Modeling and Control of Nonlinear and Distributed Parameter Systems: the Port Hamiltonian Approach

Modeling, existence of solutions

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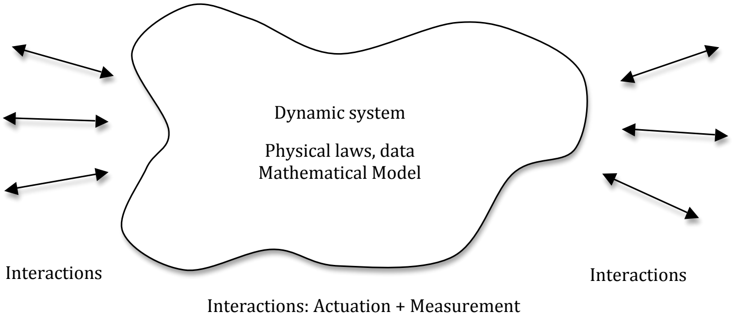
Outline

Dynamic systems

Modeling and control of (deterministic) dynamic systems



Two approaches:

Lumped parameters systems, distributed parameters systems.

Recent technological progresses and physical knowledges allow to go toward the use of complex systems:

- Highly nonlinear.
- Involving numerous physical domains and possible heterogeneity.
- With distributed parameters or organized in network.

Recent technological progresses and physical knowledges allow to go toward the use of complex systems:

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New issue for system control theory

Modeling step is important $\rightarrow$ the physical properties can be advantageously used for analysis, control or simulation purposes

Example 1: inverted pendulum system

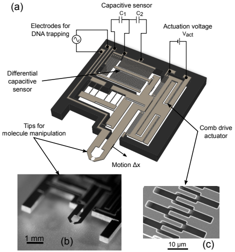
Example: Segway, Gyroskate, Self balancing scooter ...

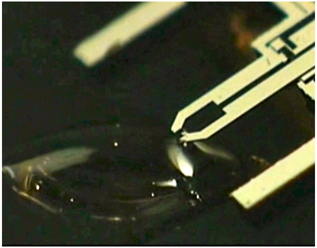
Example 1: inverted pendulum system

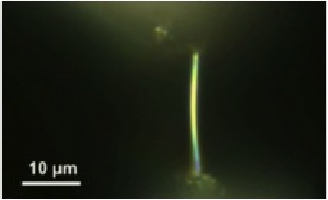
Non linear mechanical system:

- Two natural equilibria.
- Control: insure $\Theta = 0$

Example 2: Nanotweezer for DNA manipulation







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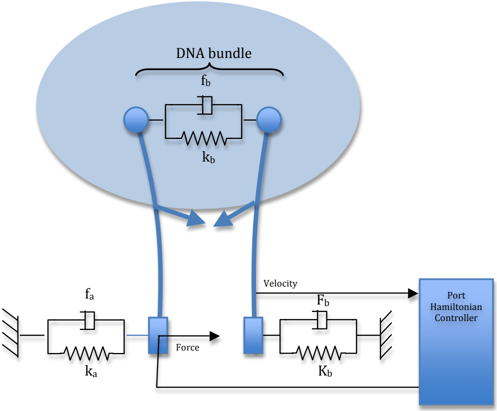
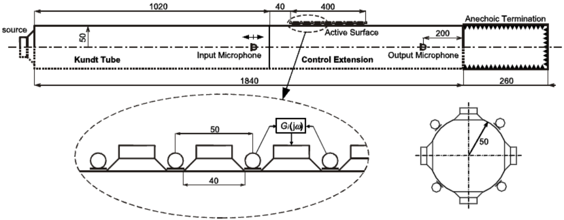


Example 3: Active skin for vibro-acoustic control



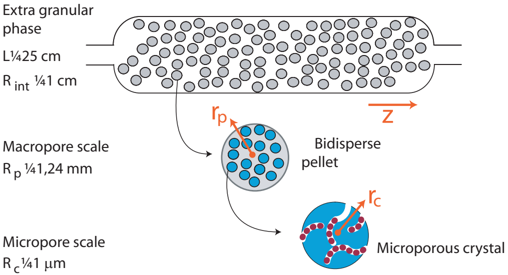
$$\frac{d}{dt}\begin{bmatrix}\theta \\ \Gamma\end{bmatrix} = \begin{bmatrix} 0 & -\overrightarrow{\text{grad}} \\ -\text{div} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\rho_0} & 0 \\ 0 & \frac{1}{\chi_s} \end{bmatrix} \begin{bmatrix}\theta \\ \Gamma\end{bmatrix}$$

2-D case:

- 2-D wave equation
- Non linear finite dimensional system : loudspeakers/microphones
- Power preserving interconnection

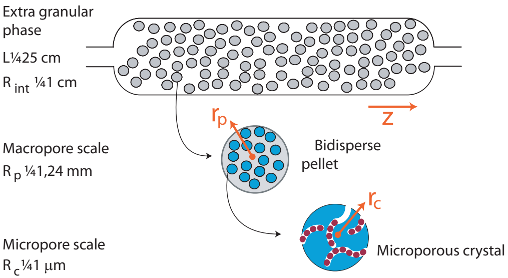
Toward a more complex actuation system with elastodynamic components

Example 4: Adsorption process



- Multiscale heterogeneous system.
- Dynamic behavior driven by irreversible thermodynamic laws

Example 4: Adsorption process



- Multiscale heterogeneous system.
- Considered phenomena:
  - Fluid scale: convection, dispersion.
  - Pellet scale: diffusion (Stephan-Maxwell).
  - Microscopic scale: Knudsen law.

Example 5: Ionic Polymer Metal Composite

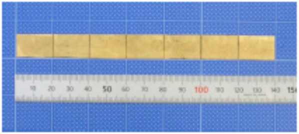
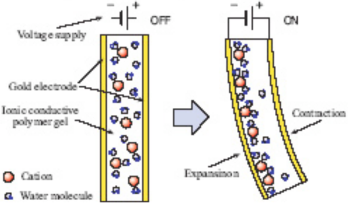


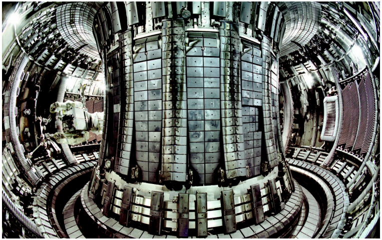
Figure 3. Beam-shaped IPMC actuator

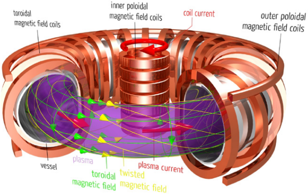


- Electromechanical system.
- 3 scales : Polymer-electrode interface, diffusion in the polymer, beam bending.

Toward more complex systems ...

Tokamak nuclear reactor





Models and Complexity

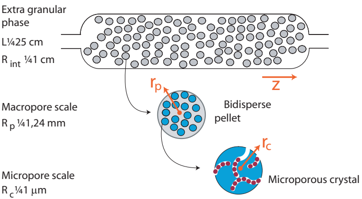
- A model is always an approximation of reality.
- A model depends on the problem context.
- A model has to be tractable.

Purpose

Derive a mathematical model based on Physics useful for:

- Simulation (model reduction)
- Analysis
- Control design

Models and Complexity (illustration)



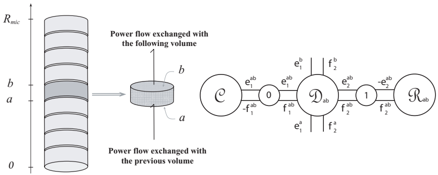
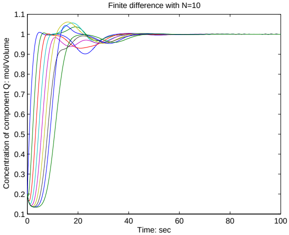
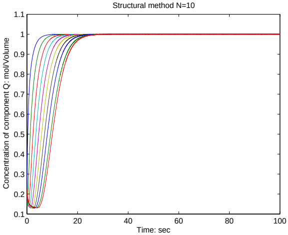


Figure 4: Principle of the spatial discretization





Outline

Port Hamiltonian framework

1833 - W. R. Hamilton

$$\begin{cases} \dot{q} & = & +\frac{\partial H}{\partial q}(q,p) \\ \dot{p} & = & -\frac{\partial H}{\partial p}(q,p) \end{cases}$$

- q vector of generalized coordinates.
- p vector of generalized momenta.
- $H(q,p)$ Hamiltonian function, total energy.

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Port Hamiltonian systems

Class of non linear dynamic systems derived from an extension to open physical systems (1992) of Hamiltonian and Gradient systems. This class has been generalized (2001) to distributed parameter systems.

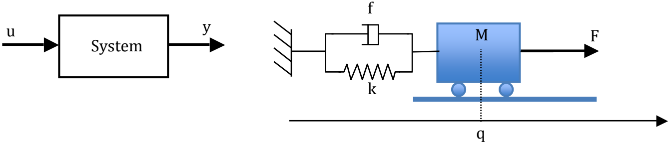
$$x(t) : \left\{ \begin{array}{l} \dot{x} = (J(x) - R(x)) \frac{\partial H(x)}{\partial x} + B(x)u \\ y = B(x)^T \frac{\partial H(x)}{\partial x} \end{array} \right. \qquad x(t,z) : \left\{ \begin{array}{l} \dot{x} = (\mathcal{J}(x) - \mathcal{R}(x)) \frac{\delta \mathcal{H}(x)}{\delta x} \\ \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{\delta \mathcal{H}(x)}{\delta x}|_\partial \end{array} \right.$$

- Central role of the energy.
- Additional information coming from the geometric structure.
- Multi-physic framework.

Outline

Finite dimensional example ...

Let consider the mass spring damper system:



From the second Newton's law:

$$M\ddot{q} = -kq - f\dot{q} + F$$

which is usually treated using the canonical state space representation:

$$\begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{M} & -f \end{pmatrix} \begin{pmatrix} q \\ \dot{q} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F$$

Finite dimensional example ...

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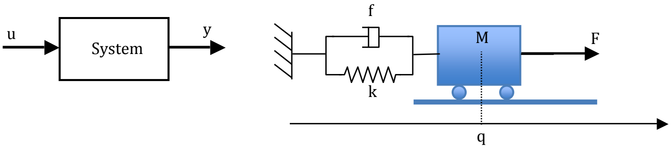


From the second Newton's law:

$$M\ddot{q} = -kq - f\dot{q} + F$$

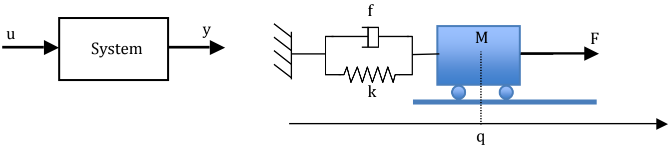
An alternative representation consist in choosing the energy variables (extensives variables) as state variables *i.e* $(q, p = M\dot{q})$

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -f \end{pmatrix}}_{J-R} \underbrace{\begin{pmatrix} kq \\ \dot{q} \end{pmatrix}}_{\partial_q H} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} F$$

with $H(q,p) = \frac{1}{2}\left(kq^2 + \frac{1}{M}p^2\right)$

Finite dimensional example ...

Let consider the mass spring damper system:



From the second Newton's law:

$$M\ddot{q} = -kq - f\dot{q} + F$$

Defining y s.t.:

$$\begin{cases} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -f \end{pmatrix} \begin{pmatrix} \partial_q H(q,p) \\ \partial_p H(q,p) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F \\ y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_q H(q,p) \\ \partial_p H(q,p) \end{pmatrix} \end{cases}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}^T \frac{dx}{dt} = \frac{\partial H}{\partial x}^T (J - R) \frac{\partial H}{\partial x} + \frac{\partial H}{\partial x}^T Bu \leq y^T u$$

Outline

Back to the modeling

The previous model can be written from the interconnection of a subset of basic mechanical elements:

- A moving inertia.
- A spring.
- A damper.
- A source and some interconnection relations.

Structured modeling

Each element is characterized by a set of power conjugated variables, the flow variables and the effort variables (intensive variables). The state variable is derive from the time integration of the flow variables (extensive variables). When the component is purely dissipative there is no associated state variable.

Moving inertia

Set of power conjugated variables:

- Flow variable: Force

$$\frac{dp}{dt} = F$$

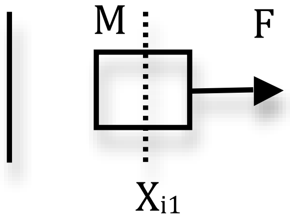
- Effort variable: velocity

$$v_i(p) = \frac{1}{m}p$$

State variable and energy

- Extensive variable: kinetic momentum p
- Energy

$$E(p) = \frac{1}{2}\frac{p^2}{m}$$



Spring

Set of power conjugated variables:

- Flow variable: Velocity

$$\frac{dq}{dt} = v_s$$

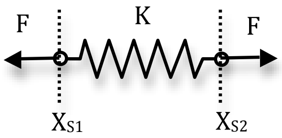
- Effort variable: Force

$$F(q) = kq$$

State variable and energy

- Extensive variable: position q
- Energy

$$E(q) = \frac{1}{2}kq^2$$



Damper

Set of power conjugated variables:

- Flow variable: Velocity

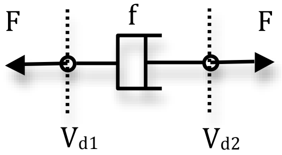
$$v_d$$

- Effort variable: Force

$$F = fv_d$$

Dissipated (co)energy:

$$D(v_d) = fv_d^2$$



Transformers and sources

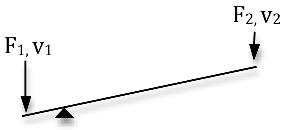
Power preserving transformations:

- Relation between velocities

$$v_2 = n v_1$$

- Relation between forces

$$F_1 = n F_2$$



There exist different kind of sources

- Velocity sources

$$v(t) = v_s(t)$$

- Forces sources,

$$F(t) = F_s(t)$$

Interconnection

When two or more mechanical subsystems are interconnected one can write at the interconnection point:

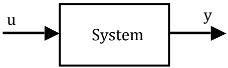
- Equality of the velocities,

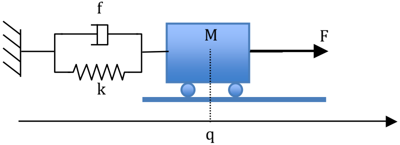
$$v_d = v_s = v_i = v$$

- Forces balance,

$$F_i + F_s + F_d = F$$

Back to the example





- Equality of the velocities,

$$v_d = v_s = v_i = v$$

- Forces balance,

$$F_i + F_s + F_d = F$$

States variables: $(x\ p)^T$

$$\begin{aligned}\frac{dq}{dt} &= v_s = v\\ \frac{dp}{dt} &= F - F_s - F_d = F - kq - fv\end{aligned}$$

$$\begin{pmatrix} \frac{dq}{dt} \\ \frac{dp}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -f \end{pmatrix} \begin{pmatrix} kq \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F$$

Port based modeling of physical systems

Port Hamiltonian formulation

The idea is to generalize what has been proposed for mechanical and electrical systems to other class of systems.

Why ?

- We have pointed out some common properties: storage, dissipation and transformation.
- Engineering systems are a combination of subsystems related to possible different physical domains and interconnection has to be consistent. See for example Adsorption processes.
- Decomposition in basic elements helps in modeling of complex dynamic systems (coming from different areas).
- Modeling is attached to the notion of graph.

Port based modeling of physical systems

Much more fundamental reasons:

- Central role of the energy can be used for control purposes. Lyapunov based approaches.
- More information are taken into account in the model through symmetries.
- The model is a knowledge based model that takes the non linearities and the distributed aspects into account.

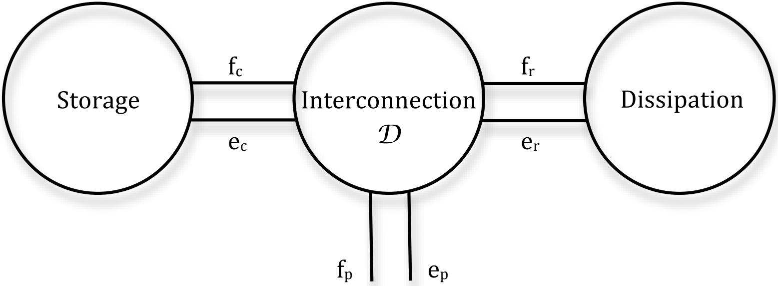
Generalized Bond Graph

Decomposition in basic elements is linked to Generalized Bond Graph (Paytner, Breedveld):

- Systems are decomposed in elements with specific energetic behavior: storage, dissipation and transformation.
- Each element is characterized by a pair of power conjugated variables: the flow variables $f \in \mathcal{F}$ and the effort variables $e \in \mathcal{E}$. The associated power port is given by:

$$P = f^T e$$

Port based modeling



$\mathcal{F} = \mathcal{F}_c \times \mathcal{F}_R \times \mathcal{F}_p$ and $\mathcal{E} = \mathcal{E}_c \times \mathcal{E}_R \times \mathcal{E}_p$

Dynamic relations : storage element

In case of storage elements:

- The state variable x is the extensive variable of Thermodynamics. It is linked to the flow variables through the balance equation:

$$\frac{dx}{dt} = -f_c$$

- The effort variable is linked to the energy variable through the relation:

$$e_c = e_c(x) = \frac{dE}{dx}$$

- The Energy balance is given by

$$\frac{dE}{dt} = \left(\frac{dE}{dx}\right)^T \left(\frac{dx}{dt}\right) = -e_c^T f_c$$

Dissipation

In the case of dissipation:

$$e_r = -e(f); \;\; f = f_r$$

or

$$f_r = -f(e); \;\; e = e_r$$

Such that

$$e^T f(e) \geq 0, \;\; e(f)^T f \geq 0$$

Examples:

$$u = Ri, \; D = u^T i = Ri^2$$

$$F = f\dot{x}, \; D = \dot{x}F = f\dot{x}^2$$

Then

$$e_R^T f_R \leq 0$$

Interconnexion

- 1 Junction (flow junction):
  - Equality of effort variables
  - Balance on the flow variables

  Example: Kirchhoff's voltage law
- 0 Junction (flow junction):
  - Equality of flow variables
  - Balance on the effort variables

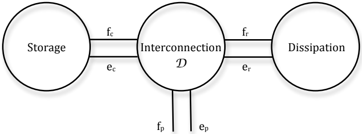
  Example: Kirchhoff's current law
- Ideal transformer "TF":

$$\begin{pmatrix} e_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ e_2 \end{pmatrix}, \quad e_1^T f_1 = e_2^T f_2$$

- Ideal gyrator "TF":

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad e_1^T f_1 = e_2^T f_2$$

Interconnection structure and power balance



The power balance is given by:

$$e_c^T f_c + e_R^T f_R^T + e_p^T f_p = 0$$

And

$$\frac{dE}{dt} = \left(\frac{dE}{dx}\right)^T \frac{dx}{dt} = -e_c^T f_c = e_R^T f_R^T + e_p^T f_p$$

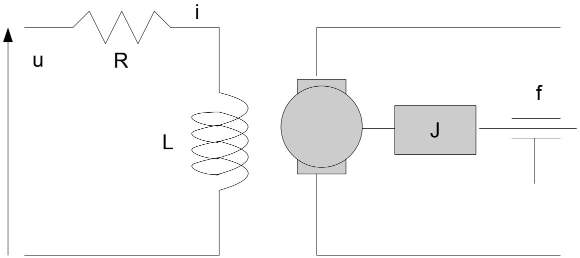
and then

$$E(t) = E(0) + \underbrace{\int_t e_R^T f_R^T dt}_{dissipated\ energy} + \underbrace{\int_t e_p^T f_p dt}_{exchanged\ energy}$$

Physical domain	flow $f \in \mathcal{F}$	effort $e \in \mathcal{E}$	state
potential translation	velocity	force	displacement
kinetic translation	force	velocity	momentum
potential rotation	angular velocity	torque	angle
kinetic rotation	torque	angular velocity	angular momentum
electric	current	voltage	charge
magnetic	voltage	current	flux linkage
potential hydraulic	volume flow	pressure	volume
kinetic hydraulic	pressure	volume flow	flow momentum
chemical	molar flow	chemical potential	number of moles
thermal	entropy flow	temperature	entropy

Exercice

Propose a port Hamiltonian model of the DC motor



Dirac structures and Port Hamiltonian systems

To summarize, the overall system is defined from pairs of flow variables, effort variables and state variables x. They are made up with:

- Energy storing elements:

$$f_c = -\frac{dx}{dt}, \; e_c = \frac{\partial E}{\partial x}$$

- Power dissipating elements

$$R(f_R, e_R) = 0, \; e_R^T f_R \geq 0$$

- Power preserving transformers, gyrators.
- Power preserving junctions.

$\Rightarrow$ Interconnexion structure = Dirac structure

Geometric structure

Dirac structure

A constant Dirac structure on a finite dimensional space $\mathcal{V}$ is subspace

$$\mathcal{D} \subset \mathcal{V} \times \mathcal{V}^*$$

such that

1. $e^T f = 0$ for all $(f, e) \in \mathcal{D}$
2. $dim\mathcal{D} = dim\mathcal{V}$

For any skew-symmetric map $J : \mathcal{V}^* \rightarrow \mathcal{V}$ its graph $\{(f, e) \in \mathcal{V} \times \mathcal{V}^* | f = Je\}$ is a Dirac structure.

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Geometric structure

Dirac structure 2

A constant Dirac structure on a finite dimensional space $\mathcal{V}$ is subspace

$$\mathcal{D} \subset \mathcal{V} \times \mathcal{V}^*$$

such that

$$\mathcal{D} = \mathcal{D}^{\perp}$$

where $\perp$ denotes orthogonal complement with respect to the bilinear form $\ll, \gg$ defined as:

$$\ll (f_1, e_1), (f_2, e_2) \gg = \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle$$

with $\langle e | f \rangle = e^T f$ the natural power product.

Geometric structure

Port Hamiltonian system

The dynamical system defined by DAEs such that:

$$(f_c, e_c, f_p, e_P) \in \mathcal{D}, \;\; t \in \mathbb{R}$$

with $f_c = \frac{\partial E}{\partial} e_c = \frac{\partial E}{\partial}$ is called port Hamiltonian system.

Outline

Infinite dimensional case

In what follows we focus on boundary controlled systems. In the general case, port Hamiltonian systems have been extended to distributed parameter systems by the use of differential geometry:

- Energy variables α_p and α_q are p and q differential forms defined on an n-dimensional manifold Z (with boundary ∂Z).
- $H := \int_Z \mathcal{H} \in \mathbb{R}$
- Port Hamiltonian system is defined by:

$$\begin{pmatrix} -\frac{\partial \alpha_p}{\partial t} \\ -\frac{\partial \alpha_q}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & (-1)^r d \\ d & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta p} \\ \frac{\delta H}{\delta q} \end{pmatrix}$$

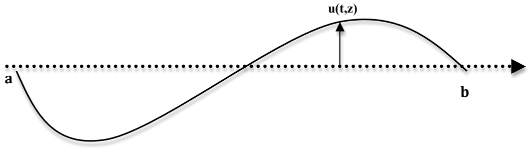
$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(-1)^{n-q} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta p}|_\partial \\ \frac{\delta H}{\delta q}|_\partial \end{pmatrix}$$

The main advantage of such formulation is that it is not depending on coordinates, applicable for nD systems.

In order to apply some functional analysis tools we focus on the 1D linear case.

Example 1 : the vibrating string

Let consider a string of length $[a, b]$:



The classical modelling is based on the wave equation : Newton's law + Hooke's law (restoring force proportional to the deformation)

$$\frac{\partial^2 u(z,t)}{\partial t^2} = \frac{1}{\mu(z)} \frac{\partial}{\partial z} \left(T(z) \frac{\partial u(z,t)}{\partial z} \right)$$

The structure of the model is not apparent. How to choose the boundary conditions ???

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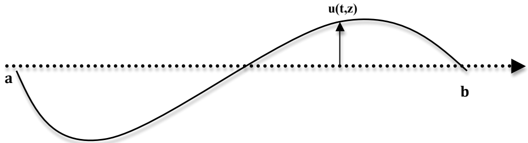

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$$\frac{\partial^2 u(z,t)}{\partial t^2} = \frac{1}{\mu(z)} \frac{\partial}{\partial z}\left(T(z) \frac{\partial u(z,t)}{\partial z} \right)$$

The structure of the model is not apparent. How to choose the boundary conditions ???

Usually: $x = \begin{bmatrix} u \\ \dot{u} \end{bmatrix} \rightarrow \begin{bmatrix} \dot{u} \\ \ddot{u} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{\mu(z)} \frac{\partial}{\partial z}\left(T(z) \frac{\partial .}{\partial z} \right) & 0 \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \end{bmatrix}$ first order diff equation in time

Vibrating string

Let choose as state variables the energy variables:

- the strain $\varepsilon = \frac{\partial u(z,t)}{\partial z}$
- the elastic momentum $p = \mu(z)v(z,t)$

The **total energy** is given by : $H(\varepsilon, p) = U(\varepsilon) + K(p)$

- $U(\varepsilon)$ is the **elastic potential energy**:

$$U(\varepsilon) = \int_a^b \frac{1}{2} T(z) \left(\frac{\partial u(z,t)}{\partial z}\right)^2 = \int_a^b \frac{1}{2} T\varepsilon(z,t)^2$$

  where $T(z)$ denotes the elastic modulus.

- $K(v)$ is the **kinetic energy**:

$$K(p) = \int_a^b \frac{1}{2}\mu(z)v(z,t)^2 = \int_a^b \frac{1}{2}\frac{1}{\mu(z)}p^2(z,t)$$

  where $\mu(z)$ denotes the string mass.

Example 1 : the vibrating string

From the conservation laws:

$$\frac{\partial}{\partial t}\begin{pmatrix} \varepsilon \\ p \end{pmatrix} + \frac{\partial}{\partial z}\begin{pmatrix} \mathcal{N}_\varepsilon \\ \mathcal{N}_p \end{pmatrix} = 0$$

The vector of fluxes β may be expressed in term of the generating forces :

$$\begin{pmatrix} \mathcal{N}_\varepsilon \\ \mathcal{N}_p \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}_{\substack{\text{canonical} \\ \text{interdomain coupling}}} \underbrace{\begin{pmatrix} \frac{\delta H}{\delta \varepsilon} \\ \frac{\delta H}{\delta p} \end{pmatrix}}_{\substack{\text{generating} \\ \text{forces}}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \sigma(z,t) \\ v(z,t) \end{pmatrix}$$

where $v(z,t)$ is the velocity and $\sigma(z,t) = T(z)\varepsilon(z,t)$ the stress. Consequently

$$\frac{\partial}{\partial t}\begin{pmatrix} \varepsilon \\ p \end{pmatrix} = -\frac{\partial}{\partial z}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\delta H}{\delta \varepsilon} \\ \frac{\delta H}{\delta p} \end{pmatrix}$$

Example 1 : the vibrating string

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where $v(z,t)$ is the velocity and $\sigma(z,t) = T(z)\varepsilon(z,t)$ the stress.

PDEs:

$$\frac{\partial}{\partial t}\begin{pmatrix} \varepsilon \\ p \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix}\begin{pmatrix} \frac{\delta H}{\delta \varepsilon} \\ \frac{\delta H}{\delta p} \end{pmatrix} \Leftrightarrow \frac{\partial^2 u(z,t)}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 u(z,t)}{\partial z^2} \text{ if } c = cte$$

+BC

Example 1: the vibrating string

Underlying structure:

$$\underbrace{\frac{\partial}{\partial t}\begin{pmatrix}\varepsilon\\ p\end{pmatrix}}_{f} = \underbrace{\begin{pmatrix}0 & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial z} & 0\end{pmatrix}}_{\substack{\mathcal{J} = \text{matrix}\\ \text{differential operator}}} \underbrace{\begin{pmatrix}T(z) & 0\\ 0 & \frac{1}{\mu(z)}\end{pmatrix}\begin{pmatrix}\varepsilon\\ p\end{pmatrix}}_{\substack{e = \text{driving}\\ \text{force}}}$$

Hamiltonian operator $\mathcal{J}$ is **skew-symmetric only for function with compact domain strictly** in Z :

$$\int_a^b \begin{pmatrix} e_1 & e_2 \end{pmatrix} \mathcal{J} \begin{pmatrix} e_1' \\ e_2' \end{pmatrix} + \begin{pmatrix} e_1' & e_2' \end{pmatrix} \mathcal{J} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \left[e_1 e_2' + e_2 e_1' \right]_a^b$$

Power balance equation :

$$\begin{aligned} \frac{d}{dt} H(\varepsilon, p) &= \int_a^b \left(\frac{\delta \mathcal{H}}{\delta \varepsilon} \frac{\partial \varepsilon}{\partial t} + \frac{\delta \mathcal{H}}{\delta p} \frac{\partial p}{\partial t} \right) dz \\ &= \int_a^b \left(\frac{\delta \mathcal{H}}{\delta \varepsilon} \frac{\partial}{\partial z} \frac{\delta \mathcal{H}}{\delta p} + \frac{\delta \mathcal{H}}{\delta p} \frac{\partial}{\partial z} \frac{\delta H}{\delta \varepsilon} \right) dz = \left[\frac{\delta \mathcal{H}}{\delta \varepsilon} \frac{\delta \mathcal{H}}{\delta p} \right]_a^b \end{aligned}$$

If driving forces are zero at the boundary, the total energy is conserved, else there is a **flow of power at the boundary**. Define two **port boundary variables** as follows :

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \frac{\delta H}{\delta \varepsilon} \\ \frac{\delta H}{\delta p} \end{pmatrix} \Big|_{a,b}$$

Example 1: the vibrating string

The linear space $\mathcal{D} \ni (f_1, f_2, e_1, e_2, f_\partial, e_\partial)$

- $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$
- $\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} |_{a,b}$

defines a **Dirac structure:**$\mathcal{D} = \mathcal{D}^\perp$ with respect to the pairing :

$$\int_a^b e_1 f_1 dz + \int_a^b e_2 f_2 dz - f_\partial^T e_\partial$$

Port Hamiltonian system

$$\left(\frac{\partial}{\partial t}\alpha, \frac{\delta H}{\delta \alpha}, f_\partial, e_\partial \right) \in \mathcal{D}$$

Example 1: the vibrating string

The linear space $\mathcal{D} \ni (f_1, f_2, e_1, e_2, f_\partial, e_\partial)$

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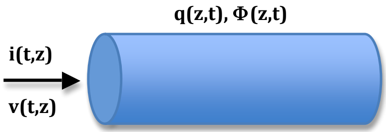
Port Hamiltonian system

$$\left(\frac{\partial}{\partial t}\alpha, \frac{\delta H}{\delta \alpha}, f_\partial, e_\partial \right) \in \mathcal{D}$$

$$\frac{dH}{dt} = f_\partial^T e_\partial$$

The lossless transmission line



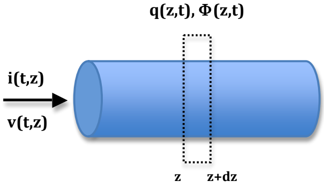
Consider an ideal lossless transmission line with spatial domain $Z = [a, b] \subset \mathbb{R}$. There are two conserved variables:

- the charge on the interval Z: $Q_{(a,b)}(t) = \int_a^b q(t,z)dz$ where $q(t,z)$ denotes the charge density,
- the flux on the interval Z : $\Phi_{(a,b)}(t) = \int_a^b \phi(t,z)dz$ where $\phi(t,z)$ denotes the flux density.

Then $q(t,z)$ and $\phi(t,z)$ are the two extensive variables that will be used for the modeling.

The lossless transmission line

Let consider an infinitesimal piece of the transmission line:



One can write the following 2 *conservation laws* in differential form:

- conservation of charge:

$$\frac{d}{dt} q(t,z) = -\frac{\partial}{\partial z} i(t,z) \tag{1}$$

  where $i(t,z)$ denotes the current at z
- conservation of flux:

$$\frac{d}{dt} \phi(t,z) = -\frac{\partial}{\partial z} v(t,z) \tag{2}$$

  where $v(t,z)$ denotes the voltage at z

The lossless transmission line

The electromagnetic properties gives the two *closure equations* for the functions $i(t,z)$ and $v(t,z)$:

- the current is given by:

$$i(t,z) = \frac{\phi(t,z)}{L(z)} \tag{3}$$

  where $L(z)$ denotes the distributed inductance of the line
- the voltage is given by:

$$v(t,z) = \frac{q(t,z)}{C(z)} \tag{4}$$

  where $C(z)$ denotes the distributed capacitance of the line

and the total electromagnetic energy of the system can be written:

$$H = \int_a^b \mathcal{H}(q,\phi)dz = \frac{1}{2}\int_a^b \left(\frac{q^2(t,z)}{C(z)} + \frac{\phi^2(t,z)}{L(z)} \right) dz \tag{5}$$

The lossless transmission line

The preceding closure equations may be written in matrix form:

$$\begin{pmatrix} i(t,z) \\ v(t,z) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H(q,\phi)}{\delta q} \\ \frac{\delta H(q,\phi)}{\delta \phi} \end{pmatrix} \tag{6}$$

where $H(q,\phi) = \int_a^b \mathcal{H}(q,\phi)dz$ and $\mathcal{H}(q,\phi)$ denotes the electromagnetic energy density:

$$\mathcal{H}(q,\phi) = \frac{1}{2}\left(\frac{q^2(t,z)}{C(z)} + \frac{\phi^2(t,z)}{L(z)}\right) \tag{7}$$

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The lossless transmission line

Combining the conservation laws and the closure equations one obtains the Hamiltonian system:

$$\frac{\partial}{\partial t}\begin{pmatrix} q \\ \phi \end{pmatrix} = \mathcal{J}\begin{pmatrix} \frac{\delta H(q,\phi)}{\delta q} \\ \frac{\delta H(q,\phi)}{\delta \phi} \end{pmatrix} \tag{8}$$

where $\mathcal{J}$ is a formally skew symmetric differential operator defined as:

$$\mathcal{J} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{pmatrix} \tag{9}$$

Take two effort densities $e(t,z)$ and $e'(t,z)$ and compute their bracket with respect to $\mathcal{J}$:

$$\begin{aligned}
\int_a^b (e_q, e_\phi)\, \mathcal{J} \begin{pmatrix} e'_q \\ e'_\phi \end{pmatrix} dz &= -\int_a^b \left(e_q \frac{\partial}{\partial z}\, e'_\phi + e_\phi\, \frac{\partial}{\partial z} e'_q \right) dz \\
&= \int_a^b \left(e'_q\, \frac{\partial}{\partial z} e_\phi + e'_\phi\, \frac{\partial}{\partial z} e_q \right) dz - \left[e'_q\, e_\phi + e'_\phi\, e_q \right]_0^1 \\
&= -\int_a^b \left(e'_q, e'_\phi \right) \mathcal{J} \begin{pmatrix} e_q \\ e_\phi \end{pmatrix} dz - \left[e'_q\, e_\phi + e'_\phi\, e_q \right]_a^b
\end{aligned}$$

We can see that it is skew symmetric for *densities that vanish at the boundary*!

The lossless transmission line

The resulting port-Hamiltonian system is given by the telegraph equations

$$\begin{pmatrix} \frac{\partial Q}{\partial t} \\ \frac{\partial \varphi}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} v \\ i \end{pmatrix}$$

together with the boundary variables

$$\begin{array}{rclrcl} f_\partial^a(t) & = & v(t,0), & f_\partial^b(t) & = & v(t,1) \\ e_\partial^a(t) & = & i(t,0), & e_\partial^b(t) & = & -i(t,1) \end{array}$$

The resulting energy-balance is

$$\frac{dH}{dt} = f_\partial^T e_\partial = -i(t,1)v(t,1) + i(t,0)v(t,0),$$

Outline

Considered class of systems

We first consider lossless systems defined on 1-D spatial domain $[a, b]$ by the PDE:

$$\frac{dx}{dt}(t,z) = \mathcal{J}\mathcal{L}(z)x(t,z),\ x(0,z) = x_0(z),$$

where $\mathcal{J}$ is a formally skew symmetric differential operator and $\mathcal{L}(z)$ a coercive operator.

For example

$$\underbrace{\frac{\partial}{\partial t}\begin{pmatrix}\epsilon \\ p\end{pmatrix}}_{f} = \underbrace{\begin{pmatrix}0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0\end{pmatrix}}_{\mathcal{J}} \underbrace{\begin{pmatrix}T(z) & 0 \\ 0 & \frac{1}{\mu(z)}\end{pmatrix}\begin{pmatrix}\epsilon \\ p\end{pmatrix}}_{e = \mathcal{L}(z)}$$

$$\Leftrightarrow$$

$$f = \mathcal{J}e$$

Outline

Bond space

The system is defined by :

$$f = \mathcal{J} e$$

and we first consider homogeneous boundary conditions.

• Let the space of flow variables, $\mathcal{F}$, and the space of effort variables,$\mathcal{E}$, be real Hilbert spaces.

• Define the space of bond variables as $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ endowed by the natural inner product

$$\left\langle b^1, b^2 \right\rangle = \left\langle f^1, f^2 \right\rangle_{\mathcal{F}} + \left\langle e^1, e^2 \right\rangle_{\mathcal{E}}, \quad b^1 = \left(f^1, e^1\right), b^2 = \left(f^2, e^2\right) \in \mathcal{B}.$$

In order to define a Dirac structure, let us moreover endow the bond space $\mathcal{B}$ with a *canonical symmetric pairing*, i.e., a bilinear form defined as follows:

$$\left\langle b^1, b^2 \right\rangle_+ = \left\langle f^1, r_{\mathcal{E},\mathcal{F}} e^2 \right\rangle_{\mathcal{F}} + \left\langle e^1, r_{\mathcal{F},\mathcal{E}} f^2 \right\rangle_{\mathcal{E}}, \; b^1 = \left(f^1, e^1\right), b^2 = \left(f^2, e^2\right) \in \mathcal{B}. \tag{10}$$

Dirac structure

Denote by $\mathcal{D}^{\perp}$ the orthogonal subspace to $\mathcal{D}$ with respect to the symmetric pairing:

$$\mathcal{D}^{\perp} = \left\{ b \in \mathcal{B} | \left\langle b, b' \right\rangle_{+} = 0 \text{ for all } b' \in \mathcal{D} \right\}. \tag{11}$$

Definition [?] :

A Dirac structure $\mathcal{D}$ on the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ is a subspace of $\mathcal{B}$ which is maximally isotropic with respect to the canonical symmetric pairing, i.e.,

$$\mathcal{D}^{\perp} = \mathcal{D}. \tag{12}$$

$$\begin{pmatrix} f \\ e \end{pmatrix} \in \mathcal{D} \Longleftrightarrow \textbf{Power conservation}$$

Port Hamiltonian Systems

PHS $\rightsquigarrow$ Definition based on Dirac structure and Hamiltonian function (total energy of the system).

Definition :

Let $\mathcal{B} = \mathcal{E} \times \mathcal{F}$ be the bound space defined above and consider the Dirac structure $\mathcal{D}$ and the Hamilonian function $\mathcal{H}(x)$ with x the energy variables. Define the flow variables, $f \in \mathcal{F}$ as the time variation of the energy variables and the effort variables $e \in \mathcal{E}$ as the variational derivative of $\mathcal{H}(x)$. The system

$$(f, e) = \left(\frac{\partial x}{\partial t}, \frac{\delta \mathcal{H}}{\delta x} \right) \in \mathcal{D}$$

is a Port Hamiltonian system with total energy $\mathcal{H}(x)$

Let us now see how to include non homogeneous boundary conditions:

$$\frac{d\mathcal{H}}{dt} = \int_a^b \frac{\delta \mathcal{H}}{\delta x}^T \frac{dx}{dt} dz = \int_a^b \frac{\delta \mathcal{H}}{\delta x}^T \mathcal{J} \frac{\delta \mathcal{H}}{\delta x} dz = \left[\Xi \left(\frac{\delta \mathcal{H}}{\delta x} \right) \right]_a^b$$

$$\langle f, e \rangle = f_\partial^T e_\partial$$

Extension to non homogeneous BC

$\rightsquigarrow$ We define the symmetric pairing (not depending on $\mathcal{J}$) and the port variables associated with $\mathcal{J}$. ([**?**])
Let $\mathcal{F} = \mathcal{E} = L^2((a,b);\mathbb{R}^n) \times \mathbb{R}^{nN}$ and define $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ with the following canonical symmetric pairing :

$$\begin{aligned}\left\langle (f^1, f^1_\partial, e^1, e^1_\partial)\ (f^2, f^2_\partial, e^2, e^2_\partial) \right\rangle_+ \\ = \langle e^1, f^2 \rangle_{L^2} + \langle e^2, f^1 \rangle_{L^2} - \langle e^1_\partial, f^2_\partial \rangle - \langle e^2_\partial, f^1_\partial \rangle,\end{aligned}$$

Definition :

Let $\mathcal{B} = \mathcal{E} \times \mathcal{F}$ be the bound space defined above and consider the Dirac structure $\mathcal{D}$ and the Hamilonian function $\mathcal{H}(x)$ with x the energy variables. Define the flow variables, $f \in \mathcal{F}$ as the time variation of the energy variables and its extension to the boundary and the effort variables $e \in \mathcal{E}$ as the variational derivative of $\mathcal{H}(x)$ and its extension to the boundary. The system

$$((f, f_\partial), (e, e_\partial)) = \left(\left(\frac{\partial x}{\partial t}, f_\partial \right), \left(\frac{\delta \mathcal{H}}{\delta x}, e_\partial \right) \right) \in \mathcal{D}_{\mathcal{J}}$$

is a Port Hamiltonian system with total energy $\mathcal{H}(x)$

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Parametrization of 1D differential operators

Parametrization ([**?**, **?**]):

$$\mathcal{J}e = \sum_{i=0}^{N} P(i)\frac{d^i e}{dz^i}(z) \qquad z \in [a,b]\,,$$

where $e \in H^N((a,b);\mathbb{R}^n)$ and $P(i)$, $i = 0,\dots,N$, is a $n \times n$ real matrix with P_N non singular and $P_i = P_i^T(-1)^{i+1}$. Let define

$$Q = \begin{pmatrix} P_1 & P_2 & \cdots & P_N \\ -P_2 & -P_3 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ (-1)^{N-1}P_N & 0 & \cdots & 0 \end{pmatrix}$$

Back to the **Vibrating string**

$$\underbrace{\frac{\partial}{\partial t}\begin{pmatrix} \epsilon \\ p \end{pmatrix}}_{f} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{P_1}\frac{\partial}{\partial z}\underbrace{\begin{pmatrix} T(z) & 0 \\ 0 & \frac{1}{\mu(z)} \end{pmatrix}\begin{pmatrix} \epsilon \\ p \end{pmatrix}}_{e}, \; Q = P_1$$

Port Variables

Definition :

The port variables $(e_\partial, f_\partial) \in \mathbb{R}^{nN}$ associated with $\mathcal{J}$ are defined by :

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} e(b) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(b) \\ e(a) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(a) \end{pmatrix}, \quad R_{\text{ext}} = \frac{U}{\sqrt{2}} \begin{pmatrix} Q & -Q \\ I & I \end{pmatrix}$$

where U is a unitary matrix such that:

$$U^T \Sigma U = \Sigma \text{ with } \Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

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Port Variables

Back to the **Vibrating string**

$$\underbrace{\frac{\partial}{\partial t}\begin{pmatrix} \epsilon \\ p \end{pmatrix}}_{f} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{P_1} \frac{\partial}{\partial z} \underbrace{\begin{pmatrix} T(z)\epsilon \\ \frac{1}{\mu(z)}p \end{pmatrix}}_{e}, Q = P_1$$

The boundary port variables are defined by:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}\begin{pmatrix} e(b) \\ e(a) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{p(b)}{\mu(b)} - \frac{p(a)}{\mu(a)} \\ T(b)\epsilon(b) - T(a)\epsilon(a) \\ T(a)\epsilon(a) + T(b)\epsilon(b) \\ \frac{p(a)}{\mu(a)} + \frac{p(b)}{\mu(b)} \end{pmatrix}$$

Dirac structure

Theorem :

The subspace $\mathcal{D}_{\mathcal{J}}$ of $\mathcal{B}$ defined as

$$\mathcal{D}_{\mathcal{J}} = \left\{ \begin{pmatrix} f \\ f_\partial \\ e \\ e_\partial \end{pmatrix} \middle| \; e \in H^N((a,b);\mathbb{R}^n),\, \mathcal{J}e = f,\, \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} e(b) \\ \vdots \\ \partial_z^{N-1} e(a) \end{pmatrix} \right\}$$

is a Dirac structure, that means that $\mathcal{D} = \mathcal{D}^{\perp}$.

Outline

Extension to systems with dissipation

Let us extend the previous results to systems defined by:

$$\frac{dx}{dt}(t,z) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}(z)x(t,z), \;\; x(0,z) = x_0(z),$$

$$\Updownarrow$$

$$\begin{pmatrix} f \\ f_p \end{pmatrix} = \mathcal{J}_e \begin{pmatrix} e \\ e_p \end{pmatrix} = \begin{pmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{pmatrix} \begin{pmatrix} e \\ e_p \end{pmatrix}$$

with $e_p = Sf_p$ where S is a coercive operator

$\begin{pmatrix} f \\ f_p \end{pmatrix} \in \mathcal{F}, \begin{pmatrix} e \\ e_p \end{pmatrix} \in \mathcal{E}$ and $\mathcal{E} = \mathcal{F} = L_2((a,b),\mathbb{R}^n) \times L_2((a,b),\mathbb{R}^n)$

Covers models of: beams, wave, plates, (with or without damping) and also systems of diffusion/convection, chemical reactors ...

A simple example: the heat equation

1D Heat conduction is usually known on the following form:

$$\frac{\partial T(z,t)}{\partial t} = D\frac{\partial^2}{\partial z^2}\left(T(z,t)\right)$$

but is in fact derived from balance equation on the energy *i.e:*

$$\frac{\partial\left(c_v T(z,t)\right)}{\partial t} = -\frac{\partial}{\partial z}\left(-\lambda\frac{\partial T(z,t)}{\partial z}\right)$$

with c_v constant and positive. This equation can be written:

$$\begin{pmatrix} \frac{\partial}{\partial t}T(z,t) \\ f_p \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix}\begin{pmatrix} T(z,t) \\ e_p \end{pmatrix} \quad \textit{with} \;\; e_p = \frac{\lambda}{c_v} f_p$$

In this case:

$$\mathcal{J} = 0,\; \mathcal{G}_R = \frac{\partial}{\partial z},\; S = \frac{\lambda}{c_v} > 0$$

Outline

Parametrization of the extended operator

$\mathcal{J}_e$ is formally skew symmetric and can be parametrized by:

$$\mathcal{J}_e \tilde{e} = \Sigma_1^N \tilde{P}_k \frac{\partial^k}{\partial z^k} \tilde{e} \text{ with } \tilde{P}_k = (-1)^{k+1} \tilde{P}_k^T$$

In this case $\tilde{P}_N$ can be not full rank and the bilinear product is defined on quotient space. The extended boundary port variables are defined by:

$$\begin{pmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{Q}_1 & -\tilde{Q}_1 \\ I & I \end{pmatrix} \begin{pmatrix} M_Q & 0 \\ 0 & M_Q \end{pmatrix} \begin{pmatrix} \tilde{e}(b) \\ \tilde{e}(a) \end{pmatrix}$$

M spanning the column of $\tilde{Q}$, $\tilde{Q}_1 = M^T \tilde{Q} M$ and $M_Q = (M^T M)^{-1} M^T$ with

$$\tilde{Q} = \begin{pmatrix} \tilde{P}_1 & \tilde{P}_2 & \cdots & \tilde{P}_N \\ -\tilde{P}_2 & -\tilde{P}_3 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ (-1)^{N-1} \tilde{P}_N & 0 & \cdots & 0 \end{pmatrix}$$

Back to the vibrating string

We consider now the vibrating string with structural damping (dissipation of the form $k_s \frac{\partial}{\partial z}\left(\frac{p}{\mu}\right)$ is given by a system of 2 conservation laws:

$$\frac{\partial}{\partial t}\begin{pmatrix} \varepsilon \\ p \end{pmatrix} = \frac{\partial}{\partial z}\begin{pmatrix} \frac{p}{\mu} \\ T\,\varepsilon + k_s \frac{\partial}{\partial z}\left(\frac{p}{\mu}\right) \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \left(\frac{\partial}{\partial z} k_s \frac{\partial}{\partial z}\right) \end{pmatrix}\begin{pmatrix} \frac{\delta H_0}{\delta \varepsilon} \\ \frac{\delta H_0}{\delta p} \end{pmatrix}$$

The extended Hamiltonian operator is:

$$\mathcal{J}_e = \begin{pmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} & 0 \\ \frac{\partial}{\partial z} & 0 & +\frac{\partial}{\partial z} \\ 0 & +\frac{\partial}{\partial z} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\frac{\partial}{\partial z}$$

and

$$S = k_s > 0$$

Boundary port variables

A matrix M spanning the columns of P_1 can be chosen as:

$$\widetilde{P}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$$

$$\text{then} \quad \widetilde{Q}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } M_Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \widetilde{e} = \begin{pmatrix} T\,\varepsilon + e_R \\ \mu^{-1} p \end{pmatrix}$$

It thus follows that the port-variables become:

$$\begin{pmatrix} \tilde{f}_\partial \\ \tilde{e}_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{Q}_1 & -\tilde{Q}_1 \\ I & I \end{pmatrix} \begin{pmatrix} \tilde{e}(b) \\ \tilde{e}(a) \end{pmatrix} = \begin{pmatrix} \frac{p}{\mu}(b) - \frac{p}{\mu}(a) \\ (T\,\varepsilon + e_R)(b) - (T\,\varepsilon + e_R)(a) \\ (T\,\varepsilon + e_R)(a) + (T\,\varepsilon + e_R)(b) \\ \frac{p}{\mu}(a) + \frac{p}{\mu}(b) \end{pmatrix}$$

Outline

C_0 Semi group

In finite dimension, linear systems can be described using first-order differential equation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

with solutions expressed through:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

The idea of semigroups ([**?**, **?**]) is to generalize the notion of e^{At} to abstract systems defined on Hilbert space by:

$$\dot{x}(z,t) = Ax(z,t),\ x(z,t) \in D(A),\ x(z,0) = x_0$$

In what follows the semigroup associated to the generator A is noted $T(t)$.

C_0 Semigroups

Definition of C_0 Semi group

Let X be a Hilbert space. $(T(t))_{t\geq 0}$ is called a strongly continuous semigroup (or C_0) semigroup if the following Holds:

1. For all $t \geq 0$, $T(t)$ is a bounded linear operator on X, *i.e.*, $T(t) \in \mathcal{L}(X)$;
2. $T(0) = I$;
3. $T(t+\tau) = T(t)T(\tau)$ for all $t, \tau \geq 0$;
4. For all $x_0 \in X$, we have that $\|T(t)x_0 - x_0\|_X$ converges to zero, when $t \downarrow 0$ *i.e.* $t \mapsto T(t)$ is strongly continuous at zero.

Even if it has been defined for infinite dimensional systems it can be used in $\mathbb{R}^n$. In this case $T(t) = e^{At}$. Properties can be checked using

$$T(t)x = \sum_{n=1}^{\infty} e^{\lambda_n t}\langle x, \phi_n\rangle \phi_n$$

C_0 Semi group

Properties of C_0 Semi group

A strongly continuous semigroup $(T(t))_{t\geq 0}$ on X has the following properties:

1. $\|T(t)\|$ is bounded on every finite sub-interval of $[0, \infty)$;
2. The mapping $t \mapsto T(t)$ is strongly continuous on the interval $[0, \infty)$;
3. For all $x \in X$ we have that $\frac{1}{t}\int_0^t T(s)xds \to x$ as $t \downarrow 0$;
4. If $\omega_0 = inf\left(\frac{1}{t}log\|T(t)\|\right)$ then $\omega_0 = lim\left(\frac{1}{t}log\|T(t)\|\right) < \infty$
5. For every $\omega > \omega_0$, $\exists\ M_\omega$ such that for every $t \geq 0$ we have $\|T(t)\| \leq M_\omega e^{\omega t}$.

Definition of infinitesimal generator

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space X. If the following limit exists

$$\lim_{t\downarrow 0}\frac{T(t)x_0 - x_0}{t} \Rightarrow x_0 \in D(A)$$

we define A the infinitesimal generator of the strongly continuous semigroup by $Ax_0 = \lim_{t\downarrow 0}\frac{T(t)x_0 - x_0}{t}$

C_0 Semi group

Theorem

Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on X with infinitesimal generator A. Then the following results hold:

1. For $x_0 \in D(A)$ and $t \geq 0$ we have $T(t)x_0 \in D(A)$;
2. $\frac{d}{dt}(T(t)x_0) = AT(t)x_0 = T(t)Ax_0$ for $x_0 \in D(A), t \geq 0$;
3. A is a closed linear operator;

It means that for $x_0 \in D(A)$ the function $x(t) = T(t)x_0$ is a solution of the abstract differential equation:

$$\dot{x}(t) = Ax(t), \; x(0) = x_0 \tag{13}$$

Definition

A differentiable function $x : [0, \infty) \to X$ is called classical solution of (13) if $\forall t \geq 0$ we have $x(t) \in D(A)$ and equation (13) is satisfied.

Lemma

Let A be the infinitesimal generator of C_0 semigroup $(T(t))_{t\geq 0}$. Then for every $x_0(t) \in D(A)$ the map $t \mapsto T(t)x_0$ is the unique classical solution of (13).

C_0 Semi group

Definition

Let $(T(t))_{t\geq 0}$ be a C_0 semigroup on X. Then $(T(t))_{t\geq 0}$ is called a contraction semigroup if $\|T(t)\| \leq 1$ and unitary semigroup if $\|T(t)\| = 1$ for every $t \geq 0$.

Definition

A linear operator $A : D(A) \subset X \to X$ is called dissipative if

$$Re\langle Ax, x\rangle \leq 0, \;\; x \in D(A)$$

Lumer-Phillips Theorem

Let A be a linear operator with domain $D(A)$ on X. Then A is the infinitesimal generator of a contraction semigroup $(T(t))_{t\geq 0}$ on X if and only if A is dissipative and $ran(I - A) = X$

Theorem

Let A be a linear, densely defined and closed operator on X. Then A is the infinitesimal generator of a contraction semigroup $(T(t))_{t\geq 0}$ on X if and only if A and A^* are dissipative.

Outline

Port Hamiltonian systems

Port Hamiltonian systems

Let W be a $n \times 2n$ real matrix. If W has full rank and satisfies $W\Sigma W^\top \geq 0$ ($W\Sigma W^\top = 0$), then the operator $\mathcal{A}x = P_1(\partial/\partial z)(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain

$$D(\mathcal{A}) = \left\{ \mathcal{L}x \in H^1(a, b; \mathbb{R}^n) \;\middle|\; \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix} \in \ker W \right\}$$

generates a contraction semigroup(unitary semigroup) on X.

Sketch of proof

We use the property

$$\langle e, \mathcal{J}e \rangle + \langle \mathcal{J}e, e \rangle = \begin{pmatrix} f_\partial^T & e_\partial^T \end{pmatrix}^T \Sigma \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}$$

to prove that with $D(\mathcal{A})$ with rank and positivity condition the operator and its adjoint are dissipative.

Outline

Boundary control systems

We are interested in abstract control systems of the form:

$$\begin{aligned} \dot{x} &= \mathscr{A}x(t), \quad x(0) = x_0, \\ \mathscr{B}x(t) &= u(t), \end{aligned} \tag{14}$$

Definition

The control system 14 is a boundary control system if the following hold:

1. The operator $A : D(A) \to X$ with $D(A) = D(\mathscr{A}) \cap \ker(\mathscr{B})$ and

   $$\mathscr{A}x = Ax \text{ for } x \in D(A)$$

   is the infinitesimal generator of a C_0 semigroup.
2. There exists an operator $B \in \mathcal{L}(U, X)$ such that for all $u \in U$ we have $Bu \in D(\mathscr{A})$, $\mathscr{A}B \in \mathcal{L}(U, X)$ and

   $$\mathscr{B}Bu = u, \;\; u \in U$$

Boundary controlled port Hamiltonian systems

Boundary controlled port Hamiltonian systems

Let W be a $n \times 2n$ real matrix. If W has full rank and satisfies $W \Sigma W^\top \geq 0$, then the system $\frac{\partial x}{\partial t} = \mathcal{A}x$ with $\mathcal{A}x = P_1(\partial/\partial z)(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain

$$D(\mathcal{A}) = \left\{ \mathcal{L}x \in H^1(a, b; \mathbb{R}^n) \;\middle|\; \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix} \in \ker W \right\}$$

and input

$$u(t) = W \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix}$$

is a Boundary Control System on X.

Sketch of proof

The operator $\mathcal{A}x = P_1(\partial/\partial z)(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain $D(\mathcal{A})$ generates a contraction semigroup on X. It remains to show that $\exists \mathscr{B}$ such that $\mathscr{B}Bu = u, \;\; u \in U$

Boundary controlled port Hamiltonian systems

Boundary controlled port Hamiltonian systems [?]

Let $\tilde{W}$ be a full rank matrix of size $n \times 2n$ with $\begin{bmatrix} W \\ \tilde{W} \end{bmatrix}$ invertible and let $P_{W,\tilde{W}}$ be given by

$$P_{W,\tilde{W}} = \left(\begin{bmatrix} W \\ \tilde{W} \end{bmatrix} \Sigma \begin{bmatrix} W \\ \tilde{W} \end{bmatrix}^\top \right)^{-1} = \begin{bmatrix} W\Sigma W^\top & W\Sigma \tilde{W}^\top \\ \tilde{W}\Sigma W^\top & \tilde{W}\Sigma \tilde{W}^\top \end{bmatrix}^{-1}.$$

Define the output of the system as the linear mapping $\mathcal{C} : \mathcal{L}^{-1} H^1(a,b;\mathbb{R}^n) \to \mathbb{R}^n$,

$$y = \mathcal{C}x(t) := \tilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix}.$$

Then for $u \in C^2(0,\infty;\mathbb{R}^k)$, $\mathcal{L}x(0) \in H^1(a,b;\mathbb{R}^n)$, and $u(0) = W \begin{bmatrix} f_{\partial,\mathcal{L}x}(0) \\ e_{\partial,\mathcal{L}x}(0) \end{bmatrix}$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} = \frac{1}{2} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^\top P_{W,\tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} - \langle G_0 \mathcal{L}x(t), \mathcal{L}x(t) \rangle \leq \frac{1}{2} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^\top P_{W,\tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}.$$

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Specific cases

Using

$$W = S(I + V, I - V)$$
$$\tilde{W} = \tilde{S}(I - V, -I - V)$$

We obtain for:

$$V = 0 \left\{ \begin{array}{ll} \dot{x}(t) & = \mathcal{J} x(t), \\ u(t) & = \frac{1}{2}\left(f_\partial(t) + e_\partial(t)\right) \\ y(t) & = \frac{1}{2}\left(f_\partial(t) - e_\partial(t)\right) \end{array} \right. \implies$$

Scattering system:
Boundary control system with the associated semigroup a contraction

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2.$$

$$V = I \left\{ \begin{array}{ll} \dot{x}(t) & = \mathcal{J} x(t) \\ u(t) & = f_\partial(t) \\ y(t) & = -e_\partial(t) \end{array} \right. \implies$$

Impedance passive system
Boundary control system with the associated semigroup unitary

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = u(t)^T y(t)$$

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Back to the vibrating string

The PDE is given by:

$$\underbrace{\frac{\partial}{\partial t}\begin{pmatrix} \epsilon \\ p \end{pmatrix}}_{f} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{P_1} \frac{\partial}{\partial z} \underbrace{\begin{pmatrix} T(z)\epsilon \\ \frac{1}{\mu(z)} p \end{pmatrix}}_{e}, Q = P_1$$

The boundary port variables are defined by:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} e(b) \\ e(a) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p(b)}{\mu(b)} - \frac{p(a)}{\mu(a)} \\ T(b)\epsilon(b) - T(a)\epsilon(a) \\ T(a)\epsilon(a) + T(b)\epsilon(b) \\ \frac{p(a)}{\mu(a)} + \frac{p(b)}{\mu(b)} \end{pmatrix}$$

By using the transformation

$$U = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \text{ s.t. } U^T \Sigma U = \Sigma$$

Back to the vibrating string

One can also choose:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} U \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} e(b) \\ e(a) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} T(b)\epsilon(b) \\ T(a)\epsilon(a) \\ \frac{p(b)}{\mu(b)} \\ -\frac{p(a)}{\mu(a)} \end{pmatrix}$$

Impedance passive case:

$$V = I \Rightarrow u = \frac{1}{\sqrt{2}} \begin{pmatrix} T(b)\epsilon(b) \\ T(a)\epsilon(a) \end{pmatrix} \text{ and } y = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{p(b)}{\mu(b)} \\ \frac{p(a)}{\mu(a)} \end{pmatrix}$$

$$\frac{dH(t)}{dt} = y(t)^T u(t)$$

Systems with dissipation

Let now consider systems with dissipation:

$$\frac{dx}{dt}(t,z) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}x(t,z),\; x(0,z) = x_0(z),$$

$$\Updownarrow$$

$$\begin{pmatrix} f \\ f_p \end{pmatrix} = \mathcal{J}_e \begin{pmatrix} e \\ e_p \end{pmatrix} = \begin{pmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{pmatrix} \begin{pmatrix} e \\ e_p \end{pmatrix}$$

with $e_p = Sf_p$ where S is a coercive operator

$\begin{pmatrix} f \\ f_p \end{pmatrix} \in \mathcal{F}, \begin{pmatrix} e \\ e_p \end{pmatrix} \in \mathcal{E}$ and $\mathcal{E} = \mathcal{F} = L_2((a,b),\mathbb{R}^n) \times L_2((a,b),\mathbb{R}^n)$

Systems with dissipation

From geometrical point of view:

$$f_e = \mathcal{J}_e e_e$$

$\mathcal{J}_e$ is formally skew symmetric and can be parametrized by:

$$\mathcal{J}_e \widetilde{e} = \Sigma_1^N \widetilde{P}_k \frac{\partial^k}{\partial z^k} \widetilde{e} \ \text{ with } \ \widetilde{P}_k = (-1)^{k+1} \widetilde{P}_k^T$$

In this case $\widetilde{P}_N$ is not full rank and the bilinear product is defined on quotient space. The extended boundary port variables are defined by:

$$\begin{pmatrix} \widetilde{f}_\partial \\ \widetilde{e}_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{Q}_1 & -\tilde{Q}_1 \\ I & I \end{pmatrix} \begin{pmatrix} M_Q & 0 \\ 0 & M_Q \end{pmatrix} \begin{pmatrix} \widetilde{e}(b) \\ \widetilde{e}(a) \end{pmatrix}$$

M spanning the column of $\widetilde{Q}$, $\widetilde{Q}_1 = M^T \widetilde{Q} M$ and $M_Q = (M^T M)^{-1} M^T$ with

$$\widetilde{Q} = \begin{pmatrix} \widetilde{P}_1 & \widetilde{P}_2 & \cdots & \widetilde{P}_N \\ -\widetilde{P}_2 & -\widetilde{P}_3 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ (-1)^{N-1} \widetilde{P}_N & 0 & \cdots & 0 \end{pmatrix}$$

BCS

Let W be full rank such that $W \Sigma W^T \geq 0$,

$$\frac{dx}{dt}(t) = \mathcal{J}_e x(t)$$

with input

$$u(t) = W \begin{pmatrix} \widetilde{f}_\partial \\ \widetilde{e}_\partial \end{pmatrix}$$

is a **boundary control system**. The operator $A_{\text{ext}} = \mathcal{J}_e$ with domain

$$D(A_{\text{ext}}) = \left\{ \begin{pmatrix} \widetilde{e} \\ \widetilde{e}_r \end{pmatrix} \in \begin{pmatrix} H^N((a,b), \mathbb{R}^n) \\ H^N((a,b), \mathbb{R}^n) \end{pmatrix} \;\middle|\; \begin{pmatrix} \widetilde{f}_\partial \\ \widetilde{e}_\partial \end{pmatrix} \in \ker W \right\}, \qquad (15)$$

generates a **contraction semigroup**.

Balance equation

Let $\tilde{W}$ be full rank such that $\begin{pmatrix} W \\ \tilde{W} \end{pmatrix}$ invertible. Let define $\mathcal{C} : H^N((a,b),\mathbb{R}^{2n}) \to \mathbb{R}^{2nN}$ ast,

$$\mathcal{C}x(t) := \tilde{W} \begin{pmatrix} f_{e,\partial}(t) \\ e_{e,\partial}(t) \end{pmatrix} \tag{16}$$

and the output as

$$y(t) = \mathcal{C}x(t), \tag{17}$$

then for $u \in C^2((0,\infty);\mathbb{R}^{2nN})$, $x(0) \in H^N((a,b),\mathbb{R}^{2n})$, and $\mathcal{B}x(0) = u(0)$:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \frac{1}{2} \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_{W,\tilde{W}} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \tag{18}$$

where

$$P_{W,\tilde{W}}^{-1} = \begin{pmatrix} W\Sigma W^T & W\Sigma\tilde{W}^T \\ \tilde{W}\Sigma W^T & \tilde{W}\Sigma\tilde{W}^T \end{pmatrix}. \tag{19}$$

Dissipative operator

Now the feedback is closed *i.e.*

$$f = \mathcal{J} e - \mathcal{G}_R S \mathcal{G}_R^* e,$$

The port variables become :

$$\begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} e(b) \\ (-S\mathcal{G}_R^* e)(b) \\ \vdots \\ \frac{d^{N-1} e}{dz^{N-1}}(b) \\ \frac{d^{N-1}(-S\mathcal{G}_R^* e)}{dz^{N-1}}(b) \\ e(a) \\ (-S\mathcal{G}_R^* e)(a) \\ \vdots \\ \frac{d^{N-1}(-S\mathcal{G}_R^* e)}{dz^{N-1}}(a) \end{pmatrix}, \tag{20}$$

Dissipative operator

Consider the operator

$$A = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$$

with domain

$$D(A) = \left\{ e \in H^N((a,b);\mathbb{R}^n) \;\middle|\; S\mathcal{G}_R^* e \in H^N((a,b);\mathbb{R}^n), \right. \tag{21}$$

$$\left. \begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} \in \ker W \right\}. \tag{22}$$

If W has full rank and satisfies $W\Sigma W^T \geq 0$, then A generates a contraction semigroup.

BCS

Let W be full rank and satisfies $W\Sigma W^T \geq 0$, then

$$\frac{dx}{dt}(t) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)x(t) \tag{23}$$

with input

$$u(t) = \mathcal{B}x(t) = W \begin{pmatrix} g_{f,\partial}(t) \\ g_{e,\partial}(t) \end{pmatrix} \tag{24}$$

is a boundary control system. Furthermore, the operator $A = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$ with domain

$$D(A) = \left\{ e \in H^N((a,b);\mathbb{R}^n) \,\middle|\, S\mathcal{G}_R^* e \in H^N((a,b);\mathbb{R}^n), \right. \tag{25}$$

$$\left. \begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} \in \ker W \right\}. \tag{26}$$

generates a contraction semigroup.

Dissipative operator

Let $\tilde{W}$ be full rank such that $\begin{pmatrix} W \\ \tilde{W} \end{pmatrix}$ invertible. Define the linear mapping $\mathcal{C} : H^N((a,b), \mathbb{R}^{2n}) \to \mathbb{R}^{2nN}$ as,

$$\mathcal{C}x(t) := \tilde{W} \begin{pmatrix} g_{f,\partial}(t) \\ g_{e,\partial}(t) \end{pmatrix} \tag{27}$$

and the output as

$$y(t) = \mathcal{C}x(t), \tag{28}$$

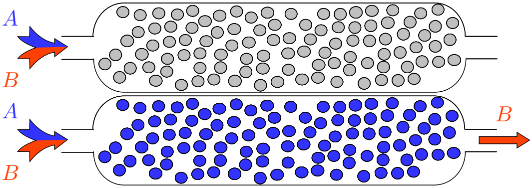
then for $u \in C^2((0,\infty); \mathbb{R}^{2nN})$, $x(0) \in H^N((a,b), \mathbb{R}^{2n})$, and $\mathcal{B}x(0) = u(0)$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 \leq \frac{1}{2} \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_{W,\tilde{W}} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \tag{29}$$

where

$$P_{W,\tilde{W}}^{-1} = \begin{pmatrix} W\Sigma W^T & W\Sigma \tilde{W}^T \\ \tilde{W}\Sigma W^T & \tilde{W}\Sigma \tilde{W}^T \end{pmatrix}. \tag{30}$$

Chemical reactor



Let consider a chemical tubular reactor $z \in [a, b]$ with reaction $A \to B$

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial z}\left(-D_a \frac{\partial C}{\partial z} + vC\right) - k_0 C + \text{Boundary conditions}$$

where $D_a > 0$ and v is a positive constant.

Chemical reactor

Let consider a chemical tubular reactor $z \in [a, b]$ with reaction $A \to B$

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial z}\left(-D_a \frac{\partial C}{\partial z} + vC\right) - kC + \text{Boundary conditions}$$

where $D_a > 0$ and v is a positive constant. By choosing

$$\mathcal{J} = -\frac{\partial}{\partial z}, \quad \mathcal{G} = \frac{\partial}{\partial z} + \sqrt{\frac{k}{D_a}}, \quad \mathcal{G}^* = -\frac{\partial}{\partial z} + \sqrt{\frac{k}{D_a}}, \quad S = \frac{D_a}{v}$$

$$\begin{pmatrix} \frac{\partial C}{\partial t} \\ f \end{pmatrix} = \begin{pmatrix} -\frac{\partial}{\partial z} & \frac{\partial}{\partial z} + \sqrt{\frac{k}{D_a}} \\ \frac{\partial}{\partial z} - \sqrt{\frac{k}{D_a}} & 0 \end{pmatrix} \begin{pmatrix} vC \\ e \end{pmatrix}$$

Boundary port variables

$$\begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} vC(b) \\ D_a \frac{\partial}{\partial z} C(b) - \sqrt{D_a k} C(b) \\ vC(a) \\ D_a \frac{\partial}{\partial z} C(a) - \sqrt{D_a k} C(a) \end{pmatrix}$$

Then

$$\begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} vC(a) - vC(b) + D_a \left(\frac{\partial}{\partial z} C(b) - \frac{\partial}{\partial z} C(a) \right) - \sqrt{D_a k} \left(C(b) - C(a) \right) \\ v \left(C(b) - C(a) \right) \\ v \left(C(b) + C(a) \right) \\ D_a \left(\frac{\partial}{\partial z} C(b) + \frac{\partial}{\partial z} C(a) \right) - \sqrt{D_a k} \left(C(b) + C(a) \right) \end{pmatrix}$$

Dankwert conditions

Usually the boundary conditions for tubular reactors are chosen as Dankwert Boundary Conditions:

- input total flow is imposed,
- output dispersive flow is equal to zero.

Dankwert conditions can be written as :

$$vC(t,a) - D_a \frac{\partial C}{\partial z}(t,a) = vC_{\mathrm{in}}(t), \quad \text{and} \quad D_a \frac{\partial C}{\partial z}(t,b) = 0, \tag{31}$$

$$\Leftrightarrow$$

$$\begin{pmatrix} vC_{in} \\ 0 \end{pmatrix} = W \begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix}$$

$$W = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & \sqrt{\frac{k_0 D_a}{v}} & 1 - \sqrt{\frac{k_0 D_a}{v}} & -1 \\ 1 & 1 + \sqrt{\frac{k_0 D_a}{v}} & \sqrt{\frac{k_0 D_a}{v}} & 1 \end{pmatrix},$$

Conclusion

One can check that

$$W\Sigma W^T \geq 0$$

iif

$$\sqrt{\frac{kD_a}{v}} \leq \frac{1}{2}$$

It means that he system is a Boundary Control System with associated C_0 semigroup unitary or a contraction if and only if the condition is satisfied.

Otherwise it is not a contraction semigroup.

Conclusion

In this first part we have:

- shown that PDEs are obtained from balances equation on extensives variables and can be related to power exchanges within the system through geometric considerations,
- in the 1D case defined:
  - the boundary port variables associated to the differential operator $\mathcal{J}$
  - Dirac structures on real Hilbert spaces
- parametrized all the boundary port variables for a large class of differential operators.
- provided some simple conditions (matrix ones) to prove existence of solutions.
- defined BCS.

In what follows we discuss stability and stabilization of boundary controlled port Halmiltonian systems.

Thank you for your attention !

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