Exercise Port-Hamiltonian systems

The shallow water equations are given as

$$\partial_t \begin{bmatrix} h \\ v \end{bmatrix} + \begin{bmatrix} v & h \\ g & v \end{bmatrix} \partial_z \begin{bmatrix} h \\ v \end{bmatrix} = 0$$

with $h(z,t)$ the water level at position z in the canal, and $v(z,t)$ the velocity of the water (g the gravitational constant).

(a) Show that this can be described as a port-Hamiltonian system with total energy

$$H(h,v) = \int_a^b \mathcal{H}(h,v)dz = \int_a^b \left(\frac{1}{2}hv^2 + \frac{1}{2}gh^2\right) dz$$

and corresponding co-energy variables

$$\begin{aligned} e_h &= \tfrac{\partial \mathcal{H}}{\partial h} &= \tfrac{1}{2}v^2 + gh \quad \text{(Bernoulli function)} \\ e_v &= \tfrac{\partial \mathcal{H}}{\partial v} &= hv \quad \text{(mass flow)} \end{aligned}$$

resulting in the port-Hamiltonian system

$$\begin{aligned} \tfrac{\partial h}{\partial t}(z,t) &= -\tfrac{\partial}{\partial z}\tfrac{\partial \mathcal{H}}{\partial v} \\ \tfrac{\partial v}{\partial t}(z,t) &= -\tfrac{\partial}{\partial z}\tfrac{\partial \mathcal{H}}{\partial h} \end{aligned}$$

with boundary variables $hv_{|a,b}$ (water flow through both ends of the canal) and $(\frac{1}{2}v^2 + gh)_{|a,b}$ (Bernoulli function, or hydrodynamic pressure, at both ends).

(b) Apply the boundary control strategy

$$h(b)v(b) = 0, \quad h(a)v(a) = -\frac{1}{2}v^2(a) - gh(a)$$

(This corresponds to closing the canal at the right-hand side b (no mass flow), and adding a linear damping at the left-hand side by letting the mass-flow at b to be negatively proportional to the Bernoulli function.) Argue that this boundary control strategy stabilizes the system around the zero-state $h(z) = 0, v(z) = 0, z \in [a,b]$.

(c) Consider the same situation as in part (a), where now the right-hand side of the canal (at the point b) will be interconnected to an infinite water reservoir with constant height h^*. This corresponds to the interconnection of scalar port-Hamiltonian system

$$\begin{aligned} \dot{\xi} &= u_c \\ y_c &= \tfrac{\partial H_c}{\partial \xi} (= gh^*) \end{aligned}$$

with the linear Hamiltonian function $H_c(\xi) = gh^*\xi$, via the feedback interconnection

$$u_c = y = h(b)v(b), \quad y_c = -u = \frac{1}{2}v^2(b) + gh(b)$$

What is the resulting closed-loop port-Hamiltonian system ? Show that it has the state $h(z,t) = h^*, v(z,t) = 0, z \in [a,b]$ as an equilibrium state.

(d) Prove that the quantity

$$\int_a^b h(z,t)dz + \xi$$

is a *conserved quantity* for the closed-loop system, i.e., its time-derivative is equal to zero. What is the physical interpretation of this ?

(e) Show that the interconnection of k channels in a common linking node A can be described as

$$e_v^1 + e_v^2 + \cdots + e_v^k = 0$$

$$e_h^1 = e_h^2 = \cdots = e_h^k$$

with e_v^i the water flow of canal i entering node A, and e_h^i the hydrodynamic pressure of canal i at node A. Describe the resulting interconnected port-Hamiltonian system.