial^2 w}{\partial \zeta^2}(\zeta, t) d\zeta \\
&= \left[w(\zeta, t) c\frac{\partial w}{\partial \zeta}(\zeta, t)\right]_0^1 - c\int_0^1 \left[\frac{\partial w}{\partial \zeta}(\zeta, t)\right]^2 d\zeta \\
&= -c\int_0^1 \left[\frac{\partial w}{\partial \zeta}(\zeta, t)\right]^2 d\zeta, \qquad (7.34)
\end{aligned}$$

where we have used the PDE, integrating by parts, and the boundary conditions. From this we see that V is non-increasing, but we don't have yet the inequality to conclude exponential stability. The last expression equals the squared norm of the spatial derivative of the state, and not of the state itself. That would be needed for an inequality like (7.31). To obtain this, we need to do some more work.

Since $w(0,t) = 0$, we have that for $p \in [0,1]$

$$w(p,t) = \int_0^p \frac{\partial w}{\partial \zeta}(\zeta,t) d\zeta.$$

Using now the Cauchy-Schwarz inequality on the interval $[0,p]$, we find that

$$|w(p,t)| \le \sqrt{\int_0^p 1^2 d\zeta} \sqrt{\int_0^p \frac{\partial w}{\partial \zeta}(\zeta,t)^2 d\zeta}$$

Thus

$$|w(p,t)|^2 \le p \int_0^p \frac{\partial w}{\partial \zeta}(\zeta,t)^2 d\zeta \le p \int_0^1 \frac{\partial w}{\partial \zeta}(\zeta,t)^2 d\zeta. \tag{7.35}$$

Integrating this expression over p gives

$$\int_0^1 |w(p,t)|^2 dp \le \frac{1}{2} \int_0^1 \frac{\partial w}{\partial \zeta}(\zeta,t)^2 d\zeta. \tag{7.36}$$

Combining this with (7.34), we find that

$$\begin{aligned} \frac{dV}{dt}(x(t)) &\le -c \int_0^1 \left[\frac{\partial w}{\partial \zeta}(\zeta,t) \right]^2 d\zeta \\ &\le -2c \int_0^2 |w(p,t)|^2 dp = -4cV(x(t)). \end{aligned} \tag{7.37}$$

From Theorem 7.3.2 we conclude that the system is exponentially stable.□

It is good to note that that constant $-4c$ in (7.37) is not the optimal constant. Hence the exponential decay of the solution will be more that $-4c$. To find the optimal constant, i.e., to find the precise decay can be very hard. Here we can improve (7.37) by improving (7.35). For instance, using the other boundary condition, we can replace (7.35) by

$$|w(p,t)|^2 \le \min\{p, 1-p\} \int_0^1 \frac{\partial w}{\partial \zeta}(\zeta,t)^2 d\zeta.$$

However, even this would not give the optimal decay rate. Knowing the location of the eigenvalues gives an exact estimate of the decay rate. This is explained in the following section.

7.4 Eigenvalues, eigenfunctions and stability

For matrices it is well-known that stability is closely linked to eigenvalues, i.e., (complex) λ's for which $Av = \lambda v$ is solvable for a non-zero vector v. For abstract state equations this is partly true. We begin by defining eigenvalues and eigenfunctions. Since eigenvalues can be complex numbers, λv will normally no longer lie in a real valued state space. Therefore we have to "complexitfy" the space. We assume that we are working with with a complex valued spaces. For instance, $L^2((a,b);\mathbb{R}^n)$ is replaced by $L^2((a,b);\mathbb{C}^n)$ and the inner product has changed from $\langle f, g\rangle = \int_a^b f(\zeta)^T g(\zeta)\, d\zeta$ to $\int_a^b g(\zeta)^* f(\zeta)\, d\zeta$.

Definition 7.4.1. Let A be an operator from $D(A) \subseteq X$ to X. We call $\lambda \in \mathbb{C}$ an *eigenvalue* if there exists an $f \in D(A)$ with $f \neq 0$, such that

$$Af = \lambda f.$$

If such a f exists, then we call it the *eigenfunction* associated to λ. □

The link with stability is explained next.

Lemma 7.4.2. *If $\lambda \in \mathbb{C}$ is an eigenvalue of the operator A with eigenfunction f, then a (classical) solution of*

$$\dot{x}(t) = Ax(t) \qquad x(0) = f \tag{7.38}$$

is given $x(t) = e^{\lambda t} f$. When we know that solutions of (7.38) are unique, then it is the only solution.

Proof: We can easily check that $x(t) = e^{\lambda t} f$ is a solution of (7.38). Namely,

$$\frac{dx}{dt}(t) = \lambda e^{\lambda t} f = e^{\lambda t}\lambda f = e^{\lambda t} A f = A\left(e^{\lambda t} f\right) = Ax(t),$$

where have used the linearity of A. Since $x(0) = f$, we see that $e^{\lambda t} f$ is a classical solution of (7.38). □

The above lemma induces the following important observations.

If there exists an eigenvalue λ with its real part larger or equal to zero, then the system cannot be asymptotically stable.

Unfortunately in general the converse does not hold, i.e., all eigenvalues in the open left half-plane does in general not imply that the system is stable as is shown in the following example.

Example 7.4.3 Consider the partial differential equation

$$\frac{\partial w}{\partial t}(\zeta,t) = -\frac{\partial w}{\partial \zeta}(\zeta,t) + w(\zeta,t), \quad \zeta, t \geq 0 \tag{7.39}$$

with boundary condition $w(0,t) = 0$. We can write this in the abstract form $\dot{x}(t) = Ax(t)$ on the state space $X = L^2(0,\infty)$ and A given by

$$Af = -\frac{df}{d\zeta} + f$$

with domain

$$D(A) = \{f \in L^2(0,\infty) \mid \frac{df}{d\zeta} \in L^2(0,\infty) \text{ and } f(0) = 0\}.$$

We will show that this PDE is unstable but has no eigenvalues. We start by showing the last assertion. For $\lambda \in \mathbb{C}$ to be an eigenvalue there should exists an $f \in D(A)$ with $f \neq 0$ and $Af = \lambda f$. Using our expression of A this equation becomes

$$-\frac{df}{d\zeta} + f = \lambda f \tag{7.40}$$

which is equivalent to $\frac{df}{d\zeta} = (1-\lambda)f$. The solution of this differential equation is well-known, namely

$$f(\zeta) = f_0 e^{(1-\lambda)\zeta}.$$

However, f should be an element of $D(A)$ and thus in particular, $f(0) = 0$. Using the latter, we find $f_0 = 0$, and thus $f = 0$, which is not allowed for an eigenfunction. So this A has no eigenvalues.

To show that the system is unstable we can look at the solution, but we can also use a kind of Lyapunov argument. Let $w(\zeta,t)$ be a classical solution of (7.39), and thus satisfying the boundary condition. Then

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty w(\zeta,t)^2 d\zeta &= \int_0^\infty 2\frac{\partial w(\zeta,t)}{\partial t} w(\zeta,t) d\zeta \\
&= \int_0^\infty 2\left[-\frac{\partial w}{\partial \zeta}(\zeta,t) + w(\zeta,t)\right] w(\zeta,t) d\zeta \\
&= -2\int_0^\infty \frac{\partial w(\zeta,t)}{\partial \zeta} w(\zeta,t) d\zeta + \int_0^\infty 2w(\zeta,t)^2 d\zeta \\
&= -\left[w(\zeta,t)^2\right]_0^\infty + 2\int_0^\infty w(\zeta,t)^2 d\zeta \\
&= 0 + 2\int_0^\infty w(\zeta,t)^2 d\zeta,
\end{aligned}$$

where we have used the boundary condition. Hence if we define $V(t) = \int_0^\infty w(\zeta,t)^2 d\zeta$, then by the above

$$\dot{V}(t) = 2V(t)$$

and thus $V(t) = e^{2t}V(0)$. Using the fact that $V(t)$ is the norm of the solution squared, we immediately see that this norm will grow beyond bound, and thus the system is unstable.

Note that we did not showed that the PDE possesses a unique weak solution for every initial condition. This is the topic of Exercise 7.2. □

The example shows that care should be taken in relating eigenvalues to stability. To conclude stability from the eigenvalues, we need sufficiently many of them, and the eigenfunctions should form a basis. This we define next.

Definition 7.4.4. Let X be our state space with inner product $\langle \cdot, \cdot \rangle$. The set of functions $\{\phi_n, n \in \mathbb{N}\}$ forms an *orthonormal basis* when the following is satisfied.

1. For all $n, m \in \mathbb{N}$ there holds $\langle \phi_n, \phi_m \rangle = 0$ whenever $n \neq m$ and $\langle \phi_n, \phi_m \rangle = 1$ whenever $n = m$;
2. There is no function orthogonal to the span of $\{\phi_n, n \in \mathbb{N}\}$. That is, if for all $N > 0$ and $\alpha_1, \cdots, \alpha_N$ there holds
$$\langle w, \sum_{n=0}^{N} \alpha_n \phi_n \rangle = 0,$$
then $w = 0$. □

If $\{\phi_n, n \in \mathbb{N}\}$ forms an orthonormal basis, then for any $z \in X$ the following holds

$$z = \sum_{n=0}^{\infty} \alpha_n \phi_n, \tag{7.41}$$

where the α_n are given by

$$\alpha_n = \langle z, \phi_n \rangle. \tag{7.42}$$

Furthermore,

$$\|z\|^2 = \sum_{n=0}^{\infty} |\alpha_n|^2. \tag{7.43}$$

Theorem 7.4.5. *Consider the abstract (linear) differential equation* $\dot{x}(t) = Ax(t), x(0) = x_0$ *the state space* X. *If* A *possesses an orthonormal basis of eigenfunctions* $\{\phi_n, n \in \mathbb{N}\}$, *then*

1. *The (weak) solution of* $\dot{x}(t) = Ax(t), x(0) = x_0$ *is given by*

$$x(t) = \sum_{n=0}^{\infty} \alpha_n e^{\lambda_n t} \phi_n, \tag{7.44}$$

   where ϕ_n *is the eigenfunction corresponding to the eigenvalue* λ_n, *and* $\alpha_n = \langle x_0, \phi_n \rangle$;

2. *The system is exponentially stable if and only if there exists a* $\rho > 0$ *such that* $\mathrm{Re}(\lambda_n) \leq -\rho$ *for all* n;

3. *The system is asymptotically stable if and only if* $\mathrm{Re}(\lambda_n) < 0$ *for all* n.

Proof: We skip the proof of the first item, and focus our attention to the second and third item.

Combining (7.41) with (7.42) and (7.43), we find for $z \in X$

$$\|z\|^2 = \left\| \sum_{n=0}^{\infty} \alpha_n \phi_n \right\|^2 = \sum_{n=0}^{\infty} |\alpha_n|^2 = \sum_{n=0}^{\infty} |\langle z, \phi_n \rangle|^2. \tag{7.45}$$

Furthermore, using (7.44) and the first condition from Definition 7.4.4,

$$\langle x(t), \phi_m \rangle = \alpha_m e^{\lambda_m t}.$$

Combining these two gives

$$\|x(t)\|^2 = \sum_{n=0}^{\infty} |\langle x(t), \phi_n \rangle|^2 = \sum_{n=0}^{\infty} |\alpha_m e^{\lambda_m t}|^2 = \sum_{n=0}^{\infty} |\alpha_n|^2 e^{2\mathrm{Re}(\lambda_n)t}. \tag{7.46}$$

Suppose now that there exists an $\rho > 0$ such that $\mathrm{Re}(\lambda_n) \leq -\rho$, then by the above equation we find for $t \geq 0$

$$\begin{aligned} \|x(t)\|^2 &= \sum_{n=0}^{\infty} |\alpha_n|^2 e^{2\mathrm{Re}(\lambda_n)t} \\ &\leq \sum_{n=0}^{\infty} |\alpha_n|^2 e^{-2\rho t} = e^{-2\rho t} \sum_{n=0}^{\infty} |\alpha_n|^2 = e^{-2\rho t} \|x_0\|^2, \end{aligned}$$

where we have used item 1 and (7.43). Hence the system is exponentially stable.

If the system is exponentially stable, then there exists $M \geq 1$ and $\omega < 0$ such that $\|x(t)\| \leq Me^{\omega t}\|x_0\|$, see Definition 7.1.2. Suppose that there would exist an eigenvalue λ_{n_0} such that $\mathrm{Re}(\lambda_{n_0}) > \omega$, then we choose $x_0 = \phi_{n_0}$ and by Lemma 7.4.2 we have that the corresponding solution equals

$$x(t) = e^{\lambda_{n_0} t}\phi_{n_0}.$$

Hence $\|x(t)\|^2 = \|e^{\lambda_{n_0} t}\phi_{n_0}\|^2 = e^{2\mathrm{Re}(\lambda_{n_0})t}$, see also (7.46). Combining all the estimates we find

$$e^{2\mathrm{Re}(\lambda_{n_0})t} = \|x(t)\|^2 \leq M^2 e^{2\omega t}\|x_0\|^2 = M^2 e^{2\omega t}\|\phi_{n_0}\|^2 = M^2 e^{2\omega t}.$$

However, since $\mathrm{Re}(\lambda_{n_0}) > \omega$ this cannot hold for all $t \geq 0$. Thus we have a contradiction.

In a similar way we can show that if there is an eigenvalues λ_{n_0} such that $\mathrm{Re}(\lambda_{n_0}) \geq 0$, then the PDE cannot be asymptotically stable. The other implication follows from (7.46), although not straightforward. □

With this result we return to Example 7.3.3.

Example 7.4.6 The state space we consider for the PDE of Example 7.3.3 is $L^2(0,1)$. The A associated to the PDE is given by

$$Af = c\frac{d^2 f}{d\zeta^2} \tag{7.47}$$

with domain

$$D(A) = \{f \in L^2(0,1) \mid \frac{d^2 f}{d\zeta^2} \in L^2(0,1) \text{ and } f(0) = 0 = f(1)\} \tag{7.48}$$

To calculate the eigenvalues/eigenfunctions we have to solve $Af = \lambda f$ for $\lambda \in \mathbb{C}$ and $0 \neq f \in X = L^2(0,1)$. Using the expression of A, this becomes

$$c\frac{d^2 f}{d\zeta^2} = \lambda f \Leftrightarrow c\frac{d^2 f}{d\zeta^2}(\zeta) = \lambda f(\zeta), \quad \zeta \in [0,1]. \tag{7.49}$$

For $\lambda \neq 0$, the solution of this ODE is given by

$$f(\zeta) = \alpha e^{\mu\zeta} + \beta e^{-\mu\zeta}, \quad \text{where } \mu^2 = \frac{\lambda}{c}. \tag{7.50}$$

The second derivative of these $f's$ are clearly in $L^2(0,1)$, and so we have to check the boundary conditions. $f(0) = 0$ gives that $\beta = -\alpha$. Thus the second boundary condition is equivalent to

$$\alpha e^{\mu 1} - \alpha e^{-\mu 1} = f(1) = 0$$

This gives $\alpha = 0$ (not allowed) or

$$\begin{aligned} e^{\mu 1} - e^{-\mu 1} = 0 \quad &\Leftrightarrow \\ e^{2\mu} = 1 \quad &\Leftrightarrow \\ 2\mu = 2n\pi i, \quad n \in \mathbb{Z} \quad &\Leftrightarrow \\ \mu = n\pi i, \quad n \in \mathbb{Z}. & \end{aligned}$$

Since $\mu^2 = \frac{\lambda}{c}$, we find the following lambda's; $\lambda_n = -n^2\pi^2 c$ for $n \in \mathbb{Z}$. However, since n and $-n$ give the same lambda, we may exclude the negative indexes, and since we assumed that $\lambda \neq 0$, we have to exclude $n = 0$ as well. So we have found the eigenvalues $\lambda_n = -n^2\pi^2 c$, $n = 1, 2, \cdots$. If we take $\lambda = 0$ in (7.49), then $f(\zeta) = \alpha + \beta\zeta$ is the general solution. Using the two boundary conditions, we find that $\alpha = \beta = 0$, and so $\lambda = 0$ is not an eigenvalue.

Given our eigenvalues, we have to calculate the corresponding eigenfunctions. From the above and equation (7.49) we find that

$$f_n(\zeta) = \alpha e^{in\pi\zeta} - \alpha e^{-in\pi\zeta}.$$

Using Euler's formula, we can write this more conveniently as

$$f_n(\zeta) = 2i\alpha \sin(n\pi\zeta) = b \sin(n\pi\zeta).$$

If these should form an orthonormal basis, then $\|f_n\|^2 = \langle f_n, f_n \rangle = 1$. So

$$1 = \int_0^1 f_n(\zeta)^2 d\zeta = \int_0^1 b^2 \sin(n\pi\zeta)^2 d\zeta = \frac{b^2}{2}.$$

So the eigenvalues and (normalised) eigenfunctions are given by

$$\lambda_n = -n^2\pi^2 c \text{ and } \phi_n(\zeta) = \sqrt{2}\sin(n\pi\zeta), \quad n = 1, 2, \cdots. \tag{7.51}$$

From the Fourier sine-series, we know that these eigenfunctions form an orthonormal basis of $L^2(0,1)$. Using Theorem 7.4.5 we conclude that the system is exponentially stable, and the optimal (why?) growth rate equals $-\pi^2 c$. This is an improvement over $-4c$ as found in Example 7.3.3. □

7.5 Exercises

7.1. Given the PDE of equation (7.4),

$$\frac{\partial x}{\partial t}(\zeta,t) = \frac{\partial x}{\partial \zeta}(\zeta,t), \quad \zeta, t \geq 0.$$

(a) Rewrite this PDE as the abstract differential equation $\dot{x}(t) = Ax(t)$ on the state space $X = L^2(0,\infty)$.

(b) Prove that the differential equation of part (a) possesses a unique weak solution for every initial condition $x_0 \in X$.

(c) Prove that the expression of (7.5) is the unique weak solution of (7.4).

7.2. In this exercise we study the PDE (7.39) and its candidate solution in more detail.

(a) Assume that the initial condition w_0 is continuously differentiable on $[0,\infty)$, and $w_0(0) = 0$, $\dot{w}_0(0) = 0$. Show that under these conditions on w_0, the $w(\zeta,t)$ as given by

$$w(\zeta,t) = \begin{cases} e^t w_0(\zeta - t) & \zeta \geq t, \\ 0 & \zeta \in [0,t). \end{cases} \tag{7.52}$$

is a classical solution of (7.39).

(b) For arbitrary $w_0 \in L^2(0,\infty)$ show that for every fixed $t \geq 0$, the $w(\zeta,t)$ as given in (7.52) is an element of $L^2(0,\infty)$.

(c) Show that for $w_0 \in L^2(0,\infty)$, the $w(\zeta,t)$ as given in (7.52) is a weak solution of (7.39).

(d) Conclude on basis of the solution that the PDE (7.39) is unstable.

7.3. Wave with damping.

7.4. Homogeneous wave with damping. Use Lyapunov function.

7.5. Heat equation with no heat flux at $\zeta = 0$ and temperature zero at $\zeta = 1$.

7.6 Notes and references

The proof of Lemma 7.2.2 is based on an idea of Cox and Zuazua in [5], and its proof can be found in [28].

Chapter 8

Stability and Stabilizability, Frequency Domain

8.1 Introduction

In the previous chapter we have studied the stability of the system in time-domain. The main emphasis was on the behaviour of the state when time goes to infinity. In this chapter we study stability from a frequency point of view, and thus focus more on the input output behaviour.

We begin by defining the poles and zeros of a transfer function.

Definition 8.1.1. Let $G(s)$ be the transfer taking values in $\mathbb{C}^{k\times m}$. A point $s \in \mathbb{C}$ for which $G(s)$ is not defined is called a *singularity*.

Let s_0 be a singularity. If the following limit exists and is non-zero for some $r \in \{1, 2, \cdots\}$

$$\lim_{s\to s_0} (s - s_0)^r G(s), \tag{8.1}$$

then s_0 is called a *pole* of *order* r.

A *zero* of $G(s)$ is a point in $\mathbb{C}$ for which the rank of $G(s)$ decreases. $\square$

To illustrate this definition we look at two functions. The rational transfer function

$$G(s) = \frac{1}{s^2(s-1)}$$

has clearly the singularities $s = 0$ and $s = 1$. It is not hard to show that $s = 0$ is a pole of order 2, and that that $s = 1$ is a pole of order 1.

If we consider the transfer function

$$\frac{1}{\sqrt{s}},$$

then this has a singularity at $s = 0$, but it is not a pole, since the limit (8.1) gives zero for every integer $r \geq 1$. Such a singularity id called an *essential singularity*.

For scalar transfer functions we can have a more direct way of defining poles and zeros

Lemma 8.1.2. *Let $G(s)$ be a scalar transfer function, and suppose that s_0 is a pole. If we can write*

$$G(s) = \frac{N(s)}{D(s)} \tag{8.2}$$

with $N(s_0) \neq 0$, and $\frac{dD}{ds}(s_0) \neq 0$, then s_0 is a pole of order one.

The complex number $z \in \mathbb{C}$ is a zero of $G(s)$ if and only if $G(z) = 0$. If $D(z) \neq 0$, then z is a zero of $G(s)$ if and only if $N(z) = 0$.

We illustrate this with the transfer function from Example 6.1.4.

Example 8.1.3 Consider the transfer function from Example 6.1.4 given by

$$G(s) = \begin{cases} \frac{1}{16} & s = 0 \\ 4\frac{(\cosh(\frac{\sqrt{s}}{2})-1)^2}{s\sqrt{s}\sinh(\sqrt{s})} & s \neq -r^2\pi^2, r = 1, 2, \cdots \end{cases} \tag{8.3}$$

We begin by finding the poles and their order. From the first line of (8.3) we know that $s = 0$ is not a pole. So to find the singularities, we want to find those $s \in \mathbb{C}$ for which $\sinh(\sqrt{s}) = 0$. Since $\sinh(w) = -i\sin(iw)$, we get that $s_r := -r^2\pi^2, r = 1, 2, \cdots$ are the singularities. We will check whether these are poles. By (8.3) we choose the N and D of (8.2) as

$$N(s) = 4\left(\cosh(\frac{\sqrt{s}}{2}) - 1\right)^2 \text{ and } D(s) = s\sqrt{s}\sinh(\sqrt{s}).$$

The derivative of D with respect to s equals

$$\begin{aligned} \frac{dD}{ds}(s) &= \frac{3}{2}\sqrt{s}\sinh(\sqrt{s}) + s\sqrt{s}\cosh(\sqrt{s})\frac{1}{2\sqrt{s}} \\ &= \frac{3}{2}\sqrt{s}\sinh(\sqrt{s}) + \frac{1}{2}s\cosh(\sqrt{s}). \end{aligned}$$

Using this expression we find

$$\frac{dD}{ds}(s_r) = 0 + \frac{1}{2}s_r(-1)^r = \frac{(-1)^r}{2}s_r.$$

Furthermore, we have that

$$N(s_r) = 4\left(\cos(\frac{r\pi}{2}) - 1\right)^2 = \begin{cases} 4 & \text{for } r \text{ odd,} \\ 0 & \text{for } r = 4k, k \in \mathbb{N}, \\ 16 & \text{elsewhere.} \end{cases}$$

From Lemma 8.1.2 we find that s_r is a pole of order one whenever r is not a multiple of 4.

We want to know what the status is of the singularities $s_{4k}, k = 1, 2, \cdots$. Therefore, we consider the limit when s approaches an s_{4k}.

$$\begin{aligned} \lim_{s \to s_{4k}} G(s) &= \lim_{s \to s_{4k}} 4\frac{(\cosh(\frac{\sqrt{s}}{2}) - 1)^2}{s\sqrt{s}\sinh(\sqrt{s})} \\ &= \lim_{\sqrt{s} \to 2ik\pi} 4\frac{(\cosh(\frac{\sqrt{s}}{2}) - 1)^2}{s\sqrt{s}\sinh(\sqrt{s})} \\ &= \lim_{z \to 2ik\pi} 4\frac{(\cosh(\frac{z}{2}) - 1)^2}{z^3\sinh(z)} = \frac{1}{16}, \end{aligned}$$

where in the last equality we used l'Hopital rule. So we see that the limit exists, and in this case we have a *removable singularity*. Thus we have a pole zero cancellation. □

8.2 Stability in frequency domain

In the previous chapter we have seen that some systems don't have eigenvalues. Similarly, we can have systems without poles, as is shown in the following example.

Example 8.2.1 Consider the system of Example 6.1.7 for which we have shown that the transfer function is given by

$$G(s) = \begin{cases} \frac{\sinh(\sqrt{s})}{\sqrt{s}} & s \neq 0 \\ 1 & s = 0 \end{cases} \tag{8.4}$$

It is clear that $G(s)$ has no singularities and thus no poles. □

The above example shows that defining stability via the poles is not a wise idea, therefor we choose another approach.

Definition 8.2.2. Given the transfer function $G(s)$ we say that it is *stable* if $G(s)$ has no singularities in the right half plane, $\mathbb{C}^+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$ and there exists a $M < \infty$ such that

$$\|G(s)\| \leq M \tag{8.5}$$

for all $s \in \mathbb{C}^+$. □

The space of all stable transfer function is also known as H^∞.

We use this definition to check the stability of the integrator, i.e., $G(s) = \frac{1}{s}$. It is clear that there is no pole with real part larger than zero. So it is stable if it is bounded on the right half plane. This is clearly not the case. The closer we get to $s = 0$, the large $G(s)$ will become, and so we cannot find an M such that (8.5) is satisfied for all $s \in \mathbb{C}^+$, and hence the integrator is an unstable system (in frequency domain).

Similarly, we can show that if there is a pole on the imaginary axis, then the transfer function is unstable.

Next we apply this definition to the transfer function from Example 8.2.1.

Example 8.2.3 As shown in Example 8.2.1 the transfer function $G(s)$ as given in (8.4) has no singularities, and so to check whether or not it is stable we have to see whether or not (8.5) holds for some M.

We know that $2\sinh(x) = e^x - e^{-x}$, and so for s real and in $\mathbb{C}^+$, we see that

$$G(s) = \frac{e^{\sqrt{s}} + e^{-\sqrt{s}}}{2\sqrt{s}}$$

which grows beyond bound when $s \to \infty$. Thus we conclude that $G(s)$ is not stable. □

In Example 6.1.6 we found another transfer function without poles. Namely, $G(s) = e^{-\frac{s}{c}}$ where c is a positive constant. It is not hard to show that this transfer function is stable, see Exercise 8.1.

Now we know when a transfer function is stable, we can define stabilizabilty in frequency domain.

8.3 Stabilizability in frequency domain

Given our definition of a stable transfer function, i.e, a stable system in frequency domain, we can define the stability of the standard feedback configuration as shown in Figure 8.1

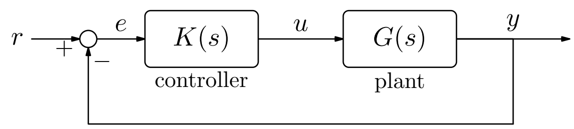

Figure 8.1: Standard feedback interconnection.

Definition 8.3.1. Given the plant/system with transfer function $G(s)$. The controller $K(s)$ stabilizes the system when the closed loop transfer as shown in Figure 8.1 is stable. That is when $G_{cl}(s) = (I + G(s)K(s))^{-1}G(s)K(s)$ is stable. □

It is easy to check that $G_{cl}(s)$ is the transfer function from r to y, see Figure 8.1.

Since the transfer function for PDE's can be very complicated, checking whether or not it is bounded on the right half plane can be very cumbersome. The following theorem gives a nice time domain condition for the stabilization via a static, i.e., proportional controller.

Theorem 8.3.2. *Consider a system with state, $x(t) \in X$, input $u(t) \in \mathbb{R}^m$ and output $y(t) \in \mathbb{R}^m$. Assume that for all classical solutions the following inequality holds*

$$\frac{d}{dt}\|x(t)\|^2 \leq u^T(t)y(t), \quad t \geq 0 \tag{8.6}$$

then for any $k > 0$ the proportional controller $u(t) = ke(t)$, see Figure 8.1, stabilizes the system in frequency domain.

Proof: So we have to show that the transfer function from r to y is bounded in the right-half plane. We do our proof only for SISO systems.

In Chapter 6 we have defined the transfer function using the (complex) signals e^{st}. So we have to extend (8.6) to complex valued input and outputs. It becomes for scalar complex valued signals;

$$\frac{d}{dt}\|x(t)\|^2 \leq \frac{1}{2}\left[u(t)^*y(t) + u(t)y(t)^*\right], \quad t \geq 0. \tag{8.7}$$

Using the exponential solution $r(t) = r_0e^{st}$, $y(t) = y_0e^{st}$, $x(t) = x_0e^{st}$ and $u(t) = k(r(t) - y(t))$ (see Figure 8.1), we find

$$\begin{aligned}\frac{d}{dt}\|x_0e^{st}\|^2 &\leq \frac{1}{2}\left[kr(t)^*y(t) + kr(t)y(t)^* - 2k|y(t)|^2\right] \\ &= \frac{1}{2}\left[kr_0^*e^{s^*t}y_0e^{st} + kr_0(t)e^{st}y_0^*e^{s^*t}\right] - k|y_0|^2e^{2\mathrm{Re}(s)t}.\end{aligned}$$

Since $|e^{st}|^2 = e^{s^*t}e^{st} = e^{2\mathrm{Re}(s)t}$ and since this term is never zero, it can be removed and we find

$$2\mathrm{Re}(s)\|x_0\|^2 \leq \frac{1}{2}\left[kr_0^*y_0 + kr_0y_0^*\right] - k|y_0|^2. \tag{8.8}$$

Note that $y_0 = G_{cl}(s)r_0$, and since we want to know something of $G_{cl}(s)$ we choose $r_0 = 1$. With this choice (8.8) becomes

$$2\mathrm{Re}(s)\|x_0\|^2 \leq \frac{1}{2}k\left[y_0 + y_0^*\right] - k|y_0|^2 = k\mathrm{Re}(y_0) - k|y_0|^2.$$

For $\mathrm{Re}(s) \geq 0$ we find

$$0 \leq 2\mathrm{Re}(s)\|x_0\|^2 \leq k\mathrm{Re}(y_0) - k|y_0|^2 \leq k|y_0| - k|y_0|^2.$$

or equivalently,

$$k|y_0|^2 \leq k|y_0|.$$

So $y_0 = 0$ or $|y_0| \leq 1$. Since r_0 is chosen to be one, we have $y_0 = G_{cl}(s)$. Concluding, we find that $|G_{cl}(s)| \leq 1$ for all s in the right-half plane, and thus the system is stabilised (in frequency domain). □

8.4 Exercises

8.1. Show that the transfer function $G(s) = e^{-\frac{s}{c}}$ where c is a positive constant, is stable.

Chapter 9
Notation

In this chapter we list some notation.

$\mathbb{N}$	$\{0, 1, 2, \cdots\}$
ODE	ordinary differential equation
pH-system	port-Hamiltonian system
PDE	partial differential equation
U	general input space
X	general state space
Y	general output space
$\langle \cdot, \cdot \rangle$	general inner product
$\| \cdot \|_{\mathcal{H}}$	energy norm for pH-system with density $\mathcal{H}$
$\langle \cdot, \cdot \rangle_{\mathcal{H}}$	"energy" inner product

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