Modeling of Distributed Parameter Systems: The Port Hamiltonian Approach

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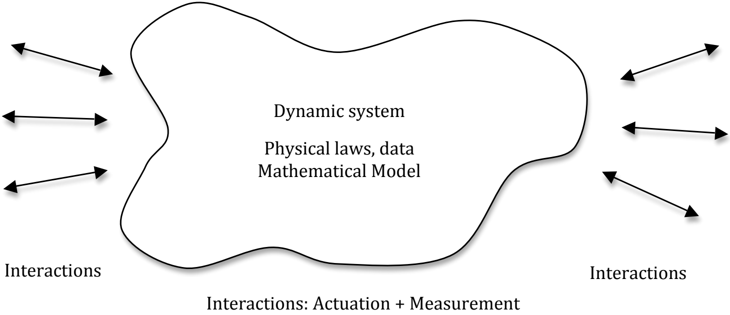
Outline

Dynamic systems

Modeling and control of (deterministic) dynamic systems



Dynamic system

Physical laws, data
Mathematical Model

Interactions

Interactions

Interactions: Actuation + Measurement

Two approaches:

Lumped parameters systems, distributed parameters systems.

Recent technological progresses and physical knowledges allow to go toward the use of complex systems:

- Highly nonlinear.
- Involving numerous physical domains and possible heterogeneity.
- With distributed parameters or organized in network.

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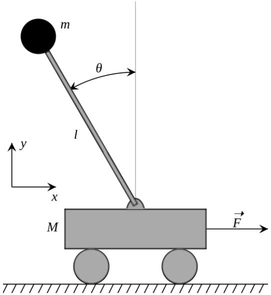
New issue for system control theory

Modeling step is important → the physical properties can be advantageously used for analysis, control or simulation purposes

Example 1: inverted pendulum system

Example: Segway system

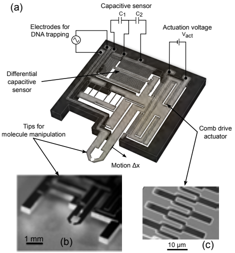
Example 1: inverted pendulum system

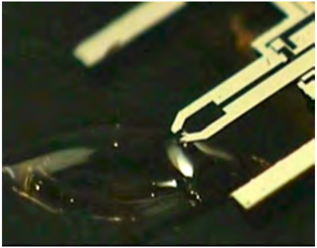


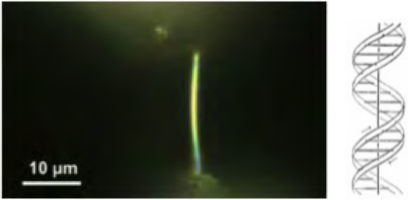
Non linear mechanical system:

- Two natural equilibria.
- Control: insure $\Theta = 0$

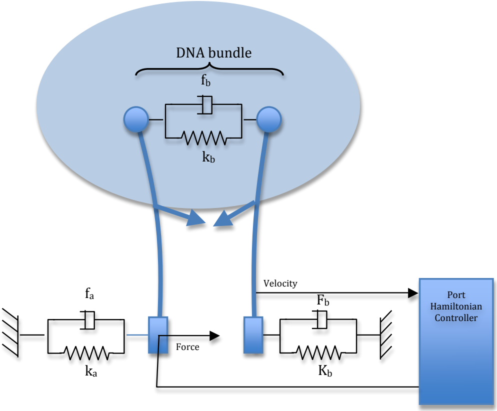
Example 2: Nanotweezer for DNA manipulation







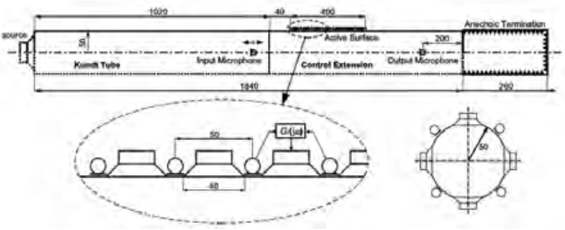
Example 2: Nanotweezer for DNA manipulation



Non linear electro mechanical

- Linear or Non linear ODEs.
- Linear PDEs.

Example 3: Active skin for vibro-acoustic control



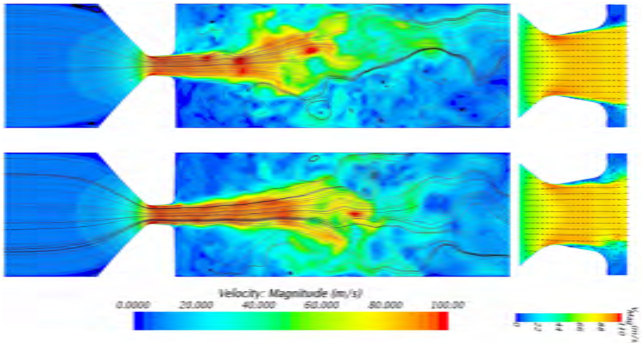
$$\frac{d}{dt}\begin{bmatrix}\theta\\ \Gamma\end{bmatrix} = \begin{bmatrix} 0 & -\overrightarrow{\mathrm{grad}} \\ -\mathrm{div} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\rho_0} & 0 \\ 0 & \frac{1}{\chi_s} \end{bmatrix} \begin{bmatrix}\theta\\ \Gamma\end{bmatrix}$$

2-D case:

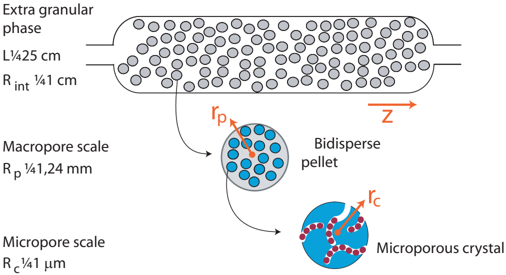
- 2-D wave equation
- Non linear finite dimensional system : loudspeakers/microphones
- Power preserving interconnection

Toward a more complex actuation system with elastodynamic components

Modeling and control of interglotal air flows (coll. USM Chile)

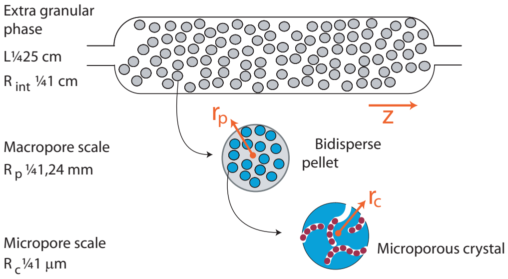


Example 4: Adsorption process



- Multiscale heterogeneous system.
- Dynamic behavior driven by irreversible thermodynamic laws

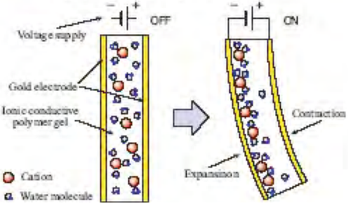
Example 4: Adsorption process



- Multiscale heterogeneous system.
- Considered phenomena:
  - Fluid scale: convection, dispersion.
  - Pellet scale: diffusion (Stephan-Maxwell).
  - Microscopic scale: Knudsen law.

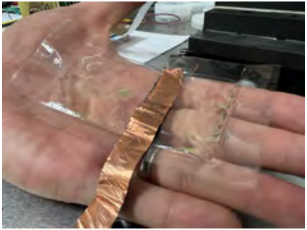
Example 5: Ionic Polymer Metal Composite

Figure 3. Beam-shaped IPMC actuator



- Electromechanical system.
- 3 scales : Polymer-electrode interface, diffusion in the polymer, beam bending.

Soft robotics (FEMTO-ST France)



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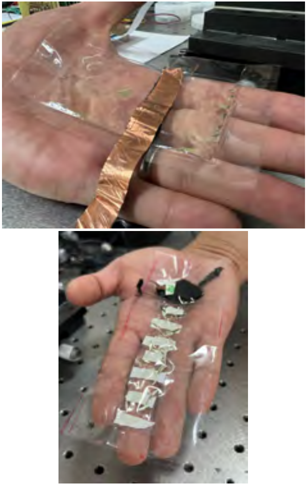
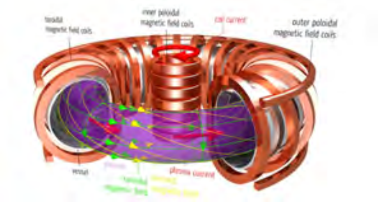


- Artificial aorta for blood pressure control (coll. EPFL Swizerland)

Toward more complex systems ...

Tokamak nuclear reactor



Models and Complexity

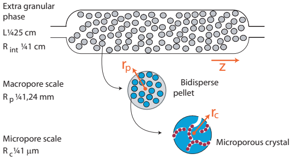
- A model is always an approximation of reality.
- A model depends on the problem context.
- A model has to be tractable.

Purpose

Derive a mathematical model based on Physics useful for:

- Simulation (model reduction)
- Analysis
- Control design

Models and Complexity (illustration)



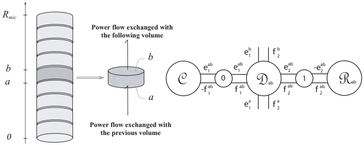
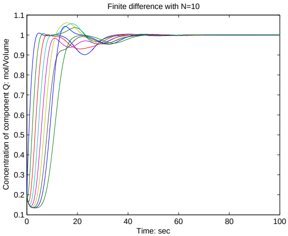
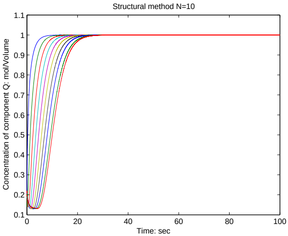


Figure 4: Principle of the spatial discretization



Finite difference with N=10



Structural method N=10

Outline

1. A unified approach

2. Finite dimensional systems

3. Distributed parameter systems
   - Example 1: the vibrating string
   - Example 2: the lossless transmission line
   - Considered class of systems

4. Port Hamiltonian Systems defined on Hilbert Spaces
   - Dirac structure
   - Port Hamiltonian Systems
   - Parametrization of 1D differential operators

5. Extension to systems with dissipation

Port Hamiltonian framework

1833 - W. R. Hamilton

$$\begin{cases} \dot{q} = +\frac{\partial H}{\partial p}(q,p) \\ \dot{p} = -\frac{\partial H}{\partial q}(q,p) \end{cases}$$

- q vector of generalized coordinates.
- p vector of generalized momenta.
- $H(q,p)$ Hamiltonian function, total energy.

Port Hamiltonian framework

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- $H(q,p)$ Hamiltonian function, total energy.

Port Hamiltonian systems

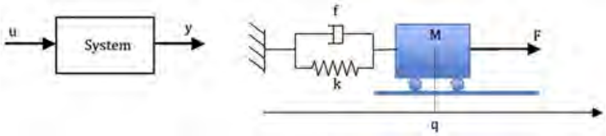
Class of non linear dynamic systems derived from an extension to open physical systems (1992) of Hamiltonian and Gradient systems. This class has been generalized (2001) to distributed parameter systems.

$$x(t):\left\{\begin{array}{l} \dot{x} = (J(x)-R(x))\frac{\partial H(x)}{\partial x} + B(x)u \\ y = B(x)^T\frac{\partial H(x)}{\partial x} \\ \\ \frac{dH}{dt} \leq y^T u \end{array}\right. \qquad x(t,\zeta):\left\{\begin{array}{l} \dot{x} = (\mathcal{J}(x)-\mathcal{R}(x))\frac{\delta H(x)}{\delta x} + \mathcal{B}_d u_d \\ y_d = \mathcal{B}_d^*\frac{\delta H(x)}{\delta x} \\ \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{\delta H(x)}{\delta x}|_\partial, \\ \\ \frac{dH}{dt} \leq y_d^T u_d + f_\partial^T e_\partial \end{array}\right. ,$$

- Central role of the energy.
- Additional information coming from the geometric structure.
- Multi-physic framework.

Finite dimensional example ...

Let consider the mass spring damper system:



From the second Newton's law:

$$M\ddot{q} = -kq - f\dot{q} + F$$

which is usually treated using the canonical state space representation:

$$\begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{f}{M} \end{pmatrix} \begin{pmatrix} q \\ \dot{q} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{M} \end{pmatrix} F$$

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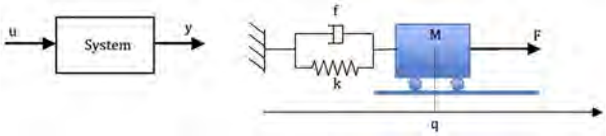


From the second Newton's law:

$$M\ddot{q} = -kq - f\dot{q} + F$$

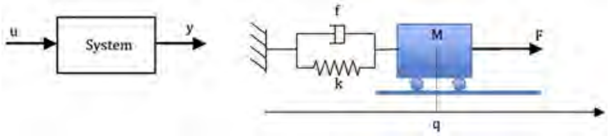
An alternative representation consists in choosing the energy variables (extensives variables) as state variables *i.e* $(q, p = M\dot{q})$

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -f \end{pmatrix}}_{J-R} \underbrace{\begin{pmatrix} kq \\ \dot{q} \end{pmatrix}}_{\partial_x H} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} F$$

with $H(x,p) = \frac{1}{2}\left(kq^2 + \frac{1}{M}p^2\right)$

Finite dimensional example ...

Let consider the mass spring damper system:



From the second Newton's law:

$$M\ddot{q} = -kq - f\dot{q} + F$$

Defining y s.t.:

$$\left\{ \begin{array}{ccl} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} & = & \begin{pmatrix} 0 & 1 \\ -1 & -f \end{pmatrix} \begin{pmatrix} \partial_q H(q,p) \\ \partial_p H(q,p) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F \\ y & = & \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_q H(q,p) \\ \partial_p H(q,p) \end{pmatrix} \end{array} \right.$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}^T \frac{dx}{dt} = \frac{\partial H}{\partial x}^T (J - R) \frac{\partial H}{\partial x} + \frac{\partial H}{\partial x}^T Bu \leq y^T u$$

Infinite dimensional case

In what follows we focus on boundary controlled systems. In the general case, port Hamiltonian systems have been extended to distributed parameter systems by the use of differential geometry:

- Energy variables α_p and α_q are p and q differential forms defined on an n-dimensional manifold Z (with boundary ∂Z).
- $H := \int_Z \mathcal{H} \in \mathbb{R}$
- Port Hamiltonian system is defined by:

$$\begin{pmatrix} -\frac{\partial \alpha_p}{\partial t} \\ -\frac{\partial \alpha_q}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & (-1)^r d \\ d & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta p} \\ \frac{\delta H}{\delta q} \end{pmatrix}$$

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -(-1)^{n-q} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta p}|_\partial \\ \frac{\delta H}{\delta q}|_\partial \end{pmatrix}$$

The main advantage of such formulation is that it is not depending on coordinates, applicable for *nD* systems.

In order to apply some functional analysis tools we focus on the 1D linear case.

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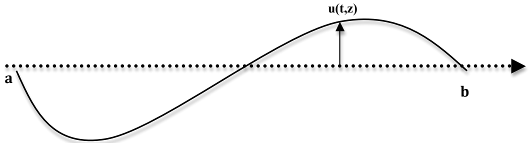
Example 1 : the vibrating string

Let consider a string of length $[a, b]$:



The classical modelling is based on the wave equation : Newton's law + Hooke's law (restoring force proportional to the deformation)

$$\frac{\partial^2 u(z,t)}{\partial t^2} = \frac{1}{\mu(z)} \frac{\partial}{\partial z} \left(T(z) \frac{\partial u(z,t)}{\partial z} \right)$$

The structure of the model is not apparent. How to choose the boundary conditions ???

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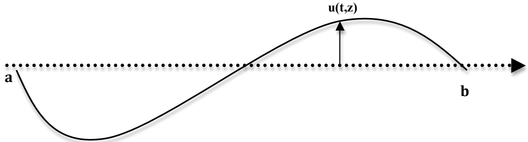

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The structure of the model is not apparent. How to choose the boundary conditions ???

Usually: $x = \begin{bmatrix} u \\ \dot{u} \end{bmatrix} \rightarrow \begin{bmatrix} \dot{u} \\ \ddot{u} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{\mu(z)} \frac{\partial}{\partial z} \left(T(z) \frac{\partial .}{\partial z} \right) & 0 \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \end{bmatrix}$ first order differential equation in time

Vibrating string

Let choose as state variables the energy variables:

- the strain $\varepsilon = \frac{\partial u(z,t)}{\partial z}$
- the elastic momentum $p = \mu(z)v(z,t)$

The **total energy** is given by : $H(\varepsilon, p) = U(\varepsilon) + K(p)$

- $U(\varepsilon)$ is the **elastic potential energy**:

  $$U(\varepsilon) = \int_a^b \frac{1}{2} T(z) \left(\frac{\partial u(z,t)}{\partial z} \right)^2 = \int_a^b \frac{1}{2} T\varepsilon(z,t)^2$$

  where $T(z)$ denotes the elastic modulus.
- $K(v)$ is the **kinetic energy**:

  $$K(p) = \int_a^b \frac{1}{2} \mu(z) v(z,t)^2 = \int_a^b \frac{1}{2} \frac{1}{\mu(z)} p^2(z,t)$$

  where $\mu(z)$ denotes the string mass.

Example 1 : the vibrating string

From the conservation laws:

$$\frac{\partial}{\partial t}\begin{pmatrix}\varepsilon\\ p\end{pmatrix}+\frac{\partial}{\partial z}\begin{pmatrix}\mathcal{N}_\varepsilon\\ \mathcal{N}_p\end{pmatrix}=0$$

The vector of fluxes β may be expressed in term of the generating forces :

$$\begin{pmatrix}\mathcal{N}_\varepsilon\\ \mathcal{N}_p\end{pmatrix}=\underbrace{\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}}_{\substack{\text{canonical}\\ \text{interdomain coupling}}}\underbrace{\begin{pmatrix}\frac{\delta H}{\delta \varepsilon}\\ \frac{\delta H}{\delta p}\end{pmatrix}}_{\substack{\text{generating}\\ \text{forces}}}=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\sigma(z,t)\\ v(z,t)\end{pmatrix}$$

where $v(z,t)$ is the velocity and $\sigma(z,t)=T(z)\varepsilon(z,t)$ the stress. Consequently

$$\frac{\partial}{\partial t}\begin{pmatrix}\varepsilon\\ p\end{pmatrix}=-\frac{\partial}{\partial z}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\frac{\delta H}{\delta \varepsilon}\\ \frac{\delta H}{\delta p}\end{pmatrix}$$

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where $v(z,t)$ is the velocity and $\sigma(z,t) = T(z)\varepsilon(z,t)$ the stress.

PDEs:

$$\frac{\partial}{\partial t}\begin{pmatrix} \varepsilon \\ p \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix}\begin{pmatrix} \frac{\delta H}{\delta \varepsilon} \\ \frac{\delta H}{\delta p} \end{pmatrix} \Leftrightarrow \frac{\partial^2 u(z,t)}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 u(z,t)}{\partial z^2} \text{ if } c = cte$$

+BC

Example 1: the vibrating string

Underlying structure:

$$\underbrace{\frac{\partial}{\partial t}\begin{pmatrix}\varepsilon\\ p\end{pmatrix}}_{f} = \underbrace{\begin{pmatrix}0 & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial z} & 0\end{pmatrix}}_{\substack{\mathcal{J} = \text{matrix}\\ \text{differential operator}}} \underbrace{\begin{pmatrix}T(z) & 0\\ 0 & \frac{1}{\mu(z)}\end{pmatrix}\begin{pmatrix}\varepsilon\\ p\end{pmatrix}}_{\substack{e = \text{driving}\\ \text{force}}}$$

Hamiltonian operator $\mathcal{J}$ is **skew-symmetric only for function with compact domain strictly** in Z :

$$\int_a^b \begin{pmatrix} e_1 & e_2 \end{pmatrix} \mathcal{J} \begin{pmatrix} e_1' \\ e_2' \end{pmatrix} + \begin{pmatrix} e_1' & e_2' \end{pmatrix} \mathcal{J} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \left[e_1 e_2' + e_2 e_1' \right]_a^b$$

Power balance equation :

$$\begin{aligned} \frac{d}{dt} H(\varepsilon, p) &= \int_a^b \left(\frac{\delta \mathcal{H}}{\delta \varepsilon} \frac{\partial \varepsilon}{\partial t} + \frac{\delta \mathcal{H}}{\delta p} \frac{\partial p}{\partial t} \right) dz \\ &= \int_a^b \left(\frac{\delta \mathcal{H}}{\delta \varepsilon} \frac{\partial}{\partial z} \frac{\delta \mathcal{H}}{\delta p} + \frac{\delta \mathcal{H}}{\delta p} \frac{\partial}{\partial z} \frac{\delta H}{\delta \varepsilon} \right) dz = \left[\frac{\delta \mathcal{H}}{\delta \varepsilon} \frac{\delta \mathcal{H}}{\delta p} \right]_a^b \end{aligned}$$

If driving forces are zero at the boundary, the total energy is conserved, else there is a **flow of power at the boundary**. Define two **port boundary variables** as follows :

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \frac{\delta H}{\delta \varepsilon} \\ \frac{\delta H}{\delta p} \end{pmatrix} \Big|_{a,b}$$

Example 1: the vibrating string

The linear space $\mathcal{D} \ni (f_1, f_2, e_1, e_2, f_\partial, e_\partial)$

- $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$
- $\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} |_{a,b}$

defines a **Dirac structure:**$\mathcal{D} = \mathcal{D}^\perp$ with respect to the pairing :

$$\int_a^b e_1 f_1 dz + \int_a^b e_2 f_2 dz - f_\partial^T e_\partial$$

Port Hamiltonian system

$$\left(\frac{\partial x}{\partial t}, \frac{\delta H}{\delta x}, f_\partial, e_\partial \right) \in \mathcal{D}$$

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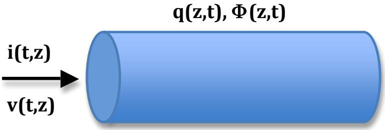
Port Hamiltonian system

$$\left(\frac{\partial x}{\partial t}, \frac{\delta H}{\delta x}, f_\partial, e_\partial\right) \in \mathcal{D}$$

Energy balance

$$\frac{dH(t)}{dt} = f_\partial^T e_\partial$$

The lossless transmission line



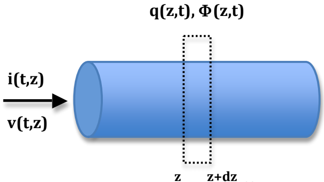
Consider an ideal lossless transmission line with spatial domain $Z = [a, b] \subset \mathbb{R}$. There are two conserved variables:

- the charge on the interval Z: $Q_{(a,b)}(t) = \int_a^b q(t, z)dz$ where $q(t, z)$ denotes the charge density,
- the flux on the interval Z : $\Phi_{(a,b)}(t) = \int_a^b \phi(t, z)dz$ where $\phi(t, z)$ denotes the flux density.

Then $q(t, z)$ and $\phi(t, z)$ are the two extensive variables that will be used for the modeling.

The lossless transmission line

Let consider an infinitesimal piece of the transmission line:



One can write the following 2 *conservation laws* in differential form:

- conservation of charge:

$$\frac{d}{dt}q(t,z) = -\frac{\partial}{\partial z}i(t,z) \tag{1}$$

  where $i(t,z)$ denotes the current at z
- conservation of flux:

$$\frac{d}{dt}\phi(t,z) = -\frac{\partial}{\partial z}v(t,z) \tag{2}$$

  where $v(t,z)$ denotes the voltage at z

The lossless transmission line

The electromagnetic properties gives the two *closure equations* for the functions $i(t,z)$ and $v(t,z)$:

- the current is given by:

$$i(t,z) = \frac{\phi(t,z)}{L(z)} \tag{3}$$

  where $L(z)$ denotes the distributed inductance of the line
- the voltage is given by:

$$v(t,z) = \frac{q(t,z)}{C(z)} \tag{4}$$

  where $C(z)$ denotes the distributed capacitance of the line

and the total electromagnetic energy of the system can be written:

$$H = \int_a^b \mathcal{H}(q,\phi)dz = \frac{1}{2}\int_a^b \left(\frac{q^2(t,z)}{C(z)} + \frac{\phi^2(t,z)}{L(z)} \right) dz \tag{5}$$

The lossless transmission line

The preceding closure equations may be written in matrix form:

$$\begin{pmatrix} i(t,z) \\ v(t,z) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H(q,\phi)}{\delta q} \\ \frac{\delta H(q,\phi)}{\delta \phi} \end{pmatrix} \tag{6}$$

where $H(q,\phi) = \int_a^b \mathcal{H}(q,\phi)dz$ and $\mathcal{H}(q,\phi)$ denotes the electromagnetic energy density:

$$\mathcal{H}(q,\phi) = \frac{1}{2}\left(\frac{q^2(t,z)}{C(z)} + \frac{\phi^2(t,z)}{L(z)}\right) \tag{7}$$

The lossless transmission line

Combining the conservation laws and the closure equations one obtains the Hamiltonian system:

$$\frac{\partial}{\partial t}\begin{pmatrix} q(t,z) \\ \phi(t,z) \end{pmatrix} = \mathcal{J}\begin{pmatrix} \frac{\delta H(q,\phi)}{\delta q} \\ \frac{\delta H(q,\phi)}{\delta \phi} \end{pmatrix} \tag{8}$$

where $\mathcal{J}$ is a formally skew symmetric differential operator defined as:

$$\mathcal{J} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{pmatrix} \tag{9}$$

Take two effort densities $e(t,z)$ and $e'(t,z)$ and compute their bracket with respect to $\mathcal{J}$:

$$
\begin{aligned}
\int_a^b (e_q, e_\phi)\, \mathcal{J} \begin{pmatrix} e'_q \\ e'_\phi \end{pmatrix} dz &= -\int_a^b \left(e_q \frac{\partial}{\partial z}\, e'_\phi + e_\phi\, \frac{\partial}{\partial z} e'_q \right) dz \\
&= \int_a^b \left(e'_q\, \frac{\partial}{\partial z} e_\phi + e'_\phi\, \frac{\partial}{\partial z} e_q \right) dz - \left[e'_q\; e_\phi + e'_\phi\; e_q \right]_0^1 \\
&= -\int_a^b \left(e'_q, e'_\phi \right) \mathcal{J} \begin{pmatrix} e_q \\ e_\phi \end{pmatrix} dz - \left[e'_q\; e_\phi + e'_\phi\; e_q \right]_a^b
\end{aligned}
$$

We can see that it is skew symmetric for *densities that vanish at the boundary*!

The lossless transmission line

The resulting port-Hamiltonian system is given by the telegraph equations

$$\begin{pmatrix} \frac{\partial Q}{\partial t} \\ \frac{\partial \varphi}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} v \\ i \end{pmatrix}$$

together with the boundary variables

$$\begin{array}{rclrcl} f_\partial^a(t) & = & v(t,0), & f_\partial^b(t) & = & v(t,1) \\ e_\partial^a(t) & = & i(t,0), & e_\partial^b(t) & = & -i(t,1) \end{array}$$

The resulting energy-balance is

$$\frac{dH}{dt} = f_\partial^T e_\partial = -i(t,1)v(t,1) + i(t,0)v(t,0),$$

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    - Port Hamiltonian Systems
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Considered class of systems

We first consider lossless systems defined on 1-D spatial domain $[a, b]$ by the PDE:

$$\frac{dx}{dt}(t,z) = \mathcal{J}\mathcal{L}(z)x(t,z),\ x(0,z) = x_0(z),$$

where $\mathcal{J}$ is a formally skew symmetric differential operator and $\mathcal{L}(z)$ a coercive operator.

For example

$$\underbrace{\frac{\partial}{\partial t}\begin{pmatrix} \epsilon \\ p \end{pmatrix}}_{f} = \underbrace{\begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix}}_{\mathcal{J}} \underbrace{\begin{pmatrix} T(z) & 0 \\ 0 & \frac{1}{\mu(z)} \end{pmatrix}\begin{pmatrix} \epsilon \\ p \end{pmatrix}}_{e = \mathcal{L}(z)}$$

$$\Leftrightarrow$$

$$f = \mathcal{J}e$$

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Bond space

The system is defined by :

$$f = \mathcal{J} e$$

and we first consider homogeneous boundary conditions.

- Let the space of flow variables, $\mathcal{F}$, and the space of effort variables,$\mathcal{E}$, be real Hilbert spaces.
- Define the space of bond variables as $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ endowed by the natural inner product

$$\left\langle b^1, b^2 \right\rangle = \left\langle f^1, f^2 \right\rangle_{\mathcal{F}} + \left\langle e^1, e^2 \right\rangle_{\mathcal{E}}, \quad b^1 = \left(f^1, e^1\right), b^2 = \left(f^2, e^2\right) \in \mathcal{B}.$$

In order to define a Dirac structure, let us moreover endow the bond space $\mathcal{B}$ with a *canonical symmetric pairing*, i.e., a bilinear form defined as follows:

$$\left\langle b^1, b^2 \right\rangle_+ = \left\langle f^1, r_{\mathcal{E},\mathcal{F}} e^2 \right\rangle_{\mathcal{F}} + \left\langle e^1, r_{\mathcal{F},\mathcal{E}} f^2 \right\rangle_{\mathcal{E}}, \; b^1 = \left(f^1, e^1\right), b^2 = \left(f^2, e^2\right) \in \mathcal{B}. \qquad (10)$$

Dirac structure

Denote by $\mathcal{D}^{\perp}$ the orthogonal subspace to $\mathcal{D}$ with respect to the symmetric pairing:

$$\mathcal{D}^{\perp} = \left\{ b \in \mathcal{B} | \left\langle b, b' \right\rangle_{+} = 0 \text{ for all } b' \in \mathcal{D} \right\}. \tag{11}$$

Definition [Courant, 1990] :

A Dirac structure $\mathcal{D}$ on the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ is a subspace of $\mathcal{B}$ which is maximally isotropic with respect to the canonical symmetric pairing, i.e.,

$$\mathcal{D}^{\perp} = \mathcal{D}. \tag{12}$$

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$$\begin{pmatrix} f \\ e \end{pmatrix} \in \mathcal{D} \Longleftrightarrow \textbf{Power conservation}$$

Port Hamiltonian Systems

PHS $\rightsquigarrow$ Definition based on Dirac structure and Hamiltonian function (total energy of the system).

Definition :

Let $\mathcal{B} = \mathcal{E} \times \mathcal{F}$ be the bound space defined above and consider the Dirac structure $\mathcal{D}$ and the Hamilonian function $\mathcal{H}(x)$ with x the energy variables. Define the flow variables, $f \in \mathcal{F}$ as the time variation of the energy variables and the effort variables $e \in \mathcal{E}$ as the variational derivative of $\mathcal{H}(x)$. The system

$$(f, e) = \left(\frac{\partial x}{\partial t}, \frac{\delta \mathcal{H}}{\delta x}\right) \in \mathcal{D}$$

is a Port Hamiltonian system with total energy $\mathcal{H}(x)$

Let us now see how to include non homogeneous boundary conditions:

$$\frac{d\mathcal{H}}{dt} = \int_a^b \frac{\delta \mathcal{H}}{\delta x}^T \frac{dx}{dt} dz = \int_a^b \frac{\delta \mathcal{H}}{\delta x}^T \mathcal{J} \frac{\delta \mathcal{H}}{\delta x} dz = \left[\Xi \left(\frac{\delta \mathcal{H}}{\delta x} \right) \right]_a^b$$

$$\langle f, e \rangle = f_\partial^T e_\partial$$

Extension to non homogeneous BC

$\rightsquigarrow$ We define the symmetric pairing (not depending on $\mathcal{J}$) and the port variables associated with $\mathcal{J}$. ([Le Gorrec et al., 2005])
Let $\mathcal{F} = \mathcal{E} = L^2((a,b);\mathbb{R}^n) \times \mathbb{R}^{nN}$ and define $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ with the following canonical symmetric pairing :

$$\begin{aligned}\langle (f^1, f^1_\partial, e^1, e^1_\partial)\ (f^2, f^2_\partial, e^2, e^2_\partial) \rangle_+ \\ = \langle e^1, f^2 \rangle_{L^2} + \langle e^2, f^1 \rangle_{L^2} - \langle e^1_\partial, f^2_\partial \rangle - \langle e^2_\partial, f^1_\partial \rangle,\end{aligned}$$

Definition :

Let $\mathcal{B} = \mathcal{E} \times \mathcal{F}$ be the bound space defined above and consider the Dirac structure $\mathcal{D}$ and the Hamilonian function $\mathcal{H}(x)$ with x the energy variables. Define the flow variables, $f \in \mathcal{F}$ as the time variation of the energy variables and its extension to the boundary and the effort variables $e \in \mathcal{E}$ as the variational derivative of $\mathcal{H}(x)$ and its extension to the boundary. The system

$$((f, f_\partial), (e, e_\partial)) = \left(\left(\frac{\partial x}{\partial t}, f_\partial \right), \left(\frac{\delta \mathcal{H}}{\delta x}, e_\partial \right) \right) \in \mathcal{D}_\mathcal{J}$$

is a Port Hamiltonian system with total energy $\mathcal{H}(x)$

Parametrization of 1D differential operators

Parametrization ([Le Gorrec et al., 2005, Villegas, 2007]):

$$\mathcal{J} e = \sum_{i=0}^{N} P(i) \frac{d^i e}{dz^i}(z) \qquad z \in [a, b]\,,$$

where $e \in H^N((a,b);\mathbb{R}^n)$ and $P(i)$, $i = 0, \ldots, N$, is a $n \times n$ real matrix with P_N non singular and $P_i = P_i^T(-1)^{i+1}$. Let define

$$Q = \begin{pmatrix} P_1 & P_2 & \cdots & P_N \\ -P_2 & -P_3 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ (-1)^{N-1} P_N & 0 & \cdots & 0 \end{pmatrix}$$

Back to the **Vibrating string**

$$\underbrace{\frac{\partial}{\partial t}\begin{pmatrix} \epsilon \\ p \end{pmatrix}}_{f} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{P_1} \frac{\partial}{\partial z} \underbrace{\begin{pmatrix} T(z) & 0 \\ 0 & \frac{1}{\mu(z)} \end{pmatrix}\begin{pmatrix} \epsilon \\ p \end{pmatrix}}_{e}, \ Q = P_1$$

Port Variables

Definition :

The port variables $(e_\partial, f_\partial) \in \mathbb{R}^{nN}$ associated with $\mathcal{J}$ are defined by :

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} e(b) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(b) \\ e(a) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(a) \end{pmatrix}, \quad R_{\text{ext}} = \frac{U}{\sqrt{2}} \begin{pmatrix} Q & -Q \\ I & I \end{pmatrix}$$

where U is a unitary matrix such that:

$$U^T \Sigma U = \Sigma \text{ with } \Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

Port Variables

Back to the **Vibrating string**

$$\underbrace{\frac{\partial}{\partial t}\begin{pmatrix} \epsilon \\ p \end{pmatrix}}_{f} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{P_1} \frac{\partial}{\partial z} \underbrace{\begin{pmatrix} T(z)\epsilon \\ \frac{1}{\mu(z)}p \end{pmatrix}}_{e}, Q = P_1$$

The boundary port variables are defined by:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} e(b) \\ e(a) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p(b)}{\mu(b)} - \frac{p(a)}{\mu(a)} \\ T(b)\epsilon(b) - T(a)\epsilon(a) \\ T(a)\epsilon(a) + T(b)\epsilon(b) \\ \frac{p(a)}{\mu(a)} + \frac{p(b)}{\mu(b)} \end{pmatrix}$$

Dirac structure

Theorem :
The subspace $\mathcal{D}_{\mathcal{J}}$ of $\mathcal{B}$ defined as

$$\mathcal{D}_{\mathcal{J}} = \left\{ \begin{pmatrix} f \\ f_\partial \\ e \\ e_\partial \end{pmatrix} \;\middle|\; e \in H^N((a,b);\mathbb{R}^n),\, \mathcal{J} e = f,\, \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_{\mathrm{ext}} \begin{pmatrix} e(b) \\ \vdots \\ \partial_z^{N-1} e(a) \end{pmatrix} \right\}$$

is a Dirac structure, that means that $\mathcal{D} = \mathcal{D}^\perp$.

Other possible choice

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}}_{U} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p(b)}{\mu(b)} - \frac{p(a)}{\mu(a)} \\ T(b)\epsilon(b) - T(a)\epsilon(a) \\ T(a)\epsilon(a) + T(b)\epsilon(b) \\ \frac{p(a)}{\mu(a)} + \frac{p(b)}{\mu(b)} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{p(a)}{\mu(a)} \\ \frac{p(b)}{\mu(b)} \\ T(a)\epsilon(a) \\ T(b)\epsilon(b) \end{pmatrix}$$

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Extension to systems with dissipation

Let us extend the previous results to systems defined by:

$$\frac{dx}{dt}(t,z) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}(z)x(t,z), \; x(0,z) = x_0(z),$$

$$\Updownarrow$$

$$\begin{pmatrix} f \\ f_p \end{pmatrix} = \mathcal{J}_e \begin{pmatrix} e \\ e_p \end{pmatrix} = \begin{pmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{pmatrix} \begin{pmatrix} e \\ e_p \end{pmatrix}$$

with $e_p = Sf_p$ where S is a coercive operator

$$\begin{pmatrix} f \\ f_p \end{pmatrix} \in \mathcal{F}, \begin{pmatrix} e \\ e_p \end{pmatrix} \in \mathcal{E} \text{ and } \mathcal{E} = \mathcal{F} = L_2((a,b),\mathbb{R}^n) \times L_2((a,b),\mathbb{R}^n)$$

Covers models of: beams, wave, plates, (with or without damping) and also systems of diffusion/convection, chemical reactors ...

A simple example: the heat equation

1D Heat conduction is usually known on the following form:

$$\frac{\partial T(z,t)}{\partial t} = D\frac{\partial^2}{\partial z^2}\left(T(z,t)\right)$$

but is in fact derived from balance equation on the energy *i.e:*

$$\frac{\partial\left(c_v T(z,t)\right)}{\partial t} = -\frac{\partial}{\partial z}\left(-\lambda\frac{\partial T(z,t)}{\partial z}\right)$$

with c_v constant and positive. This equation can be written:

$$\begin{pmatrix} \frac{\partial}{\partial t}T(z,t) \\ f_p \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix}\begin{pmatrix} T(z,t) \\ e_p \end{pmatrix} \quad \textit{with} \;\; e_p = \frac{\lambda}{c_v}f_p$$

In this case:

$$\mathcal{J} = 0,\; \mathcal{G}_R = \frac{\partial}{\partial z},\; S = \frac{\lambda}{c_v} > 0$$

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Parametrization of the extended operator

$\mathcal{J}_e$ is formally skew symmetric and can be parametrized by:

$$\mathcal{J}_e \widetilde{e} = \Sigma_1^N \widetilde{P}_k \frac{\partial^k}{\partial z^k} \widetilde{e} \ \text{ with } \ \widetilde{P}_k = (-1)^{k+1} \widetilde{P}_k^T$$

In this case $\widetilde{P}_N$ can be not full rank and the bilinear product is defined on quotient space. The extended boundary port variables are defined by:

$$\begin{pmatrix} \widetilde{f}_\partial \\ \widetilde{e}_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{Q}_1 & -\tilde{Q}_1 \\ I & I \end{pmatrix} \begin{pmatrix} M_Q & 0 \\ 0 & M_Q \end{pmatrix} \begin{pmatrix} \widetilde{e}(b) \\ \widetilde{e}(a) \end{pmatrix}$$

M spanning the column of $\widetilde{Q}$, $\widetilde{Q}_1 = M^T \widetilde{Q} M$ and $M_Q = (M^T M)^{-1} M^T$ with

$$\widetilde{Q} = \begin{pmatrix} \widetilde{P}_1 & \widetilde{P}_2 & \cdots & \widetilde{P}_N \\ -\widetilde{P}_2 & -\widetilde{P}_3 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ (-1)^{N-1} \widetilde{P}_N & 0 & \cdots & 0 \end{pmatrix}$$

Back to the vibrating string

We consider now the vibrating string with structural damping (dissipation of the form $k_s \frac{\partial}{\partial z}\left(\frac{p}{\mu}\right)$ is given by a system of 2 conservation laws:

$$\frac{\partial}{\partial t}\begin{pmatrix} \varepsilon \\ p \end{pmatrix} = \frac{\partial}{\partial z}\begin{pmatrix} \frac{p}{\mu} \\ T\,\varepsilon + k_s \frac{\partial}{\partial z}\left(\frac{p}{\mu}\right) \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \left(\frac{\partial}{\partial z} k_s \frac{\partial}{\partial z}\right) \end{pmatrix} \begin{pmatrix} \frac{\delta H_0}{\delta \varepsilon} \\ \frac{\delta H_0}{\delta p} \end{pmatrix}$$

The extended Hamiltonian operator is:

$$\mathcal{J}_e = \begin{pmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} & 0 \\ \frac{\partial}{\partial z} & 0 & +\frac{\partial}{\partial z} \\ 0 & +\frac{\partial}{\partial z} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{\partial}{\partial z}$$

and

$$S = k_s > 0$$

Boundary port variables

A matrix M spanning the columns of P_1 can be chosen as:

$$\widetilde{P}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$$

$$\text{then} \quad \widetilde{Q}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } M_Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \widetilde{e} = \begin{pmatrix} T\varepsilon + e_R \\ \mu^{-1} p \end{pmatrix}$$

It thus follows that the port-variables become:

$$\begin{pmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{Q}_1 & -\tilde{Q}_1 \\ I & I \end{pmatrix} \begin{pmatrix} \tilde{e}(b) \\ \tilde{e}(a) \end{pmatrix} = \begin{pmatrix} \frac{p}{\mu}(b) - \frac{p}{\mu}(a) \\ (T\varepsilon + e_R)(b) - (T\varepsilon + e_R)(a) \\ (T\varepsilon + e_R)(a) + (T\varepsilon + e_R)(b) \\ \frac{p}{\mu}(a) + \frac{p}{\mu}(b) \end{pmatrix}$$

Conclusion

In this first part we have:

- shown that PDEs are obtained from balances equation on extensives variables and can be related to power exchanges within the system through geometric considerations,
- in the 1D case defined:
  - the boundary port variables associated to the differential operator $\mathcal{J}$
  - Dirac structures on real Hilbert spaces
- parametrized all the boundary port variables for a large class of differential operators.

We did not pay any attention on existence of solutions.

In the next part we focus on solutions and stability properties.

Thank you for your attention !

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