Modeling and Control of Distributed Parameter Systems: The Port Hamiltonian Approach

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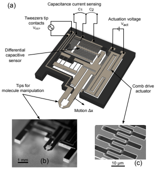
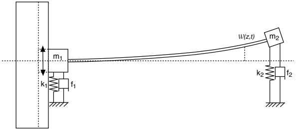
Outline

Context : control of flexible structures

- Boundary controlled systems





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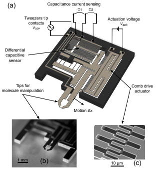
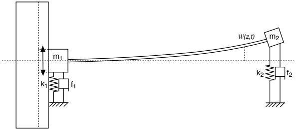
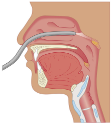
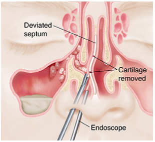
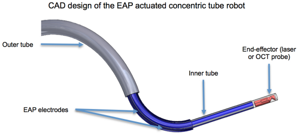




- In-domain control of distributed parameter systems







  - Exploration, imaging, diagnosis.
  - Mini invasive surgery.
  - Toward miniaturized and *smart* endoscopes.

Context : port Hamiltonian systems

- Port Hamiltonian systems:
  - The state variables are chosen as the energy variables.
  - The links between the energy function and the system dynamics is made explicit through symmetries.
  - The boundary port variables are power conjugated.

- Energy shaping consists in using the physical properties of the system to derive efficient control laws with guaranteed performances (step further stabilization).

- ”Easy” to extend to non linear or systems defined on higher dimensional spaces.

Outline

Infinite dimensional Port Hamiltonian systems (PHS)

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$$\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix} = \begin{bmatrix} 0 & \mathcal{G} \\ -\mathcal{G}^* & -R\end{bmatrix}\begin{bmatrix} \mathcal{L}_1(\zeta)x_1(\zeta,t) \\ \mathcal{L}_2(\zeta)x_2(\zeta,t)\end{bmatrix} + \begin{bmatrix} 0 \\ I\end{bmatrix} u_d(\zeta,t) \tag{1}$$

$$y_d(\zeta,t) = \begin{bmatrix} 0 & I\end{bmatrix}\begin{bmatrix} \mathcal{L}_1(\zeta)x_1(\zeta,t) \\ \mathcal{L}_2(\zeta)x_2(\zeta,t)\end{bmatrix} \tag{2}$$

$$u_\partial = \mathcal{B}\begin{bmatrix} \mathcal{L}_1(\zeta)x_1(\zeta,t) \\ \mathcal{L}_2(\zeta)x_2(\zeta,t)\end{bmatrix}, \quad y_\partial = \mathcal{C}\begin{bmatrix} \mathcal{L}_1(\zeta)x_1(\zeta,t) \\ \mathcal{L}_2(\zeta)x_2(\zeta,t)\end{bmatrix} \tag{3}$$

where $x = [x_1^T, x_2^T]^T \in X L^2([a,b], \mathbb{R}^n) \times L^2([a,b], \mathbb{R}^n)$, $\mathcal{L} = diag(\mathcal{L}_1, \mathcal{L}_2)$ and $\mathcal{L}(\zeta) = \mathcal{L}^T(\zeta)$ and $\mathcal{L}(\zeta) \geq \eta$ with $\eta > 0$ for all $\zeta \in [a,b]$, $R \in \mathbb{R}^{(n,n)}$, $R = R^T > 0$, $\mathcal{B}(\cdot)$ and $\mathcal{C}(\cdot)$ are some boundary input and boundary output mapping operators. Furthermore

$$\mathcal{G} = \sum_{i=0}^{N} G_i \frac{\partial^i}{\partial \zeta^i}, \text{ and } \mathcal{G}^* = \sum_{i=0}^{N} (-1)^i G_i^T \frac{\partial^i}{\partial \zeta^i}$$

with $G_i \in \mathbb{R}^{(n,n)}$.

Infinite dimensional Port Hamiltonian systems (PHS)

For a sake of compactness we shall use the following notation

$$P_i = \begin{bmatrix} 0 & G_i \\ (-1)^{i+1} G_i^T & 0 \end{bmatrix}, \; R_0 = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \tag{4}$$

and the formulation of (1)

$$\frac{\partial x}{\partial t}(\zeta, t) = \sum_{i=0}^{N} P_i \frac{\partial^i}{\partial \zeta^i} \left(\mathcal{L}(\zeta) x(\zeta, t)\right) - R_0 \mathcal{L}(\zeta) x(\zeta, t) + \begin{bmatrix} 0 \\ I \end{bmatrix} u_d(\zeta, t) \tag{5}$$

$$y_d(\zeta, t) = \begin{bmatrix} 0 & I \end{bmatrix} \mathcal{L}(\zeta) x(\zeta, t) \tag{6}$$

$$u_\partial = \mathcal{B}\left(\mathcal{L}(\zeta) x(\zeta, t)\right), y_\partial = \mathcal{C}\left(\mathcal{L}(\zeta) x(\zeta, t)\right) \tag{7}$$

The total energy of the system $H(x)$ is defined by

$$H(x) = \frac{1}{2} \int_a^b \left(x^T(\zeta, t) \mathcal{L}(\zeta) x(\zeta, t) \right) d\zeta$$

Boundary controlled port Hamiltonian systems

Mixed in-domain / boundary controlled port Hamiltonian systems (IDBC-PHS)

A mixed in-domain / boundary controlled port Hamiltonian system is an infinite dimensional system of the form (5-7) where

$$u_\partial = W_B \begin{bmatrix} \mathcal{L}(b)x(b,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{L}x)}{\partial\zeta^{N-1}}(b,t) \\ \mathcal{L}(a)x(a,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{L}x)}{\partial\zeta^{N-1}}(a,t) \end{bmatrix}, \text{ and } y_\partial = W_C \begin{bmatrix} \mathcal{L}(b)x(b,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{L}x)}{\partial\zeta^{N-1}}(b,t) \\ \mathcal{L}(a)x(a,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{L}x)}{\partial\zeta^{N-1}}(a,t) \end{bmatrix} \tag{8}$$

with

$$W_B = \left[\frac{1}{\sqrt{2}}\left(\Xi_2 + \Xi_1 P_e\right) \quad \frac{1}{\sqrt{2}}\left(\Xi_2 - \Xi_1 P_e\right) \right], \tag{9}$$

$$W_C = \left[\frac{1}{\sqrt{2}}\left(\Xi_1 + \Xi_2 P_e\right) \quad \frac{1}{\sqrt{2}}\left(\Xi_1 - \Xi_2 P_e\right) \right], \tag{10}$$

Boundary controlled port Hamiltonian systems

where

$$P_e = \begin{bmatrix} P_1 & \cdots & (-1)^{N-1} P_N \\ \vdots & \ddots & 0 \\ (-1)^{N-1} P_N & 0 & 0 \end{bmatrix} \tag{11}$$

and Ξ_1 and Ξ_2 in $\mathbb{R}^{k\times k}$ satisfy

$$\Xi_2^\top \Xi_1 + \Xi_1^\top \Xi_2 = 0, \text{ and } \Xi_2^\top \Xi_2 + \Xi_1^\top \Xi_1 = I \tag{12}$$

The energy balance associated to the system reads

$$\frac{dH}{dt} = \int_a^b y_d^T u_d d\zeta - \int_a^b \left(x_2^T(\zeta,t) \mathcal{L}_2^T(\zeta) R \mathcal{L}_2(\zeta) x_2(\zeta,t) \right) d\zeta + y_\partial^T u_\partial \tag{13}$$

$$\leq \int_a^b y_d^T u_d d\zeta + y_\partial^T u_\partial \tag{14}$$

Boundary controlled port Hamiltonian systems

Existence of solution [?]

The operator

$$\mathcal{J} = \sum_{i=0}^{N} P_i \frac{\partial^i}{\partial \zeta^i} \left(\mathcal{L}(\zeta)x(\zeta,t)\right) - R_0 \mathcal{L}(\zeta)x(\zeta,t)$$

with domain

$$D(\mathcal{J}) = \left\{ \mathcal{L} \in H^N\left(a,b;\mathbb{R}^n\right) \mid \begin{bmatrix} \mathcal{L}(b)x(b,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{L}x)}{\partial \zeta^{N-1}}(b,t) \\ \mathcal{L}(a)x(a,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{L}x)}{\partial \zeta^{N-1}}(a,t) \end{bmatrix} \in KerW_B \right\}$$

where W_B is defined by (9) and Ξ_1 and Ξ_2 satisfy (12), generates a contraction semigroup on X. Furthermore the system (5-7) with (9-10) and (12) defines a boundary control system.

Boundary controlled port Hamiltonian systems

The general formulation (1) allows to model a large class of systems.

For example:

- The 1D wave equation where $n = 1$, $N = 1$, $G_0 = 0, G_1 = 1$.
- The Euler Bernouilli beam equation. In this case $n = 1$, $N = 2$, $G_0 = 0, G_1 = 0, G_2 = 1$.
- The Timoshenko beam equation. In this case $n = 2$, $N = 1$, and

$$G_0 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In what follows we focus on first order differential operators

Outline

Control by interconnection

The system is interconnected with a dynamic controller in a power preserving way.

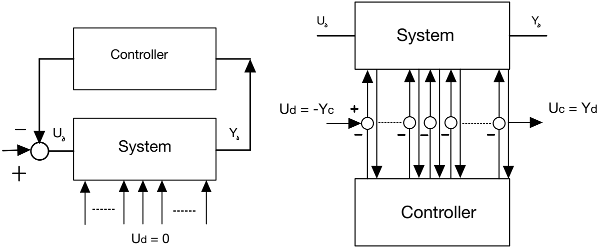

Figure: Control by interconnection. Boundary control (left), in domain control (right).

The closed loop energy is equal to the sum of the open loop energy and the controller energy.

Energy shaping

Objectives

Modification of the closed loop system's properties (energy shaping) + stabilization (damping injection).

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We first look for structural invariants $C(x, x_c)$ *i.e.* $\frac{dC}{dt} = 0$

$$C(x, x_c) = x_c + F(x) = \kappa$$

where F is a smooth function.

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where F is a smooth function. In this case the closed loop energy function reads

$$H_{cl}(x, x_c) = H_{cl}(x) = H(x) + \color{red}{H_c}(\kappa - F(x))$$

Asymptotic stability of the closed loop system in x^* is achieved using damping injection such that

$$\frac{dH_{cl}}{dt} < 0, \forall x \neq x^*.$$

Energy shaping

- Boundary control case : Asymptotic stabilisation [Macchelli et al., 2017], Exponential stabilisation [Macchelli et al., 2020] $\Rightarrow$ Control = integrals over the spatial domain.

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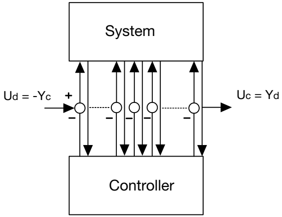
- **In this talk** : we consider in domain control



and the system is connected to the controller in a power preserving way:

$$\begin{pmatrix} u_d(\zeta,t) \\ y_d(\zeta,t) \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u_C(\zeta,t) \\ y_C(\zeta,t) \end{pmatrix} + \begin{pmatrix} u'(\zeta,t) \\ 0 \end{pmatrix}, \tag{15}$$

Control by interconnection : ideal case

• Ideal case : the control acts at each point ζ of the spatial domain. The controller is of the form

$$\begin{cases} \dfrac{\partial x_C}{\partial t}(\zeta,t) = \mathcal{J}_c \mathcal{Q}_c x_C(\zeta,t) + \mathcal{B}_c u_C(\zeta,t) \\ \quad y_C(\zeta,t) = \mathcal{B}_c{}^* \mathcal{Q}_c x_C(\zeta,t) + \mathcal{S}_c u_C(\zeta,t) \end{cases} \tag{16}$$

where $\mathcal{Q}_c(\zeta) = \mathcal{Q}_c^T(\zeta)$ and $\mathcal{Q}_c(\zeta) \geq \eta_c$ with $\eta_c > 0$ for all $\zeta \in [a,b]$, $\mathcal{S}_c$ and $\mathcal{S}_c(\zeta) = \mathcal{S}_c^T(\zeta)$ and $\mathcal{S}_c(\zeta) \geq \eta_s$ with $\eta_s > 0$ for all $\zeta \in [a,b]$ and:

$$\mathcal{B}_c = B_{c0} + B_{c1}\frac{\partial}{\partial \zeta}, \text{ and } \mathcal{J}_c = J_{c0} + J_{c1}\frac{\partial}{\partial \zeta} \tag{17}$$

with $B_{c0}, B_{c1} \in \mathbb{R}^{(n_c,1)}$, $J_{c0} = -J_{c0}^T$, $J_{c1} = J_{c1}^T \in \mathbb{R}^{(n_c,n_c)}$.

Control by interconnection : ideal case

The closed loop system reads :

$$\frac{\partial x_e}{\partial t} = \begin{pmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \\ \frac{\partial x_c}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{G} & 0 \\ -\mathcal{G}^* & -(\mathcal{S}_c + R) & -\mathcal{B}_c^* \\ 0 & \mathcal{B}_c & \mathcal{J}_c \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 x_1 \\ \mathcal{L}_2 x_2 \\ \mathcal{Q}_c x_c \end{pmatrix} \tag{18}$$

Structural invariants

The closed loop system (18) admits structural invariants of the form

$$\kappa_0 = C(x_e) = \int_a^b \Psi^T x_e d\zeta \tag{19}$$

with $\Psi = (\psi_1, \psi_2, \psi_3)$ if and only if

$$-\mathcal{G}\psi_2(\zeta) = 0 = -\mathcal{B}_c\psi_2(\zeta) + \mathcal{J}_c^*\psi_3(\zeta) \tag{20}$$

$$(\mathcal{S}_c + R)\,\psi_2(\zeta) = 0 \tag{21}$$

$$\mathcal{G}\psi_1(\zeta) + \mathcal{B}_c^*\psi_3(\zeta) = 0 \tag{22}$$

$$\left. \begin{pmatrix} 0 & G_1 & 0 \\ -G_1^T & 0 & -B_{c1} \\ 0 & B_{c1}^T & J_{c1} \end{pmatrix} \begin{pmatrix} \psi_1(\zeta) \\ \psi_2(\zeta) \\ \psi_3(\zeta) \end{pmatrix} \right|_{a,b} = 0 \tag{23}$$

Energy shaping : ideal case

Energy shaping [Trenchant et al., 2017]

Choosing $\mathcal{B}_c = \mathcal{G}$ and $\mathcal{J}_c = 0$ the closed loop system (18) admits as structural invariants the function $C(x_e)$ defined by (19) and

$$\Psi = (\Psi_1, 0, \Psi_1)$$

In this case the hyperbolic system (1) connected to the dynamic controller (29) of the form

$$\begin{cases} \frac{\partial x_C}{\partial t}(\zeta,t) = \mathcal{G} u_C(\zeta,t) \\ y_C(\zeta,t) = \mathcal{G}^* \mathcal{Q}_c x_C(\zeta,t) + \mathcal{S}_c u_C(\zeta,t) \end{cases} \tag{24}$$

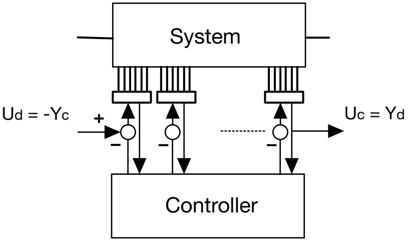
is equivalent to the system

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{G} \\ -\mathcal{G}^* & -(R+\mathcal{S}_c) \end{bmatrix} \begin{bmatrix} (\mathcal{L}_1(\zeta)+\mathcal{Q}_c(\zeta))\, x_1(\zeta,t) \\ \mathcal{L}_2(\zeta) x_2(\zeta,t) \end{bmatrix} \tag{25}$$

$$u_\partial = \mathcal{B}\begin{bmatrix} (\mathcal{L}_1(\zeta)+\mathcal{Q}_c(\zeta))\, x_1(\zeta,t) \\ \mathcal{L}_2(\zeta) x_2(\zeta,t) \end{bmatrix}, \; y_\partial = \mathcal{C}\begin{bmatrix} (\mathcal{L}_1(\zeta)+\mathcal{Q}_c(\zeta))\, x_1(\zeta,t) \\ \mathcal{L}_2(\zeta) x_2(\zeta,t) \end{bmatrix} \tag{26}$$

Control by interconnection

- Non ideal case : the distributed parameter system is actuated through piecewise constant elements.



Early lumping approach

The system is first discretized using a structure preserving method (mixed finite element method [?]) such that the approximation of (1) is again a PHS with n elements:

$$\begin{pmatrix} \dot{x}_{1d} \\ \dot{x}_{2d} \end{pmatrix} = (J_n - R_n) \begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix} + B_b u_b + \begin{pmatrix} 0 \\ B_{0d} \end{pmatrix} \mathbf{u}_d, \tag{27a}$$

$$y_b = B_b^T \begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix} + D_b u_b, \tag{27b}$$

$$\mathbf{y}_d = \begin{pmatrix} 0 & B_{0d}^T \end{pmatrix} \begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix}, \tag{27c}$$

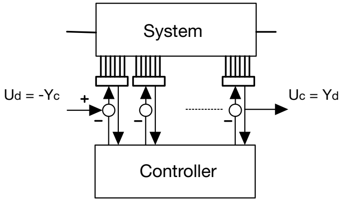
where $x_{id} = \begin{pmatrix} x_i^1 & \cdots & x_i^n \end{pmatrix}^T \in \mathbb{R}^{np \times 1}$ for $i \in \{1, \cdots, 2p\}$,

$$J_n = \begin{pmatrix} 0 & J_i \\ -J_i^T & 0 \end{pmatrix} \quad \text{and} \quad R_n = \begin{pmatrix} 0 & 0 \\ 0 & R_d \end{pmatrix},$$

The discretized energy reads:

$$H_d(x_{1d}, x_{2d}) = \frac{1}{2} \left(x_{1d}^T Q_1 x_{1d} + x_{2d}^T Q_2 x_{2d} \right). \tag{28}$$

Control by interconnection



The controller is designed as finite dimensional PHS of the form:

$$\begin{cases} \dot{x}_c = (J_c - R_c)\, Q_c x_c + B_c u_c, \\ y_c = B_c^T Q_c x_c + D_c u_c, \end{cases} \tag{29}$$

interconnected in a power preserving way through the relation

$$\begin{pmatrix} u_d \\ u_c \end{pmatrix} = \begin{pmatrix} 0 & -M \\ M^T & 0 \end{pmatrix} \begin{pmatrix} y_d \\ y_c \end{pmatrix}, \text{ where } M = \mathbb{I}_m \otimes \mathbf{1}_{k\times 1} \in \mathbb{R}^{n\times m}, \tag{30}$$

Control by interconnection

The closed loop system is given by

$$\dot{x}_{cl} = (J_{cl} - R_{cl})\, Q_{cl} x_{cl}, \tag{31}$$

where $x_{cl} = \begin{pmatrix} x_{1d}^T, & x_{2d}^T, & x_c^T \end{pmatrix}^T$, $Q_{cl} = \text{diag}\begin{pmatrix} Q_1, & Q_2, & Q_c \end{pmatrix}$,

$$J_{cl} = \begin{pmatrix} O & J_i & 0 \\ -J_i^T & 0 & -B_{0d} M B_c^T \\ 0 & B_c M^T B_{0d}^T & J_c \end{pmatrix}, \; R_{cl} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & R_d + B_{0d} M D_c M^T B_{0d}^T & 0 \\ 0 & 0 & R_c \end{pmatrix}.$$

The Hamiltonian of the controller (29) is:

$$H_c(x_c) = \frac{1}{2} x_c^T Q_c x_c. \tag{32}$$

Therefore, the closed loop Hamiltonian function reads:

$$H_{cld}(x_{1d}, x_{2d}, x_c) = H_d(x_{1d}, x_{2d}) + H_c(x_c). \tag{33}$$

Energy shaping

Approximate energy shaping [Liu et al., 2021]

Choosing $J_c = 0$, and $R_c = 0$, the closed loop system (31) admits:

$$C(x_{1d}, x_c) = B_c M^T B_{0d}^T J_i^{-1} x_{1d} - x_c \tag{34}$$

as structural invariant, *i.e.* $\dot{C}(x_{1d}, x_c) = 0$ along the closed loop trajectories. If $x_{1d}(0)$ and $x_c(0)$ satisfy $C(x_{1d}(0), x_c(0)) = 0$, the controller is a proportional-integral control, and the control law (30) is equivalent to the state feedback:

$$\mathbf{u}_d = -B_c^T Q_c B_c M^T B_{0d}^T J_i^{-1} x_{1d} - D_c M^T B_{0d}^T Q_2 x_{2d}. \tag{35}$$

Therefore, the closed loop system yields:

$$\begin{pmatrix} \dot{x}_{1d} \\ \dot{x}_{2d} \end{pmatrix} = \begin{pmatrix} 0 & J_i \\ -J_i^T & -(R_d + B_{0d} M D_c M^T B_{0d}^T) \end{pmatrix} \begin{pmatrix} \tilde{Q}_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix}, \tag{36}$$

where : $\tilde{Q}_1 = Q_1 + J_i^{-T} B_{0d} M B_c^T Q_c B_c M^T B_{0d}^T J_i^{-1}$ is the new closed loop energy matrix associated to x_{1d}.

Energy shaping

Problem

The energy related to first n elements of x_{1d} in closed loop is shaped in an optimal way if and only if $X = B_c^T Q_c B_c$ sym. sem. def. pos. minimizes $f(X) = \left\| AXA^T - Q_m \right\|_F$, where $A = (J_i)^{-T} B_{0d} M \in \mathbb{R}^{n\times m}$, $SR_0^{m\times m}$ and $Q_m = (\tilde{Q}_{1d} - Q_1)_{n\times n}$.

Solution

$f(X)$ is convex and the minimization of $f(X)$ is equivalent to the minimization of $f^2(X)$, which has a unique minimum given by

$$\hat{X} = V\Sigma_0^{-1} U_1^T Q_m U_1 \Sigma_0^{-1} V^T \tag{37}$$

with V, Σ_0 and U_1 the matrices of the singular value decomposition (SVD) of the matrix A *i.e.* $A = U\Sigma V^T = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_0 \\ 0 \end{pmatrix} V^T$, where $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ are unitary matrices, $U_1 \in \mathbb{R}^{n\times m}$, $U_2 \in \mathbb{R}^{n\times q}$, $q = n - m$, and $\Sigma_0 = \Sigma_0^T \geq 0$ is the diagonal matrix of singular values of A.

Energy shaping

- The choice of the controller matrices B_c and Q_c is not unique. It only has to satisfy the condition (37). It is done in order to modify the shape of the closed loop energy function of the system in the x_1 coordinate.
- The choice of the controller matrix D_c follows a similar procedure, with the optimization of the difference between the approximate dissipation and the desired one.
- D_c allows to add local dissipation but more interestingly it may be used to add global dissipative effect equivalent to diffusion.
- The quality of the shaping (controller bandwidth) depends on the number of patches.

Stability analysis

The controller is now connected to the infinite dimensional system leading to :

$$\dot{\mathcal{X}} = \underbrace{\begin{pmatrix} (\mathcal{J} - \mathcal{R} - \mathcal{B}D_c\mathcal{B}^*) & -\mathcal{B}B_c^T \\ B_c\mathcal{B}^* & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L} & 0 \\ 0 & Q_c \end{pmatrix}}_{\mathcal{A}_{cl}} \mathcal{X}, \tag{38}$$

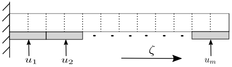
where $\mathcal{X} = \begin{pmatrix} x^T & x_c^T \end{pmatrix}^T \in X_s$ where $X_s = L_2\left([0,L], \mathbb{R}^{2p}\right) \times \mathbb{R}^m$.

Existence of solution, stability analysis

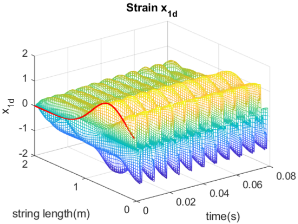
- The operator $\mathcal{A}_{cl}$ defined in (38) generates a contraction semigroup on $X_s = L_2\left([0,L], \mathbb{R}^{2p}\right) \times \mathbb{R}^m$.
- The operator $\mathcal{A}_{cl}$ has a compact resolvent.
- Asymptotic stability: For any $\mathcal{X}(0) \in L_2\left([0,L], \mathbb{R}^{2n}\right) \times \mathbb{R}^m$, the unique solution of (38) tends to zero asymptotically, and the closed loop system (38) is globally asymptotically stable.

Energy shaping : application

We consider the control of a weakly damped vibratring string using m homogeneously distributed patches (n discretization elements).



We consider the case with m patches, *i.e.* $m = 10$, $n = 50$ and $k = 5$. The initial conditions are set to a spatial distribution $x_1(\zeta, 0) = \mathcal{N}(1.5, 0.113)$ for the strain distribution and to zero for velocity distribution.



Control by interconnection

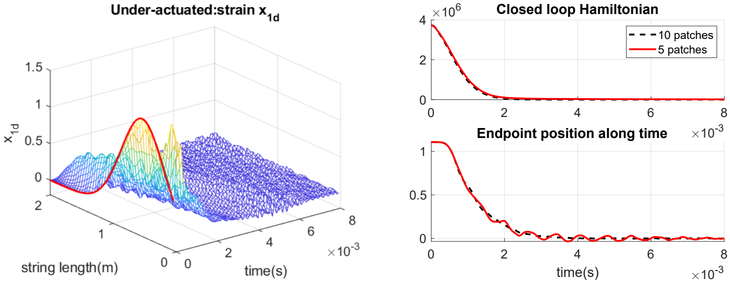
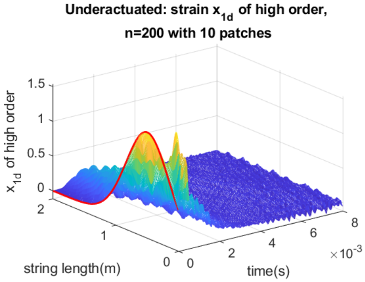


Figure: Closed loop evolution of the angular strain for $m = 10$ (a), Hamiltonian function and endpoint position (b) in the under-actuated case for $m = 10$, $m = 5$.

Control by interconnection



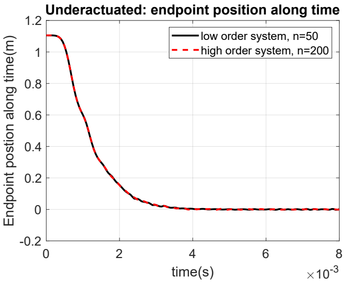
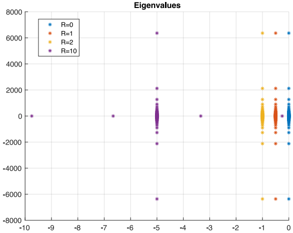


Figure: Closed loop evolution of the angular strain of the high order system (a), and comparison of the endpoint position of the low order and high order systems using the same controller (b).

Control by interconnection

Achievable performances



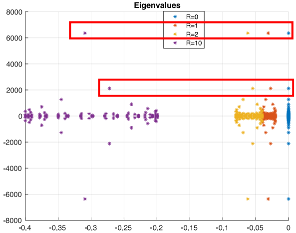


Figure: Control by interconnection. Full actuation (left), partial actuation (right).

Outline

Context and motivation

Infinite dimensional Port Hamiltonian systems (PHS)

Control by interconnection and energy shaping

Observer design

Irreversible boundary controlled port Hamiltonian Systems

Conclusions and future works

Observer design

In many cases the power conjugated variable is not (completely) measurable. In this case one has to use an observer.

$$\mathcal{U}\begin{cases}\frac{\partial x}{\partial_t}(\zeta,t)=P_1\frac{\partial}{\partial\zeta}(\mathcal{H}(\zeta)x(\zeta,t))+P_0\mathcal{H}(\zeta)x(\zeta,t),\\ W_{\mathcal{B}}\begin{pmatrix}\mathbf{f}_\partial(t)\\ \mathbf{e}_\partial(t)\end{pmatrix}=u(t),\quad x(\zeta,0)=x_0(\zeta),\\ y(t)=W_{\mathcal{C}}\begin{pmatrix}\mathbf{f}_\partial(t)\\ \mathbf{e}_\partial(t)\end{pmatrix},\\ y_m(t)=\mathcal{C}_m x(\zeta,t),\end{cases}\tag{39}$$

$$\hat{\mathcal{U}}\begin{cases}\frac{\partial\hat{x}}{\partial_t}(\zeta,t)=P_1\frac{\partial}{\partial\zeta}(\mathcal{H}\hat{x}(\zeta,t))+P_0(\mathcal{H}\hat{x}(\zeta,t)),\\ W_{\mathcal{B}}\left(\begin{smallmatrix}\hat{f}_\partial(t)\\ \hat{e}_\partial(t)\end{smallmatrix}\right)=\hat{u}(t),\qquad \hat{x}(\zeta,0)=\hat{x}_0(\zeta)\\ \hat{y}(t)=W_{\mathcal{C}}\left(\begin{smallmatrix}\hat{f}_\partial(t)\\ \hat{e}_\partial(t)\end{smallmatrix}\right),\\ \hat{y}_m(t)=\mathcal{C}_m\hat{x}(\zeta,t),\end{cases}\tag{40}$$

Since the system $\hat{\mathcal{U}}$ in (40) is virtual, the input $\hat{u}(t)$ is designed with all the available information, i.e. $\hat{u}(t)=f(u(t),y_m(t),\hat{x}(\zeta,t))$, where $u(t)$ and $y_m(t)$ are considered known from (39) and $f(\cdot)$ is a function to be designed.

Observer design

Defining

$$\tilde{x}(\zeta,t) := x(\zeta,t) - \hat{x}(\zeta,t). \tag{41}$$

Then, from (39) and (40), we obtain the error dynamics equations as follows:

$$\tilde{\mathcal{U}} \begin{cases} \frac{\partial \tilde{x}}{\partial_t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}\tilde{x}(\zeta,t)) + P_0(\mathcal{H}\tilde{x}(\zeta,t)), \\ W_{\mathcal{B}}\left(\begin{smallmatrix} \tilde{f}_\partial(t) \\ \tilde{e}_\partial(t) \end{smallmatrix}\right) = \tilde{u}(t), \quad \tilde{x}(\zeta,0) = \tilde{x}_0(\zeta), \\ \tilde{y}(t) = W_{\mathcal{C}}\left(\begin{smallmatrix} \tilde{f}_\partial(t) \\ \tilde{e}_\partial(t) \end{smallmatrix}\right). \end{cases} \tag{42}$$

We define the Hamiltonian of the error system as:

$$\tilde{H}(t) = \frac{1}{2}\|\tilde{x}(t)\|^2_{\mathcal{H}} = \frac{1}{2}\int_a^b \tilde{x}(\zeta,t)^T \mathcal{H}(\zeta)\tilde{x}(\zeta,t)d\zeta. \tag{43}$$

Since $W_{\mathcal{B}}$ and $W_{\mathcal{C}}$ are such that $W_{\mathcal{C}}\Sigma W_{\mathcal{B}}^T = I$, the time derivative of $\tilde{H}(t)$ satisfies

$$\dot{\tilde{H}}(t) = \tilde{u}(t)^T \tilde{y}(t). \tag{44}$$

Observer design

Full sensing case

Consider the BC-PHS (39)with $y_m(t) = y(t)$. The state of the observer (40) with

$$\hat{u}(t) = u(t) + L(y_m(t) - \hat{y}_m(t)), \tag{45}$$

converges exponentially to the state of the BC-PHS (39) if $0 < L + L^T \in \mathbb{R}^{n\times n}$.

Partial sensing case

Consider the BC-PHS (39) with $y_m(t) = C_m y(t)$ and $C_m = \left(I_p \ 0_{p\times n-p} \right) \in \mathbb{R}^{p\times n}$, $0 < p < n$. The states of the observer (40) with

$$\hat{u}(t) = u(t) + C_m^T L(y_m(t) - \hat{y}_m(t)) \quad \text{and} \quad L \in \mathbb{R}^{p\times p} \tag{46}$$

converges exponentially to the state of the BC-PHS (39) if L is such that $C_m^T L^T C_m + C_m^T L C_m \geq 0$, and one of the following conditions is satisfied ($\gamma > 0$)

$$\begin{aligned} &\|\mathcal{H}(b)\tilde{x}(b,t)\|_{\mathbb{R}}^2 \leq \gamma \tilde{y}(t)^T C_m^T L C_m \tilde{y}(t) \quad \text{or} \\ &\|\mathcal{H}(a)\tilde{x}(a,t)\|_{\mathbb{R}}^2 \leq \gamma \tilde{y}(t)^T C_m^T L C_m \tilde{y}(t), \end{aligned} \tag{47}$$

Observer design

Position measurement

Consider the BC-PHS (39). Assume that the measurement is on the following form:

$$y_m(t) = \int_0^t C_m y(\tau) d\tau + y_m(0), \text{ with } C_m = \begin{pmatrix} 0_{p\times n-p} & I_p \end{pmatrix}. \tag{48}$$

Assume that the BC-PHS is approximately observable with respect to the output $C_m y(t)$. The state of the observer (40) with

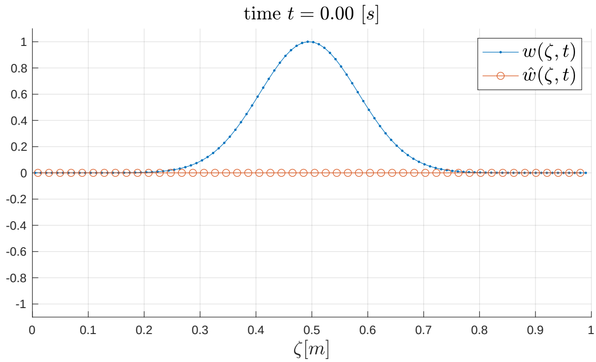
$$\begin{aligned} \hat{u}(t) &= u(t) + C_m^T L_1(y_m(t) - \hat{y}_m(t) + \theta(t)), \\ \dot{\theta}(t) &= -L_2(y_m(t) - \hat{y}_m(t) + \theta(t)), \quad \theta(0) = \theta_0. \end{aligned} \tag{49}$$

converges asymptotically to the state of the BC-PHS (39) if L_1, $L_2 \in \mathbb{R}^{p\times p}$ are both positive definite matrices.

Implementation on the elastic string example

We consider now

- The position of the end point *i.e.* $\omega(b,t)$, is measured .
- The state is reconstructed using a Luenberger PH finite dimensional observer (the control uses $\hat{\omega}(b,t)$ and $\hat{v}(b,t)$)$\Rightarrow$ the closed loop stability is guaranteed [**?**].



Outline

Irreversible systems

We consider a 1-D isentropic fluid in Lagrangian coordinates, also known as *p-system*, with $[a, b] \ni z,\ a,\ b \in \mathbb{R},\ a < b$. We choose as state variables

- the specific volume $\phi(t, z)$,
- the velocity $\upsilon(t, z)$ of the fluid.

System of two conservation laws :

$$\frac{\partial \phi}{\partial t}(t, z) = \frac{\partial \upsilon}{\partial z}(t, z)$$
$$\frac{\partial \upsilon}{\partial t}(t, z) = -\frac{\partial p}{\partial z}(t, z) - \frac{\partial \tau}{\partial z}(t, z)$$

where $p(\phi)$ is the pressure of the fluid, $\tau = -\hat{\mu}\frac{\partial \upsilon}{\partial z}$ with $\hat{\mu}$ the viscous damping coefficient. The total energy of the system is given by the sum of the kinetic energy and internal energy:

$$H(\upsilon,\ \phi) = \int_a^b \left(\frac{1}{2}\upsilon^2 + u(\phi)\right) dz$$

Irreversible systems

$$\begin{bmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial \upsilon}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial z} \left(\begin{bmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta \upsilon} \end{bmatrix} \right) + \begin{bmatrix} 0 & 0 \\ 0 & \frac{\partial}{\partial z} \left(\hat{\mu} \frac{\partial .}{\partial z} \right) \end{bmatrix} \left(\begin{bmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta \upsilon} \end{bmatrix} \right), \tag{50}$$

Or alternatively, splitting

$$\frac{\partial}{\partial z} \left(\hat{\mu} \frac{\partial \upsilon}{\partial z} \right) = \frac{\partial}{\partial z} e_r, \text{ with } e_r = \hat{\mu} f_r, \text{ and } f_r = \frac{\partial \upsilon}{\partial z}$$

One can write

$$\begin{bmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial \upsilon}{\partial t} \\ f_r \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\tilde{P}_1} \frac{\partial}{\partial z} \left(\begin{bmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta \upsilon} \\ e_r \end{bmatrix} \right), \text{ with } e_r = \hat{\mu} f_r \tag{51}$$

Irreversible systems

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- There is still an underlying Dirac structure and we can define from $\tilde{P}_1$ some boundary port variables such that $\frac{dH}{dt} \leq f_\partial^T e_\partial$
- This is a DAE system and the thermal domain is not represented.

The non-isentropic fluid: the irreversible case

We can account for the thermal domain by considering Gibbs' equation

$$du = -pd\phi + Tds$$

where s denotes the entropy density and T the temperature. The total energy of the system is still the sum of the kinetic and the internal energy but now depends on s

$$H\left(v,\,\phi,\,s\right) = \int_a^b \left(\frac{1}{2}v^2 + u\left(\phi, s\right)\right) dz$$

From the conservation of the total energy and Gibbs' equation $\frac{\partial u}{\partial s} = T$ we get

$$\frac{\partial s}{\partial t}(t,z) = \frac{\hat{\mu}}{T}\left(\frac{\partial v}{\partial z}\right)^2 (t,z)$$

The non-isentropic fluid: the irreversible case

The system of balance equations may be written as the quasi-Hamiltonian system

$$\begin{bmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial \upsilon}{\partial t} \\ \frac{\partial s}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial (\cdot)}{\partial z} & 0 \\ \frac{\partial (\cdot)}{\partial z} & 0 & \frac{\partial}{\partial z}\left(\frac{\hat{\mu}}{T}\left(\frac{\partial \upsilon}{\partial z}\right)(\cdot)\right) \\ 0 & \frac{\hat{\mu}}{T}\left(\frac{\partial \upsilon}{\partial z}\right)\frac{\partial (\cdot)}{\partial z} & 0 \end{bmatrix} \left(\begin{bmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta \upsilon} \\ \frac{\delta H}{\delta s} \end{bmatrix} \right)$$

With

$$\frac{dH}{dt} = y^T \nu$$

and

$$\frac{dS}{dt} = \int_a^b \sigma dz - y_s^T \nu_s$$

IPHS : General formulation [?]

We introduce the Boundary Controlled Irreversible Port Hamiltonian System (BC-IPHS) defined on a 1D spatial domain $z \in [a, b]$, a, $b \in \mathbb{R}$, $a < b$. The state variables of the system are the $n+1$ *extensive variables*. The following partition of the state vector $\mathbf{x} \in \mathbb{R}^{n+1}$ shall be considered: the first n variables by $x = [q_1, \ldots, q_n]^\top \in \mathbb{R}^n$ and the entropy density by $s \in \mathbb{R}$. Gibbs' equation is equivalent to the existence of an energy functional

$$H(x, s) = \int_a^b h\left(x(z), s(z)\right) dz \tag{52}$$

where $h(x, s)$ is the energy density function. The total entropy functional is denoted by

$$S(t) = \int_a^b s(z, t) dz \tag{53}$$

IPHS : General formulation

An infinite dimensional IPHS undergoing m irreversible processes is defined by

$$\frac{\partial}{\partial t}\begin{bmatrix} x(t,z) \\ s(t,z) \end{bmatrix} = \begin{bmatrix} P_0 & G_0\mathbf{R_0} \\ -\mathbf{R_0}^\top G_0^\top & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta x}(t,z) \\ \frac{\delta H}{\delta s}(t,z) \end{bmatrix} + \begin{bmatrix} P_1 \frac{\partial(.)}{\partial z} & \frac{\partial(G_1\mathbf{R_1}.)}{\partial z} \\ \mathbf{R_1}^\top G_1^\top \frac{\partial(.)}{\partial z} & g_s\mathbf{r_s}\frac{\partial(.)}{\partial z} + \frac{\partial(g_s\mathbf{r_s}.)}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta x}(t,z) \\ \frac{\delta H}{\delta s}(t,z) \end{bmatrix} \tag{54}$$

where $P_0 = -P_0^\top \in \mathbb{R}^{n\times n}$, $P_1 = P_1^\top \in \mathbb{R}^{n\times n}$, $G_0 \in \mathbb{R}^{n\times m}$, $G_1 \in \mathbb{R}^{n\times m}$ with $m \leq n$ with $\mathbf{R_l}\left(\mathbf{x}, \frac{\delta H}{\delta \mathbf{x}}\right) \in \mathbb{R}^{m\times 1}$, $l = 0, 1$, defined by

$$R_{0,i} = \gamma_{0,i}\left(x, z, \frac{\delta H}{\delta x}\right)\{S|G_0(:,i)|H\}$$

$$R_{1,i} = \gamma_{1,i}\left(x, z, \frac{\delta H}{\delta x}\right)\left\{S|G_1(:,i)\frac{\partial}{\partial z}|H\right\}$$

and

$$r_s = \gamma_s\left(x, z, \frac{\delta H}{\delta x}\right)\{S|H\}$$

and $\gamma_{k,i}\left(x, z, \frac{\delta H}{\delta x}\right) > 0,\ k = 0,\ 1;\ i \in \{1, \ldots m\}$, $\gamma_s\left(x, z, \frac{\delta H}{\delta x}\right) > 0$ and $g_s(x)$,

IPHS : General formulation

For any two functionals H_1 and H_2 of the type (52) and for any matrix differential operator $\mathcal{G}$ we define the pseudo-brackets

$$\begin{aligned} \{H_1|\mathcal{G}|H_2\} &= \begin{bmatrix} \frac{\delta H_1}{\delta x} \\ \frac{\delta H_1}{\delta s} \end{bmatrix} \begin{bmatrix} 0 & \mathcal{G} \\ -\mathcal{G}^* & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H_2}{\delta x} \\ \frac{\delta H_2}{\delta s} \end{bmatrix}, \\ \{H_1|H_2\} &= \frac{\delta H_1}{\delta s}^\top \left(\frac{\partial}{\partial z} \frac{\delta H_2}{\delta s} \right) \end{aligned} \quad (55)$$

where $\mathcal{G}^*$ denotes the formal adjoint operator of $\mathcal{G}$.

IPHS : General formulation

Definition 1
A Boundary Controlled IPHS (BC-IPHS) is an infinite dimensional IPHS

$$\frac{\partial}{\partial t}\begin{bmatrix} x(t,z) \\ s(t,z) \end{bmatrix} = \begin{bmatrix} P_0 & G_0\mathbf{R_0} \\ -\mathbf{R_0}^\top G_0^\top & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta x}(t,z) \\ \frac{\delta H}{\delta s}(t,z) \end{bmatrix} + \begin{bmatrix} P_1\frac{\partial(.)}{\partial z} & \frac{\partial(G_1\mathbf{R_1}.)}{\partial z} \\ \mathbf{R_1}^\top G_1^\top\frac{\partial(.)}{\partial z} & g_s\mathbf{r_s}\frac{\partial(.)}{\partial z} + \frac{\partial(g_s\mathbf{r_s}.)}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta x}(t,z) \\ \frac{\delta H}{\delta s}(t,z) \end{bmatrix} \tag{56}$$

Augmented with the boundary port variables

$$v(t) = W_B\begin{bmatrix} e(t,b) \\ e(t,a) \end{bmatrix}, \qquad y(t) = W_C\begin{bmatrix} e(t,b) \\ e(t,a) \end{bmatrix} \tag{57}$$

as linear functions of the modified effort variable

$$e(t,z) = \begin{bmatrix} \frac{\delta H}{\delta x}(t,z) \\ \mathbf{R}(\mathbf{x},\frac{\delta H}{\delta \mathbf{x}})\,\frac{\delta H}{\delta s}(t,z) \end{bmatrix}, \text{ with } \mathbf{R}\left(\mathbf{x},\frac{\delta H}{\delta \mathbf{x}}\right) = \begin{bmatrix} 1 & \mathbf{R_1}(\mathbf{x},\frac{\delta \mathbf{H}}{\delta \mathbf{x}}) & \mathbf{r_s}(\mathbf{x},\frac{\delta \mathbf{H}}{\delta \mathbf{x}}) \end{bmatrix}^\top \tag{58}$$

IPHS : General formulation

Furthermore

$$W_B = \left[\frac{1}{\sqrt{2}} \left(\Xi_2 + \Xi_1 P_{ep}\right) M_p \quad \frac{1}{\sqrt{2}} \left(\Xi_2 - \Xi_1 P_{ep}\right) M_p \right],$$
$$W_C = \left[\frac{1}{\sqrt{2}} \left(\Xi_1 + \Xi_2 P_{ep}\right) M_p \quad \frac{1}{\sqrt{2}} \left(\Xi_1 - \Xi_2 P_{ep}\right) M_p \right],$$

where $M_p = \left(M^\top M\right)^{-1} M^\top$, $P_{ep} = M^\top P_e M$ and $M \in \mathbb{R}^{(n+m+2)\times k}$ is spanning the columns of $P_e \in \mathbb{R}^{n+m+2}$ of rank k, defined by

$$P_e = \begin{bmatrix} P_1 & 0 & G_1 & 0 \\ 0 & 0 & 0 & g_s \\ G_1^\top & 0 & 0 & 0 \\ 0 & g_s & 0 & 0 \end{bmatrix} \tag{59}$$

and where Ξ_1 and Ξ_2 in $\mathbb{R}^{k\times k}$satisfy $\Xi_2^\top \Xi_1 + \Xi_1^\top \Xi_2 = 0$ and $\Xi_2^\top \Xi_2 + \Xi_1^\top \Xi_1 = I$.

IPHS : General formulation

First law of Thermodynamics

The total energy balance is

$$\dot{H} = y(t)^\top v(t)$$

which leads, when the input is set to zero, to $\dot{H} = 0$ in accordance with the first law of Thermodynamics.

IPHS : General formulation

First law of Thermodynamics

The total energy balance is

$$\dot{H} = y(t)^\top v(t)$$

which leads, when the input is set to zero, to $\dot{H} = 0$ in accordance with the first law of Thermodynamics.

Second law of Thermodynamics

The total entropy balance is given by

$$\dot{S} = \int_a^b \sigma_t dz - y_S^\top v_s$$

where y_s and v_s are the entropy conjugated input/output and σ_t is the total internal entropy production. This leads, when the input is set to zero, to $\dot{S} = \int_a^b \sigma_t dz \geq 0$ in accordance with the second law of Thermodynamics.

Control design : Heat equation

Balance equation on u ($z \in [0, L]$)

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial z}\left(-\lambda \frac{\partial T}{\partial z}\right)$$

where λ denotes the heat conduction coefficient. From Gibbs' equation $du = Tds$

$$\frac{\partial s}{\partial t} = \frac{\lambda}{T^2}\frac{\partial T}{\partial z}\frac{\partial}{\partial z}\left(\frac{\delta U}{\delta s}\right) + \frac{\partial}{\partial z}\left(\frac{\lambda}{T^2}\frac{\partial T}{\partial z}\left(\frac{\delta U}{\delta s}\right)\right) \qquad (60)$$

which is equivalent to (56) where $P_0 = 0$, $P_1 = 0$, $G_0 = 0$, $G_1 = 0$, $g_s = 1$ and $r_s = \gamma_s\{S|U\}$ with $\gamma_s = \frac{\lambda}{T^2}$ and $\{S|U\} = \frac{\partial T}{\partial z}$. In this case $P_e = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $n = 1$ and $m = 1$. Choosing $\Xi_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\Xi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ we have

$$v(t) = \begin{bmatrix} \left(\frac{\lambda_s}{T}\frac{\partial T}{\partial z}\right)(t, L) \\ -\left(\frac{\lambda_s}{T}\frac{\partial T}{\partial z}\right)(t, 0) \end{bmatrix}, \qquad y(t) = \begin{bmatrix} T(t, L) \\ T(t, 0) \end{bmatrix}, \qquad (61)$$

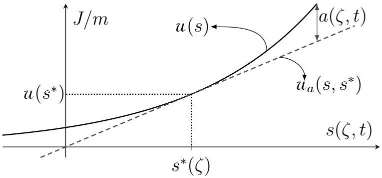
respectively the entropy flux and the temperature at each boundary.

Control design : The heat equation

Idea

- Use the Thermodynamic availability function as closed loop Lyapunov function.

$$\mathcal{A} = \int_0^L (u(s) - u_a(s))\, dz$$



- Use Entropy Assignment to guarantee the convergence of trajectories.

Control design

Availability Based Interconnection

The boundary control feedback $v = \beta(\mathbf{y}) + v'$, with v' an auxiliary boundary input, maps (60), (61) into the target system

$$\partial_t s = \overline{r}_s \partial_z \left(\delta_s \mathcal{H}\right) + \partial_z \left(\overline{r}_s \delta_s \mathcal{H}\right) \quad (62)$$

$$\tilde{\mathbf{u}} = \Xi v' \quad (63)$$

where $\mathcal{H} = U$ and

$$\Xi = \begin{bmatrix} \left.\frac{\delta_s \mathcal{A}}{T}\right|_L & 0 \\ 0 & \left.\frac{\delta_s \mathcal{A}}{T}\right|_0 \end{bmatrix} \text{ and } v' = \begin{bmatrix} \left.\lambda\left(\frac{\partial_z(\delta_s \mathcal{A})}{T}\right)\right|_L \\ \left.\lambda\left(\frac{\partial_z(\delta_s \mathcal{A})}{T}\right)\right|_0 \end{bmatrix} \quad (64)$$

and $\overline{r}_s = \gamma_s\{\mathcal{S}|\mathcal{A}\}$, if the following matching conditions are satisfied

$$\gamma_s\{\mathcal{S}|\mathcal{H}_a\}\partial_z \left(\delta_s \mathcal{H}\right) + \partial_z \left(\gamma_s\{\mathcal{S}|\mathcal{H}_a\}\delta_s \mathcal{H}\right) = 0 \quad (65)$$

$$\beta(\mathbf{y}) + \begin{bmatrix} \left.\lambda\left(\frac{\partial_z(\delta_s \mathcal{H}_a)}{T}\right)\right|_L \\ \left.\lambda\left(\frac{\partial_z(\delta_s \mathcal{H}_a)}{T}\right)\right|_0 \end{bmatrix} = 0 \quad (66)$$

Control design

Entropy Assignment

Let's consider the irreversible port-Hamiltonian system (60)-(63) with boundary control law

$$\tilde{\mathbf{u}} = -\Gamma \mathbf{y} \tag{67}$$

with $\Gamma = \Xi \Phi \Xi^\top$, and $\Phi = \Phi^\top > 0$, then the system is asymptotically stable. If Φ is defined by

$$\Phi = \begin{bmatrix} \frac{\phi_L}{T|_L} & 0 \\ 0 & \frac{\phi_0}{T|_0} \end{bmatrix} \tag{68}$$

where ϕ_L and ϕ_0 are strictly positive, the target temperature profile $T_e^* = m^* z + b^*, \forall z \in [0, L]$, is achievable from any initial condition T_0. At the end the control is

$$\mathbf{u} = \beta(\mathbf{y}) - \Phi \Xi^\top \mathbf{y} \tag{69}$$

Control design

- Initial condition $T_0 = 303.15, \forall z \in [0, 0.1]$
- Target profile $T_e^* = 150z + 313.15, \quad z \in [0, 0.1]$

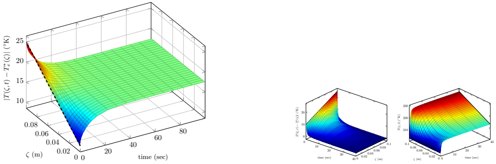


Figure: Behavior of the absolute error of temperature response with respect to desired equilibrium profile, using ABI (left) control and ABI-EA (right) control.

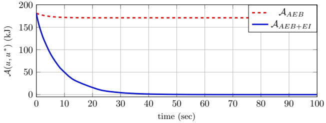


Figure: Available energy.

Outline

Conclusions and future works

Conclusion

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- In the in domain case energy shaping is achieved in an optimal way considering an early lumping approach and closed loop stability was proven.
- In the IPHS case we did not pay attention to existence of solutions.

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Future works

- Study of the impact of the distribution of the patches on the achievable performances.
- Control design for a class of non linear PDE systems.
- Extension to 2D DPS.
- Extension to a larger class of irreversible PHS.

Thank you for your attention !

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