Modeling and Control of Nonlinear and Distributed Parameter Systems: the Port Hamiltonian Approach

Stability, control design

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Outline

1. Control of finite dimensional PHS
2. Stability of BCS
3. In-domain controlled Port Hamiltonian systems
4. Conclusion and future works

Port-Hamiltonian control systems

Let us recall the state space model of a port-Hamiltonian control system

$$\begin{cases} \dot{x} = \left(J(x) - R(x)\right) \dfrac{\partial H}{\partial x}(x) + g(x)u, \\ y = g^\top(x) \dfrac{\partial H}{\partial x}(x), \end{cases}$$

where where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}^m$, $m < n$, is the control action, $H : \mathbb{R}^n \to \mathbb{R}$ is the total stored energy, $J(x) = -J(x)^\top$ is the $n \times n$ natural interconnection matrix, $R(x) = R(x)^\top \geq 0$ is the $n \times n$ damping matrix, $g(x)$, is the $n \times m$ input map and $\mathbf{u}, \mathbf{y} \in \mathbb{R}^m$, are conjugated variables whose product has units of power.

$$\begin{cases} \dot{H} = u^\top y - \dfrac{\partial H}{\partial x}^\top R \dfrac{\partial H}{\partial x}, \\ \dot{H} \leq u^\top y, \end{cases}$$

Example: pendulum (model)

The dynamic equations

Consider the pendulum with damping

$$\begin{cases} \dot{q} = \dfrac{p}{m} \\ \dot{p} = -mg\sin(q) - f\dfrac{p}{m} + u \end{cases} \tag{1}$$

with state variables $x = [p, q]$, with q the configuration and p the momentum.

The port Hamiltonian model is:

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -f \end{pmatrix}}_{J-R} \begin{pmatrix} \frac{\partial H_0}{\partial q} \\ \frac{\partial H_0}{\partial p} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{g} u \\ y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial H_0}{\partial q} \\ \frac{\partial H_0}{\partial p} \end{pmatrix} = \dfrac{p}{m} \end{cases} \tag{2}$$

with Hamiltonian : $H_0(q,\, p) = mg(1 - \cos q) + \frac{1}{2m}p^2$

Stabilization of passive systems

Let us consider systems arising from some physical energy model. We then usually have

$$H(t) = H(t_0) + \underbrace{\int_0^t u(\tau)y(\tau)d\tau}_{\text{supplied energy}} - \underbrace{\int_0^t S(x(\tau))d\tau}_{\text{dissipated energy}} .$$

So if $H(x)$ qualifies as a Lyapunov function and $S(x)$ vanishes at $x = 0$ (and only in $x = 0$), then the system is asymptotically stable!

$\boxed{\text{So why do we need the control then?}}$

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Stabilization of passive systems

- What if we want to increase the stability/rate of dissipation?: damping injection,
- What if we want to stabilize at some different equilibrium point/change the performances, $x = x^*$, $x^* \neq 0$: Energy shaping,
- What if $S(x)$ vanishes for some $x \neq 0$ or $S(x) = 0$?: damping injection + Energy shaping

Stabilization of PHS: Damping injection

Consider the energy balance equation of a passive system:

$$H(t) = H(t_0) + \underbrace{\int_0^t u(\tau)y(\tau)d\tau}_{\text{supplied energy}} - \underbrace{\int_0^t S(x(\tau))d\tau}_{\text{dissipated energy}} .$$

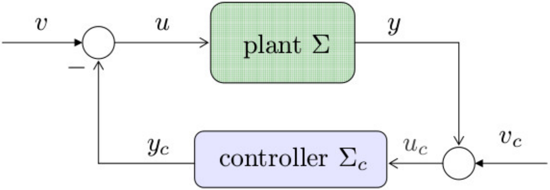
And assume that $H(x)$ qualifies as a Lyapunov function candidate. If we select the input $u = -Ky$, with K a positive definite constant matrix, then the energy balance equation becomes:

$$H(t) = H(t_0) \underbrace{-K\int_0^t y^2(\tau)d\tau}_{\text{controller}} - \underbrace{\int_0^t S(x(\tau))d\tau}_{\text{dissipated energy}} ,$$

$$H(t) = H(t_0) - \underbrace{\int_0^t \left(Ky^2(\tau)d\tau + S(x(\tau))\right) d\tau}_{\text{dissipated energy}}.$$

Control by interconnection

A controlled system may be viewed as a plant system interconnected with a control system exchanging energy



The interconnection is power continuous if

$$u = v - y_c, \qquad \text{and} \qquad u_c = y + v_c, \qquad \forall t$$

Control by interconnection

Assume that the plant and the controller are PHS

$$\Sigma : \quad \begin{cases} \dot{x} = [J(x) - R(x)]\dfrac{\partial H}{\partial x}(x) + g(x)u \\ y = g^\top(x)\dfrac{\partial H}{\partial x}(x) \end{cases} \quad x \in \mathcal{X}$$

$$\Sigma_c : \quad \begin{cases} \dot{\xi} = [J_c(\xi) - R_c(\xi)]\dfrac{\partial H_c}{\partial \xi}(\xi) + g_c(\xi)u_c \\ y_c = g_c^\top(\xi)\dfrac{\partial H_c}{\partial \xi}(\xi) \end{cases} \quad \xi \in \mathcal{X}_c$$

Booth are passive systems, so a power preserving interconnection, $u = -y_c$, $y = u_c$, yields a passive closed-loop system.

Control by interconnection

The closed-loop systems looks

$$\begin{bmatrix}\dot{x}\\ \dot{\xi}\end{bmatrix} = \left(\underbrace{\begin{bmatrix} J(x) & -g(x)g_c^\top(\xi)\\ g_c(\xi)g^\top(x) & J_c(\xi)\end{bmatrix}}_{J_{cl}(x,\xi)} - \underbrace{\begin{bmatrix} R(x) & 0\\ 0 & R_c(\xi)\end{bmatrix}}_{R_{cl}(x,\xi)}\right)\begin{bmatrix}\frac{\partial H_d}{\partial x}(x)\\ \frac{\partial H_d}{\partial \xi}(\xi)\end{bmatrix}$$

$$\begin{bmatrix} y\\ y_c\end{bmatrix} = \underbrace{\begin{bmatrix} g(x) & 0\\ 0 & g_c(\xi)\end{bmatrix}}_{g_{cl}}\begin{bmatrix}\frac{\partial H_d}{\partial x}(x)\\ \frac{\partial H_d}{\partial \xi}(\xi)\end{bmatrix}$$

With total energy function

$$\boxed{H_d(x,\xi) = H(x) + H_c(\xi)}$$

We may equivalently write the closed-loop system as

$$\dot{w} = (J_{cl} - R_{cl})\frac{\partial H_d}{\partial w}, \qquad y_{cl} = g_{cl}^\top \frac{\partial H_d}{\partial w}$$

with $w = [x\ \xi]$.

Control by interconnection

Aim

We would like to get an energy function in terms of x only: $H_d = H_d(x)$, so that we can assign the minimum at a desired point and characterize it in terms of the plant system.

In order achieve this, we must restrict the dynamics to a submanifold of the (x, ξ) space parametrized by x. This means that we are looking for a submanifold

$$\boxed{\Omega_C = (x, \xi) : \xi = F(x) - C}$$

which is dynamically invariant, i.e.,

$$\frac{dC}{dt} = \left(\frac{\partial F_i}{\partial x}^\top \dot{x} - \dot{\xi}_i \right)_{\xi = F_i(x) - C} = 0$$

Control by interconnection

Casimir functions

Let us look for structural invariants that relates each state of the controller with the states of the plant:

$$C_i(x, \xi_i) = F_i(x) - \xi_i$$

In order to relate all the states of the controller with the state of the plant we define $F(x) = [F_1(x), F_2(x), \dots, F_{n_c}(x)]$, and define the following Casimir function

$$C = \sum_{i=1}^{n} (F_i(x) - \xi_i) = \sum_{i=1}^{n} C_i(x, \xi_i)$$

C is an invariant of the system, hence

$$\dot{C} = \frac{\partial C}{\partial w}^{\top} \dot{w} = \frac{\partial C}{\partial w}^{\top} \left(J_{cl} \frac{\partial H_{cl}}{\partial w} \right) = 0$$

But furthermore, C is a structural invariant, so it should be invariant with respect to the structure of the system:

$$\frac{\partial C}{\partial w}^{\top} J_{cl} = 0$$

Control by interconnection

Casimir functions

Let us look for structural invariants that relates each state of the controller with the states of the plant:

$$C = \sum_{i=1}^{n} C_i(x, \xi_i) = \sum_{i=1}^{n} (F_i(x) - \xi_i)$$

we obtain the following matching condition

$$\underbrace{\begin{bmatrix} \frac{\partial F}{\partial x}^\top(x) & -\mathbb{I} \end{bmatrix} \begin{bmatrix} J(x) - R(x) & -g(x) g_C^\top(\xi) \\ g_C(\xi) g^\top(x) & J_C(\xi) - R_C(\xi) \end{bmatrix}}_{\text{Matching condition}} \begin{bmatrix} \frac{\partial H_d}{\partial x}(x) \\ \frac{\partial H_d}{\partial \xi}(\xi) \end{bmatrix} = 0$$

- Only the term in blue is considered in the matching condition because we want the Casimir functions to be structural invariants of the system: not depend on $H_d(x, \xi)$.

Control by interconnection

The condition for existence of Casimir functions for the closed loop system

$$\begin{bmatrix} \frac{\partial F}{\partial x}^\top(x) & -\mathbb{I} \end{bmatrix} \begin{bmatrix} J(x) - R(x) & -g(x)g_C^\top(\xi) \\ g_C(\xi)g^\top(x) & J_C(\xi) - R_C(\xi) \end{bmatrix} = 0$$

may be written out as

Matching equations

$$\frac{\partial F}{\partial x}^\top(x)J(x)\frac{\partial F}{\partial x}(x) = J_c(\xi)$$

$$R(x)\frac{\partial F}{\partial x}(x) = 0 \qquad \text{Dissipation obstacle!}$$

$$R_c(\xi) = 0$$

$$\frac{\partial F}{\partial x}^\top(x)J(x) = g_c(\xi)g^\top(x)$$

Control by interconnection

The closed-loop dynamic then takes the form

$$\dot{x} = \Big[J(x) - R(x)\Big]\frac{\partial H}{\partial x}(x) - g(x)g_C^\top(\xi)\frac{\partial H_c}{\partial \xi}(\xi)$$

Using the second and fourth M.C. we get

$$\dot{x} = \Big[J(x) - R(x)\Big]\left(\frac{\partial H}{\partial x}(x) + \frac{\partial F}{\partial x}(x)\frac{\partial H_c}{\partial \xi}(\xi)\right)$$

Since $\xi = F(x) - C$, we use the chain-rule for differentiation to establish

$$\frac{\partial F}{\partial x}(x)\frac{\partial H_c}{\partial \xi}(\xi) = \frac{\partial H_c}{\partial x}(F(x) - C)$$

Control by interconnection

Hence we obtain:

$$\dot{x} = \Big[J(x) - R(x)\Big]\left(\frac{\partial H}{\partial x}(x) + \frac{\partial H_c}{\partial x}(F(x) - C)\right)$$

Or equivalently

$$\boxed{\dot{x} = \Big[J(x) - R(x)\Big]\frac{\partial H_d}{\partial x}(x)}$$

With closed-loop energy $H_d(x) = H(x) + H_c(F(x) - C)$.

Pendulum (extended Casimir functions)

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & {\color{red}-f} \end{pmatrix}}_{J} \begin{pmatrix} \frac{\partial H_0}{\partial q} \\ \frac{\partial H_0}{\partial p} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{g} u \\ y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial H_0}{\partial q} \\ \frac{\partial H_0}{\partial p} \end{pmatrix} = \dfrac{p}{m} \end{cases} \tag{3}$$

Recall: we look for Casimir functions such as:

$$C(q, p, x_c) = F(q, p) - x_c$$

Using the M.C.s.

- From physical considerations we know that we only need to shape the q coordinate: F is only one scalar function.
- Then, from M.E.1. we obtain that $J_c = 0$, and from M.E.3 that $R_c = 0$.
- Finally from M.E.4 we have that $\frac{\partial F}{\partial q} = 1$.

hence a generating function is: $F(q, p) = q$

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Pendulum (control)

For the controller design we choose a function $H_C(x_c)$ such that

$$H_d(x) = H_0(x) + H_c(F(x))$$

has a minimum at the desired equilibrium $x_* = (x_1^*, 0)$. The simplest choice is given by

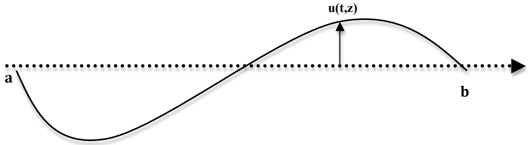
$$H_C(x_c) = -mg(1 - \cos x_c) + \frac{1}{2}\alpha mg(x_c - x_1^*)^2$$

The control is finally (with damping injection) :

$$u = -y_c - dp = -\frac{\partial H_C}{\partial x_c}(x_c)\,|_{x_c=q} - dp = mg\sin q - \alpha mg(q - x_1^*) - dp$$

which is the well-known as proportional plus gravity compensation control.

Stability and stabilization of BCS



We are now interested in stability of BCS. We consider:

- Asymptotic stability
- Exponential stability

in the case of

- Static boundary feedback
- Dynamic boundary feedback

We shalle also see how to design dynamic controllers in order to shape the closed loop energy function by using structural invariants.

Stability

We are interested in (exponential) stability of abstract systems of the form

$$\dot{x}(t) = Ax(t),\ t \geq 0,\ \ x(0) = x_0$$

i.e. when the solution tends to zero (exponentially) fast as $t \to 0$.

Definition

The C_0 semigroup $(T(t))_{t\geq 0}$ on X is exponentially stable if there exist positives constants M and α such that

$$\|T(t)\| \leq Me^{-\alpha t} \ \ \textit{for } t \geq 0$$

Stability

Theorem

Suppose that A is the infinitesimal generator of a C_0 semigroup $(T(t))_{t\geq 0}$ on X. The following are equivalent

1. $(T(t))_{t\geq 0}$ is exponentially stable
2. There exists a positive operator $P \in \mathcal{L}(X)$ such that
$$\langle Ax, Px\rangle + \langle Px, Ax\rangle = -\langle x, x\rangle \text{ for all } x \in D(A) \tag{4}$$
3. There exists a positive operator $P \in \mathcal{L}(X)$ such that
$$\langle Ax, Px\rangle + \langle Px, Ax\rangle \leq -\langle x, x\rangle \text{ for all } x \in D(A)$$

Equation (4) is called Lyapunov equation.

Stability

When there exists a positive operator $P \in \mathcal{L}(X)$ such that

$$\langle Ax, Px\rangle + \langle Px, Ax\rangle \leq 0 \text{ for all } x \in D(A)$$

one has to prove that there exists an invariant set and that this invariant set reduces to zero.
Lassale's invariant principle

Boundary controlled port Hamiltonian systems

Boundary controlled port Hamiltonian systems

Let W be a $n \times 2n$ real matrix. If W has full rank and satisfies $W\Sigma W^\top \geq 0$, then the system $\frac{\partial x}{\partial t} = P_1 \frac{\partial}{\partial z}\left(\mathcal{L}(z)x)(t,z)\right) + (P_0 - G_0)\mathcal{L}(z)x(t,z)$with input

$$u(t) = W \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix}$$

is a BCS on X. The operator $\mathcal{A}x = P_1(\partial/\partial z)(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain

$$D(\mathcal{A}) = \left\{ \mathcal{L}x \in H^1(a,b;\mathbb{R}^n) \;\middle|\; \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix} \in \ker W \right\}$$

generates a contraction semigroup on X.

Boundary controlled port Hamiltonian systems

Boundary controlled port Hamiltonian systems

Let $\tilde{W}$ be a full rank matrix of size $n \times 2n$ with $\begin{bmatrix} W \\ \tilde{W} \end{bmatrix}$ invertible and let $P_{W,\tilde{W}}$ be given by

$$P_{W,\tilde{W}} = \left(\begin{bmatrix} W \\ \tilde{W} \end{bmatrix} \Sigma \begin{bmatrix} W \\ \tilde{W} \end{bmatrix}^\top \right)^{-1} = \begin{bmatrix} W\Sigma W^\top & W\Sigma\tilde{W}^\top \\ \tilde{W}\Sigma W^\top & \tilde{W}\Sigma\tilde{W}^\top \end{bmatrix}^{-1}.$$

Define the output of the system as the linear mapping $\mathcal{C} : \mathcal{L}^{-1}H^1(a,b;\mathbb{R}^n) \to \mathbb{R}^n$,

$$y = \mathcal{C}x(t) := \tilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x}(t) \\ e_{\partial,\mathcal{L}x}(t) \end{bmatrix}.$$

Then for $u \in C^2(0,\infty;\mathbb{R}^k)$, $\mathcal{L}x(0) \in H^1(a,b;\mathbb{R}^n)$, and $u(0) = W \begin{bmatrix} f_{\partial,\mathcal{L}x}(0) \\ e_{\partial,\mathcal{L}x}(0) \end{bmatrix}$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} = \frac{1}{2}\begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^\top P_{W,\tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} - \langle G_0\mathcal{L}x(t), \mathcal{L}x(t)\rangle \le \frac{1}{2}\begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^\top P_{W,\tilde{W}} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}.$$

Outline

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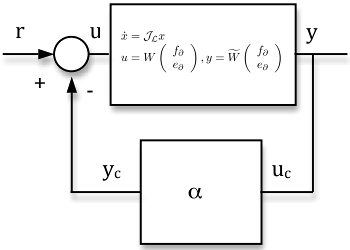
Closed loop control with static feedback

Impedance passive case

As it has been pointed out in [Villegas, 2007], if the matrices W and $\tilde{W}$ are selected such that $P_{W,\tilde{W}} = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \Sigma$, then the BCS fulfills

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} \leq u^\top(t)y(t).$$



$$\begin{cases} \dot{x} = \mathcal{J}_{\mathcal{L}} x \\ r = \left(W + \alpha \widetilde{W}\right) \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \\ y = \widetilde{W} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \end{cases}$$

Asymptotic stability

Lemma

Assume that $(\lambda - \mathcal{A})^{-1} : X \to X$ is a compact operator for $\lambda > 0$. Then the system described by:

$$\begin{cases} \dot{x} = \mathcal{J}_{\mathcal{L}} x \\ r = \left(W + \alpha \widetilde{W}\right) \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \\ y = \widetilde{W} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \end{cases}$$

with $r = 0$ and $\alpha > 0$ is asymptotically stable.

Sketch of poof

We use the energy as Lyapunov function and Lassale's invariant principle. First the closed loop system is a BCS with infinitesimal generator of a contraction semigroup as soon as $\alpha > 0$. If $u = 0$, $\frac{dH}{dt} = -y^T \alpha y$

Exponential stability

Lemma

Consider a BCS such that $W_{cl}\Sigma W_{cl}^T \geq 0$ with $u(t) = 0$, for all $t \geq 0$. Then the energy of the system $E(t) = (1/2)\|x(t)\|^2_{\mathcal{L}}$ satisfies for τ large enough

$$E(\tau) \leq c(\tau)\int_0^\tau \|(\mathcal{L}x)(t,b)\|^2_{\mathbb{R}}dt, \quad \text{and} \quad E(\tau) \leq c(\tau)\int_0^\tau \|(\mathcal{L}x)(t,a)\|^2_{\mathbb{R}}dt,$$

Theorem : exponential stability.

BCS is exponentially stable if the energy of the system satisfies

$$(dE/dt) \leq -k\|(\mathcal{L}x)(t,b)\|^2_{\mathbb{R}} \; or \; (dE/dt) \leq -k\|(\mathcal{L}x)(t,a)\|^2_{\mathbb{R}}$$

where k is a positive real constant.

Example : Timoshenko beam

As state variables we choose

$$\begin{aligned}
x_1 &= \quad \frac{\partial w}{\partial z} - \phi : && \text{shear displacement,} \\
x_2 &= \quad \rho \frac{\partial w}{\partial t} : && \text{transverse momentum distribution,} \\
x_3 &= \quad \frac{\partial \phi}{\partial z} : && \text{angular displacement,} \\
x_4 &= \quad I_\rho \frac{\partial \phi}{\partial t} : && \text{angular momentum distribution.}
\end{aligned}$$

Then the model of the beam can be rewritten as

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P_1} \frac{\partial}{\partial z}\begin{pmatrix} K\,x_1 \\ \frac{1}{\rho}\,x_2 \\ EI\,x_3 \\ \frac{1}{I_\rho}\,x_4 \end{pmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{P_0} \underbrace{\begin{pmatrix} K\,x_1 \\ \frac{1}{\rho}\,x_2 \\ EI\,x_3 \\ \frac{1}{I_\rho}\,x_4 \end{pmatrix}}_{\mathcal{L}x}.$$

Velocity feedback

One can define the boundary port variables:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\rho^{-1}x_2)(b) - (\rho^{-1}x_2)(a) \\ (Kx_1)(b) - (Kx_1)(a) \\ (I_\rho^{-1}x_4)(b) - (I_\rho^{-1}x_4)(a) \\ (EIx_3)(b) - (EIx_3)(a) \\ (Kx_1)(b) + (Kx_1)(a) \\ (\rho^{-1}x_2)(b) + (\rho^{-1}x_2)(a) \\ (EIx_3)(b) + (EIx_3)(a) \\ (I_\rho^{-1}x_4)(b) + (I_\rho^{-1}x_4)(a) \end{pmatrix}. \tag{5}$$

Let us consider stabilization by applying velocity feedback *i.e.* following BC:

$$\frac{1}{\rho(a)} x_2(a) = 0, \qquad \frac{1}{I_\rho(a)} x_4(a) = 0,$$

$$K(b)x_1(b,t) = -\alpha_1 \frac{1}{\rho(b)} x_2(b,t), \quad EI(b)x_3(b,t) = -\alpha_2 \frac{1}{I_\rho(b)} x_4(b)$$

Velocity feedback

Input mapping:

$$W_{cl} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ \alpha_1 & 1 & 0 & 0 & 1 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 1 & 0 & 0 & 1 & \alpha_2 \end{bmatrix}$$

then

$$W_{cl} \Sigma W_{cl}^T = 2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_2 \end{bmatrix} \geq 0$$

As output we can choose

$$y = \begin{pmatrix} -K(a)x_1(a) \\ -(EI)(a)x_3(a) \\ \frac{1}{\rho(b)} x_2(b) \\ \frac{1}{I_{\rho(b)}} x_4(b) \end{pmatrix}, \quad \text{with} \quad \widetilde{W} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Velocity feedback

Then

$$P_{W,\widetilde{W}}^{-1} = \begin{bmatrix} 2\alpha & I \\ I & 0 \end{bmatrix}, P_{W,\widetilde{W}} = \begin{bmatrix} 0 & I \\ I & -2\alpha \end{bmatrix}$$

Energy balance:

$$\frac{d}{dt}E(t) = \frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 = \langle u(t), y(t)\rangle_U - \langle \alpha y(t), y(t)\rangle_{\mathbb{R}}$$

where

$$\langle \alpha y(t), y(t)\rangle_{\mathbb{R}} = \alpha_1|(\rho^{-1}x_2)(b,t)|^2 + \alpha_2|(I^{-1}x_4)(b,t)|^2$$

Then

$$\begin{aligned} \| (\mathcal{L}x(b)) \|_{\mathbb{R}}^2 &= |(kx_1)(b)|^2 + |(\rho^{-1}x_2)(b)|^2 + |(EIx_3)(b)|^2 + |(I_\rho^{-1}x_4)(b)|^2 \\ &= (\alpha_1^2+1)|(\rho^{-1}x_2)(b,t)|^2 + (\alpha_2^2+1)|(I_\rho^{-1}x_4)(b)|^2 \\ &\leq \kappa\langle \alpha y(t), y(t)\rangle_{\mathbb{R}} = -\kappa\frac{d}{dt}E(t) \end{aligned}$$

$\Rightarrow$ Exponential stability

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Outline

Dynamic boundary feedback

Consider a strictly passive linear finite dimensional system

$$\dot{v} = A_c v + B_c u_c, \qquad y_c = C_c v + D_c u_c.$$

with storage function $E_c(t) = \frac{1}{2}\langle v(t) Q_c v(t)\rangle_{\mathbb{R}^m}$, $Q_c = Q_c^\top > 0 \in \mathbb{R}^m \times \mathbb{R}^m$.

Theorem [Villegas, 2007]

Let the open-loop BCS satisfy $\frac{1}{2}\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 = u(t)y(t)$. Consider a LTI strictly passive finite dimensional system with storage function $E_c(t) = \frac{1}{2}\langle v(t), Q_c v(t)\rangle_{\mathbb{R}^m}$. Then the power preserving feedback interconnection

$$u = r - y_c, \qquad\qquad y = u_c,$$

with $r \in \mathbb{R}^n$ the new input of the system is a BCS on the extended state space $\tilde{x} \in \tilde{X} = X \times V$ with inner product $\langle \tilde{x}_1, \tilde{x}_2\rangle_{\tilde{X}} = \langle x_1, x_2\rangle_{\mathcal{L}} + \langle v_1, Q_c v_2\rangle_V$. Furthermore, the operator $\mathcal{A}_e$ defined by

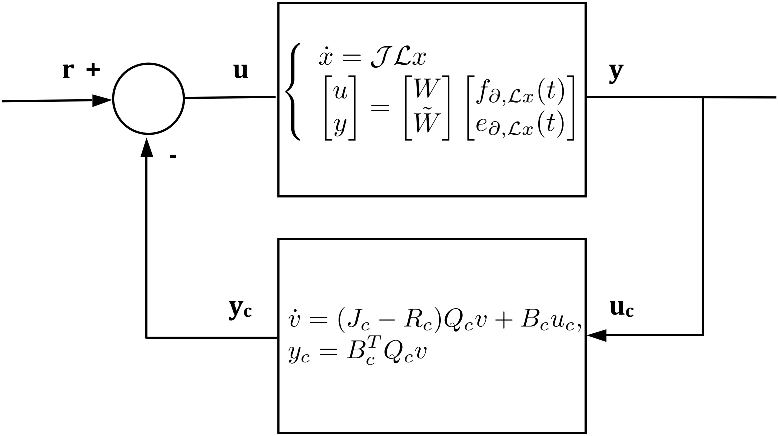
$$\mathcal{A}_e \tilde{x} = \begin{bmatrix} \mathcal{J}\mathcal{L} & 0 \\ B_c \mathcal{C} & A_c \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}, \quad D(\mathcal{A}_e) = \left\{ \begin{bmatrix} x \\ v \end{bmatrix} \in \begin{bmatrix} X \\ V \end{bmatrix} \middle| \mathcal{L}x \in H^N(a, b; \mathbb{R}^n), \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \\ v \end{bmatrix} \in \ker \tilde{W}_D \right\}$$

where

$$\tilde{W}_D = \begin{bmatrix} (W + D_c \tilde{W} & C_c) \end{bmatrix}$$

generates a contraction semigroup on $\tilde{X}$.

Dynamic boundary feedback



Asymptotic stability

Finite dimensional port Hamiltonian controller

$$\dot{v} = (J_c - R_c)Q_c v + B_c u_c, \quad y_c = B_c^\top Q_c v, \quad E_c(t) = \frac{1}{2} v(t)^\top Q_c v(t)$$

where we assume that $Q_c = Q_c^\top > 0$, $J_c = -J_c^\top$, $R_c = R_c^\top \geq 0$ and B_c are real constant matrices of proper dimensions. Furthermore, the controller is assumed to be exponentially stable, i.e., $A_c := (J_c - R_c)Q_c$ is Hurwitz.

Theorem

Consider the above controller connected to the impedance passive system through $u = r - y_c, u_c = y$. Then the operator $\mathcal{A}_e$ described in the previous theorem has compact resolvant.

Theorem

Consider the feedback system $u = r - y_c, u_c = y$ where the controller is chosen satisfying the condition above. Then the closed loop system such that $r = 0$ is globally asymptotically stable.

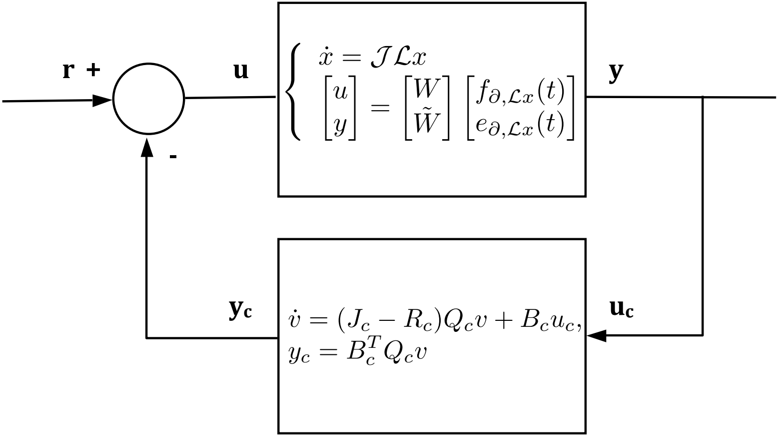
Sketch of proof

- Let first consider that $\omega(0) \in D(\mathcal{A}_e)$. By the aforementioned Theorem [Villegas, 2007], $\mathcal{A}_e$ generates a contraction semigroup.
- Let now consider the energy as Lyapunov function $E_c(t) = \frac{1}{2}\langle \omega(t), \omega(t)\rangle_{\tilde{X}}$. Since $\omega(0) \in D(\mathcal{A}_e)$ and:

$$\frac{dE_c(t)}{dt} = \langle \dot{\omega}(t), \omega(t)\rangle_{\tilde{X}} = \langle \mathcal{A}_e\omega(t), \omega(t)\rangle_{\tilde{X}} = -v^T Q_d v \tag{6}$$

  where $Q_d > 0$. Since $(\lambda I - \mathcal{A}_e)^{-1}$ is compact and the semigroup is a contraction it follows from LaSalle's invariance principle that all solutions asymptotically tend to the maximal invariant set $\mathcal{O}_c = \left\{\tilde{x} \in \tilde{X} | \dot{E}_c = 0\right\}$.
- Let $\mathcal{E}$ be the largest invariant subset of $\mathcal{O}_c$. We can prove that $\mathcal{E} = \{0\}$. From $\dot{E}_c(t) = 0$ and (6) we have $v(t) = 0$ and then $\dot{v}(t) = 0$. Let $\eta < n$ be the rank of $\ker(B_c)$. Form the controller structure $y_c = 0$ and $n - \eta > 0$ components of u_c equal 0. It follows that $\mathcal{O}_c$ reduces to the solution of a first order PDE of dimension n with $2n - \eta$ boundary variables set to zero. It follows from Holmgren's Theorem that $\tilde{x}(t) = 0$, hence the asymptotic stability. The same hold for $\omega(0) \in \tilde{X}$ by using denseness argument.

Energy shaping



Energy shaping

Idea:

Use the total energy as Lyapunov function candidate

From the power preserving interconnection:

$$\widetilde{E}(x, v) = E(x) + E_c(v)$$

We are looking for Casimir functions (structural invariants $\Rightarrow \dot{C} = 0$) on the form:

$$C(x, v) = v - F(x)$$

then

$$v - F(x) = \kappa$$

And

$$\widetilde{E}(x, v) = E(x) + E_c(F(x) + \kappa)$$

It remains to choose E_c and to add dissipation such that:

$$\frac{\partial \widetilde{E}}{\partial x}(x^*) = 0, \text{ and } \frac{dE}{dt}(x) < 0$$

Casimir

Let consider the structural invariants of the closed loop system *i.e.* Casimirs, of the form:

$$C(x(t), v(t)) = \Gamma^\top v(t) + \int_a^b \Psi^\top(z) x(t,z) dz \tag{7}$$

with $\Gamma \in \mathbb{R}^m$, $\Psi(z) \in \mathbb{R}^n$ and $\Psi^\top(z)x(t,z) \in H^1(a, b; \mathbb{R}^n)$.

Computation of Casimir functions

Let consider the previously defined boundary controlled port Hamiltonian system with $r = 0$. Then (7) is a Casimir function for the closed loop system if and only if:

$$P_1 \frac{\partial}{\partial z}\Psi(z) + (P_0 + G_0)\Psi(z) = 0, \tag{8}$$

$$(J_c + R_c)\Gamma + B_c \tilde{W} R \begin{bmatrix} \Psi(b) \\ \Psi(a) \end{bmatrix} = 0, \tag{9}$$

$$B_c^\top \Gamma + WR \begin{bmatrix} \Psi(b) \\ \Psi(a) \end{bmatrix} = 0. \tag{10}$$

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Energy shaping

Sketch of the proof

$C(x_e(t))$ is a Casimir function if and only if $\frac{dC}{dt} = 0$ independently to the energy function,

$$\frac{dC}{dt} = \left\langle \frac{\delta C}{\delta x_e}, \frac{dx_e}{dt} \right\rangle_{L^2} \tag{11}$$

$$= \left\langle \frac{\delta C}{\delta x_e}, \mathcal{A}_e \mathcal{H}_e x_e \right\rangle_{L^2} \tag{12}$$

$$= \left\langle \mathcal{A}_e^* \frac{\delta C}{\delta x_e}, \mathcal{H}_e x_e \right\rangle_{L^2} + BC \tag{13}$$

$$\tag{14}$$

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Energy shaping

Proposition [Macchelli et al., 2017]

Under the hypothesis that the Casimir functions exist, the closed-loop dynamics (when $u = y_c + u'$) is given by :

$$\begin{aligned} \frac{\partial x}{\partial t}(t,\zeta) &= P_1 \frac{\partial}{\partial \zeta} \frac{\delta H_{cl}}{\delta x}(x(t))(\zeta) + (P_0 - G_0) \frac{\delta H_{cl}}{\delta x}(x(t))(\zeta) \\ u' &= W' R \begin{pmatrix} \left(\frac{\delta H_{cl}}{\delta x}(x) \right)(b) \\ \left(\frac{\delta H_{cl}}{\delta x}(x) \right)(a) \end{pmatrix} \end{aligned} \tag{15}$$

in which δ denotes the variational derivative, while

$$\begin{aligned} H_{cl}(x(t)) = \frac{1}{2}\|x(t)\|_{cl}^2 + \frac{1}{2} \left(\int_a^b \hat{\Psi}^T(\zeta) x(t,\zeta)\, dz \right)^T \times \\ \times \hat{\Gamma}^{-1} Q_C \hat{\Gamma}^{-T} \int_a^b \hat{\Psi}(\zeta)^T x(t,\zeta)\, dz \end{aligned} \tag{16}$$

and W' is a $n \times 2n$ full rank, real matrix s.t. $W' \Sigma W'^T \geq 0$. Asymptotic stability is ensured by damping injection.

Extension to systems with dissipation

Proposition [Macchelli et al., 2017]

The feedback law $u = \beta(x) + u'$, with u' an auxiliary boundary input, maps the original system into the target dynamical system

$$\begin{aligned} \frac{\partial x}{\partial t}(t,z) &= P_1 \frac{\partial}{\partial z} \frac{\delta H_d}{\delta x}(x(t))(z) + (P_0 - G_0)\frac{\delta H_d}{\delta x}(x(t))(z) \\ u'(t) &= WR \begin{pmatrix} \left(\frac{\delta H_d}{\delta x}(x(t))\right)(b) \\ \left(\frac{\delta H_d}{\delta x}(x(t))\right)(a) \end{pmatrix} \end{aligned} \tag{17}$$

with $H_d(x) = H(x) + H_a(x)$, provided that

$$P_1 \frac{\partial}{\partial z} \frac{\delta H_a}{\delta x}(x) + (P_0 - G_0)\frac{\delta H_a}{\delta x}(x) = 0 \tag{18}$$

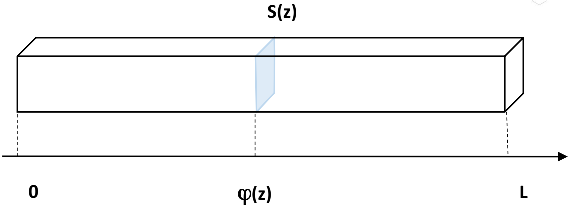
$$\beta(x) + WR \begin{pmatrix} \left(\frac{\delta H_a}{\delta x}(x)\right)(b) \\ \left(\frac{\delta H_a}{\delta x}(x)\right)(a) \end{pmatrix} = 0. \tag{19}$$

Main results

- System without dissipation (immersion reduction method)
  - Computation of the Casimir invariants: $\hat{\Gamma}$, $\hat{\Psi}^T(z)$
  - Implementation of the control: $Q_c \to H_d$
  - Stabilization using damping injection.
- System with dissipation (direct state feedback)
  - Direct computation of H_a and $\beta(x)$
  - Stabilization using damping injection.

- In the two cases we can prove asymptotic stability [Villegas et al., 2009, Ramirez et al., 2014, Macchelli et al., 2017].
- Not so many degrees of freedom but the closed loop energy function can be partially shaped.

Example: longitudinal (axial) vibration of a beam



State variables : deformation and linear momentum density

$$\varepsilon(t,\zeta) = \frac{\partial \varphi}{\partial \zeta}(t,\zeta), \quad p(t,\zeta) = \rho S(\zeta) v(t,\zeta) \tag{20}$$

Material's deformation is considered linear (Hooke's law) :

$$\rho S(\zeta)\frac{\partial^2 \varphi}{\partial t^2}(t,\zeta) = \frac{\partial}{\partial \zeta}\left[ES(\zeta)\frac{\partial \varphi}{\partial \zeta}(t,\zeta)\right] - D\frac{\partial \varphi}{\partial t}(t,\zeta)\mathrm{d}\zeta$$

The energy is given by (kinetic+potential):

$$H(p(t,\zeta),\varepsilon(t,\zeta)) = \frac{1}{2}\int_0^L \left[\frac{p^2(t,\zeta)}{\rho S(\zeta)} + ES(\zeta)\varepsilon^2(t,\zeta)\right]\mathrm{d}\zeta$$

Example: longitudinal (axial) vibration of a beam

From:

$$H(p(t,\zeta),\varepsilon(t,\zeta)) = \frac{1}{2}\int_0^L \left[\frac{p^2(t,\zeta)}{\rho S(\zeta)} + ES(\zeta)\varepsilon^2(t,\zeta)\right] \mathrm{d}\zeta$$

We define the co-energy variables:

$$\sigma_S(t,\zeta) = \frac{\delta H}{\delta \varepsilon}(\varepsilon(t,\zeta)) = ES(\zeta)\varepsilon(t,\zeta) = S(\zeta)\sigma(t,\zeta)$$
$$v(t,\zeta) = \frac{\delta H}{\delta p}(p(t,\zeta)) = \frac{p(t,\zeta)}{\rho S(\zeta)} = \frac{\partial \varphi}{\partial t}(t,\zeta)$$

Then:

$$\frac{\partial}{\partial t}\left(\rho S(\zeta)\frac{\partial \varphi}{\partial t}(t,\zeta)\right) = \frac{\partial}{\partial \zeta}\left[ES(\zeta)\frac{\partial \varphi}{\partial \zeta}(t,\zeta)\right] - D\frac{\partial \varphi}{\partial t}(t,\zeta)$$

with

$$\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial \zeta}(t,\zeta)\right) = \frac{\partial}{\partial \zeta}\left(\frac{\partial \varphi}{\partial t}(t,\zeta)\right)$$

Example: longitudinal (axial) vibration of a beam

The port-Hamiltonian formulation of the system is then

$$\frac{\partial}{\partial t}\begin{pmatrix}\varepsilon(t,\zeta)\\ p(t,\zeta)\end{pmatrix}=\begin{pmatrix}0 & \frac{\partial}{\partial\zeta}\\ \frac{\partial}{\partial\zeta} & -D\end{pmatrix}\begin{pmatrix}ES(\zeta) & 0\\ 0 & \frac{1}{\rho S(\zeta)}\end{pmatrix}\begin{pmatrix}\varepsilon(t,\zeta)\\ p(t,\zeta)\end{pmatrix}$$

which is in the form :

$$\frac{\partial x}{\partial t}(t,\zeta)=P_1\frac{\partial}{\partial\zeta}\left(\mathcal{H}(\zeta)x(t,\zeta)\right)+(P_0-G_0)\mathcal{H}(\zeta)x(t,\zeta) \tag{21}$$

with $P_0=0$ and

$$P_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\qquad G_0=\begin{pmatrix}0 & 0\\ 0 & D\end{pmatrix}\qquad \mathcal{H}(\zeta)=\begin{pmatrix}ES(\zeta) & 0\\ 0 & \frac{1}{\rho S(\zeta)}\end{pmatrix}$$

Input and output

The boundary port variables are

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} v(L) - v(0) \\ \sigma_S(L) - \sigma_S(0) \\ \sigma_S(L) + \sigma_S(0) \\ v(L) + v(0) \end{pmatrix}$$

The boundary input and output are selected as

$$u(t) = \begin{pmatrix} v(t,0) \\ \sigma_S(t,L) \end{pmatrix} \qquad y(t) = \begin{pmatrix} -\sigma_S(t,0) \\ v(t,L) \end{pmatrix} \tag{22}$$

which can be derived choosing W and $\tilde{W}$ such that:

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad \tilde{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

The energy balance is then :

$$\frac{\mathrm{d}H}{\mathrm{d}t}(t) = -\int_0^L Dv^2(t,\zeta)\,\mathrm{d}\zeta + y^{\mathrm{T}}(t)u(t) \leq y^{\mathrm{T}}(t)u(t).$$

Lossless case : Approach based on structural invariants

We consider a dynamic controller with $n_C = 2$, $R_C = 0$, $B_C = I$ and

$$J_C = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

which implies that the closed-loop system is characterized by the following Casimir functions:

$$C_1(\xi_1(t), \varepsilon(t,\cdot)) = \xi_1(t) - \int_0^L \varepsilon(t,\zeta)\,\mathrm{d}\zeta$$

$$C_2(\xi_2(t), p(t,\cdot)) = \xi_2(t) - \int_0^L p(t,\zeta)\,\mathrm{d}\zeta.$$

The controller Hamiltonian is chosen such that

$$\hat{H}_c(\xi_1, \xi_2) = \frac{1}{2}\Xi_1\xi_1^2 + \frac{1}{2}\Xi_2\xi_2^2 \tag{23}$$

Approach based on structural invariants

The closed loop energy function is:

$$H_{cl}(\varepsilon, p) = \frac{1}{2}\int_0^L \left[\frac{p^2}{\rho S(\zeta)} + ES(\zeta)\varepsilon^2\right] \mathrm{d}\zeta + \\ + \frac{1}{2}\Xi_1 \left(\int_0^L \varepsilon \, \mathrm{d}\zeta\right)^2 + \frac{1}{2}\Xi_2 \left(\int_0^L p \, \mathrm{d}\zeta\right)^2 \quad (24)$$

and the control is of the form

$$u = -y_c = -G_c \delta H_c = -\begin{pmatrix} \Xi_2 & 0 \\ 0 & \Xi_1 \end{pmatrix} \begin{pmatrix} \int_0^L p \, \mathrm{d}\zeta \\ \int_0^L \varepsilon \, \mathrm{d}\zeta \end{pmatrix}$$

System with dissipation

Due to the dissipation $D \neq 0$, the energy-Casimir method cannot be applied. The closed loop energy function cannot be shaped in the p coordinate.

Admissible H_a :

$$\hat{H}_a(\xi_1, \xi_2) = \frac{1}{2}\Xi_1\xi_1^2 + \frac{1}{2}\Xi_2\xi_2^2$$

with

$$\begin{aligned} \xi_1(\varepsilon(t,\cdot)) &= \int_0^L \varepsilon(t,\zeta)\,\mathrm{d}\zeta \\ \xi_1(\varepsilon(t,\cdot), p(t,\cdot)) &= \int_0^L [D(L-z)\varepsilon(t,\zeta) + p(t,\zeta)]\,\mathrm{d}\zeta \end{aligned} \tag{25}$$

Leading to

$$u = -\begin{pmatrix} \Xi_2 & 0 \\ 0 & \Xi_1 \end{pmatrix} \begin{pmatrix} \int_0^L [D(L-z)\varepsilon(t,\zeta) + p(t,\zeta)]\,\mathrm{d}\zeta \\ \int_0^L \varepsilon\,\mathrm{d}\zeta \end{pmatrix}$$

Achievable performances

We consider now that $D = 0$, all parameters equal 1 (simulations are provided considering a finite volume approximation)

$$u(t) = \begin{pmatrix} v(t,0) \\ \sigma_S(t,L) \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{u}(t) \end{pmatrix} \qquad y(t) = \begin{pmatrix} -\sigma_S(t,0) \\ v(t,L) \end{pmatrix} = \begin{pmatrix} \tilde{y}(t) \\ \bar{y}(t) \end{pmatrix}$$

and we plot the position at the end point of the system.

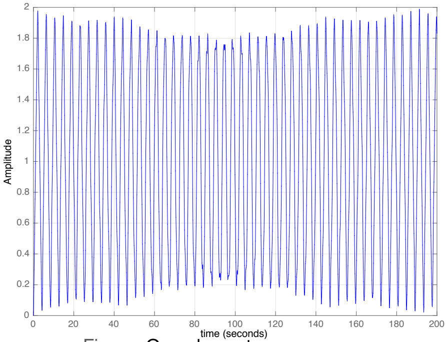


Figure: Open loop step response.

Simulation

We first consider the static feedback case *i.e.* when pure dissipation is added at the boundary:

$$u_2 = -k_d y_2$$

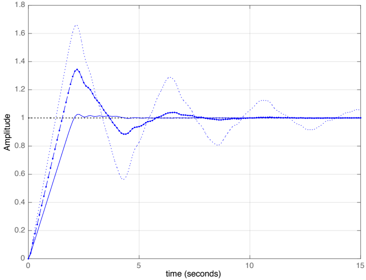


Figure: Step response of the closed loop system with pure dissipation term.

Simulation

In a second instance we consider the control law devoted to energy shaping in addition to a pure dissipation term:

$$u = -k_c\left(x_{22} - x_{01}\right) - k_d\dot{x}_{22}$$

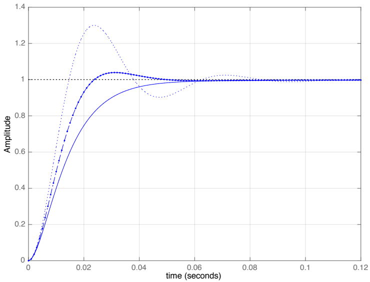


Figure: Step response of the closed loop system with state feedback.

Outline

General case: class of systems

Consider systems of two conservation laws

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{G} \\ -\mathcal{G}^* & 0 \end{pmatrix}\begin{pmatrix} \mathcal{L}_1 x_1 \\ \mathcal{L}_2 x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & I \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $x_1, x_2 \in L_2([a,b], \mathbb{R}^n)$, $\mathcal{L}_1(\zeta) > 0$ and $\mathcal{L}_2(\zeta) > 0$

$$\mathcal{G} = G_0 + G_1 \frac{\partial}{\partial \zeta} + G_2 \frac{\partial^2}{\partial \zeta^2} \qquad \mathcal{G}^* = G_0^T - G_1^T \frac{\partial}{\partial \zeta} + G_2^T \frac{\partial^2}{\partial \zeta^2}$$

with $G_0, G_1, G_2 \in \mathbb{R}^{(n,n)}$ and $\mathcal{G}^*$ is the formal adjoint of $\mathcal{G}$. This formulation allows to model a large class of systems:

- The 1D wave equation: $n = 1$, $G_0 = 0$, $G_1 = 1$, $G_2 = 0$.
- The Euler Bernouilli beam equation. In this case $n = 1$, $G_0 = 0$, $G_1 = 0$, $G_2 = 1$.
- The Timoshenko beam equation. In this case $n = 2$ and

$$G_0 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, G_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, G_2 = 0_{2,2}$$

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Distributed control

We consider the system is connected to the infinite dimensional controller

$$\frac{\partial x_c}{\partial t} = 0.\mathcal{Q}_c x_c + \mathcal{B}_c u_c \tag{26}$$

$$y_c = \mathcal{B}_c{}^* \mathcal{Q}_c x_c + \mathcal{D}_c u_c \tag{27}$$

where $\mathcal{Q}_c > 0 \in \mathbb{R}^{n \times n}$, $\mathcal{B}_c$ a differential operator operator of the form:

$$\mathcal{B}_c = B_{c0} + B_{c1} \frac{\partial}{\partial \zeta} + B_{c2} \frac{\partial^2}{\partial \zeta^2}$$

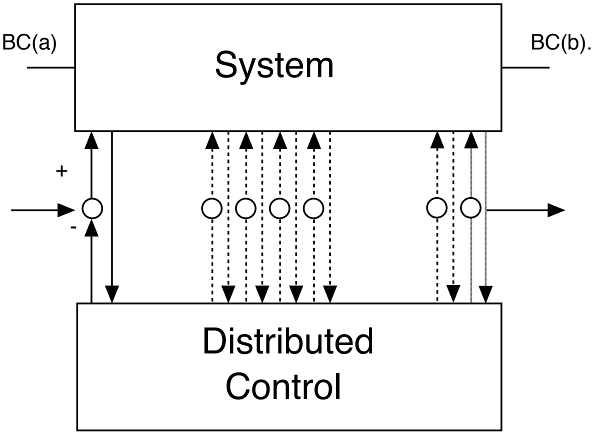
with $B_{c0}, B_{c1}, B_{c2} \in \mathbb{R}^{n \times n}$ and $\mathcal{D}_c = I \frac{\partial^2}{\partial \zeta^2}$ through the power preserving interconnection :

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u_c \\ y_c \end{pmatrix} \tag{28}$$

The energy of the controller is given by:

$$H_c(x_c) = \frac{1}{2} \int_a^b x_c^T \mathcal{Q}_c x_c d\zeta$$

Distributed control



Distributed control

Due to the power preserving interconnection

$$H_{cl}(x, x_c) = H(x) + H_c(x_c)$$

We look for closed loop structural invariants $C(x, x_c)$ to shape the closed loop energy function *i.e.*

$$\frac{dC}{dt}(x, x_c) = 0, \text{ in closed loop}$$

If these Casimir functions exist and can be written in the form

$$C(x, x_c) = \int_a^b (x_c + F(x))\, d\zeta = \kappa$$

it is possible to relate the state of the controller x_c with the state of the system x. By choosing the controller energy function, it is then possible to shape the closed loop energy function as

$$\begin{aligned} H_{cl}(x, x_c) &= H(x) + H_c(x_c) && (29) \\ &= H(x) + H_c(\kappa - F(x)) && (30) \\ &= H_{cl}(x) && (31) \end{aligned}$$

Distributed control case

The closed loop system is given by:

$$\frac{\partial x_e}{\partial t} := \begin{pmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \\ \frac{\partial x_c}{\partial t} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \mathcal{G} & 0 \\ -\mathcal{G}^* & -\mathcal{D}_c & -\mathcal{B}_c^* \\ 0 & \mathcal{B}_c & 0 \end{pmatrix}}_{\mathcal{A}_e} \begin{pmatrix} \mathcal{L}_1 x_1 \\ \mathcal{L}_2 x_2 \\ \mathcal{Q}_c x_c \end{pmatrix} \tag{32}$$

The closed loop system (32) admits structural invariants of the form

$$\kappa_0 = C(x_e) = \int_a^b \Psi^T x_e d\zeta \tag{33}$$

with $\Psi = (\psi_1, \psi_2, \psi_c)$ if and only if

Distributed control case

$$-\mathcal{G}\psi_2(\zeta) = 0 = -\mathcal{B}_c\psi_2(\zeta) \tag{34}$$

$$\mathcal{G}^*\psi_1(\zeta) - \mathcal{D}_c^*\psi_2(\zeta) + \mathcal{B}_c^*\psi_3(\zeta) = 0 \tag{35}$$

$$\left.\begin{pmatrix} 0 & G_1 & 0 \\ -G_1^T & 0 & B_{c1} \\ 0 & B_{c1}^T & 0 \end{pmatrix}\begin{pmatrix} \psi_1(\zeta) \\ \psi_2(\zeta) \\ \psi_3(\zeta) \end{pmatrix}\right|_{a,b} = 0 \tag{36}$$

$$\left.\begin{pmatrix} 0 & -G_2 & 0 \\ G_2^T & -I & -B_{c2} \\ 0 & B_{c2}^T & 0 \end{pmatrix}\begin{pmatrix} \psi_1(\zeta) \\ \psi_2(\zeta) \\ \psi_3(\zeta) \end{pmatrix}\right|_{a,b} = 0 \tag{37}$$

$$\left.\begin{pmatrix} 0 & -G_2 & 0 \\ G_2^T & -I & -B_{c2} \\ 0 & B_{c2}^T & 0 \end{pmatrix}\begin{pmatrix} \frac{d\psi_1}{d\zeta}(\zeta) \\ \frac{d\psi_2}{d\zeta}(\zeta) \\ \frac{d\psi_3}{d\zeta}(\zeta) \end{pmatrix}\right|_{a,b} = 0 \tag{38}$$

Control design

Control design

Choosing $\mathcal{B}_c = \mathcal{G}$ and appropriate initial conditions the closed loop system (32) admits as structural invariants the function $C(x_e)$ defined by (33) and

$$\Psi = (\Psi_1, 0, -\Psi_1)$$

i.e. $x_c = x_1$

The dynamic controller we consider at the end is of the form

$$\frac{\partial x_c}{\partial t} = 0.\mathcal{Q}x_c + \mathcal{G}u_c \qquad (39)$$

$$y_c = \mathcal{G}^* \mathcal{Q}_c x_c - \mathcal{D}_c u_c \qquad (40)$$

with $x_c(\zeta, 0) = x_1(\zeta, 0)$ and $\mathcal{D}_c = I\frac{\partial^2}{\partial \xi^2}$.

Control design

The closed loop system is equivalent to the system

$$\begin{pmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \\ \frac{\partial x_c}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{G} & 0 \\ -\mathcal{G}^* & \mathcal{D}_c & -\mathcal{G}^* \\ 0 & \mathcal{G} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 x_1 \\ \mathcal{L}_2 x_2 \\ \mathcal{Q}_c x_c \end{pmatrix} \tag{41}$$

- The operator $\mathcal{J}_{cl}$ defined on

$$D(\mathcal{J}_{cl}) = \left\{ x_e(\zeta, t) \in H^N\left((a,b), \mathbb{R}^{2n+1}\right) \middle| \begin{pmatrix} f_{\partial,e} \\ e_{\partial,e} \end{pmatrix} \in \ker W_e, x_c(\zeta, 0) = x_1(\zeta, 0) \right\}$$

  with $W_e^T \Sigma W_e \geq 0$ generates a contraction semigroup.
- Choosing the initial conditions such that $x_c(\zeta, 0) = x_1(\zeta, 0)$

$$\begin{pmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{G} \\ -\mathcal{G}^* & -\mathcal{D}_c \end{pmatrix} \begin{pmatrix} (\mathcal{L}_1 + \mathcal{Q}_c) x_1 \\ \mathcal{L}_2 x_2 \end{pmatrix}$$

  with boundary conditions $0 = W_e \begin{pmatrix} f_{\partial,e} \\ e_{\partial,e} \end{pmatrix}$ with $x_c(a) = x_1(a,t), x_c(b,t) = x_1(b,t)$ is asymptotically stable.

Example: 1D wave propagation of sound in a wave guide

The state variables are the kinetic momentum $\Phi(\zeta, t)$ of the air and its volumetric expansion $\Gamma(\zeta, t)$ defined on $\zeta \in [0, L]$. The total energy of the system is given by:

$$H(\Phi, \Gamma) = \frac{1}{2} \int_a^b \left(\frac{1}{\mu_0} \Phi(\zeta, t)^2 + \frac{1}{\chi_s} \Gamma(\zeta, t)^2 \right) d\zeta$$

with μ_0 the average mass density and χ_s the adiabatic compressibility factor. The co-state variables are

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \frac{\delta H}{\delta \Phi} \\ \frac{\delta H}{\delta \Gamma} \end{pmatrix} = \begin{pmatrix} \frac{1}{\mu_0} \Phi(\zeta, t) \\ \frac{1}{\chi_s} \Gamma(\zeta, t) \end{pmatrix} = \begin{pmatrix} v(\zeta, t) \\ P(\zeta, t) \end{pmatrix}$$

namely the velocity $v(\zeta, t)$ and the pressure $P(\zeta, t)$. The resulting model is given by:

$$\frac{\partial}{\partial t} \begin{pmatrix} \Gamma \\ \Phi \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial \zeta} \\ -\frac{\partial}{\partial \zeta} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\chi_s} \Gamma \\ \frac{1}{\mu_0} \Phi \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{42}$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\chi_s} \Gamma \\ \frac{1}{\mu_0} \Phi \end{pmatrix} \tag{43}$$

This model fits with the general port Hamiltonian formulation (56) with

$$P_0 = 0, P_1 = 1, P_2 = 0, \mathcal{L}_1 = \frac{1}{\mu_0}, \mathcal{L}_2 = \frac{1}{\chi_s}$$

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Example: 1D wave propagation of sound in a wave guide

The controller defined by

$$\frac{\partial x_c}{\partial t} = 0.\mathcal{L}_1 x_c + \frac{\partial u_c}{\partial \zeta}(\zeta, t) \tag{44}$$

$$y_c = -\frac{\partial\left(\mathcal{L}_1 x_c\right)}{\partial \zeta}(\zeta, t) - D\frac{\partial^2\left(\mu_0 u_c\right)}{\partial \zeta^2}(\zeta, t) \tag{45}$$

allows to transform the original system into

$$\frac{\partial \Phi}{\partial t}(\zeta, t) = D\frac{\partial^2 \Phi}{\partial \zeta^2}(\zeta, t)$$

by using the power preserving interconnection $u = -y_c, u_c = y$.

Example

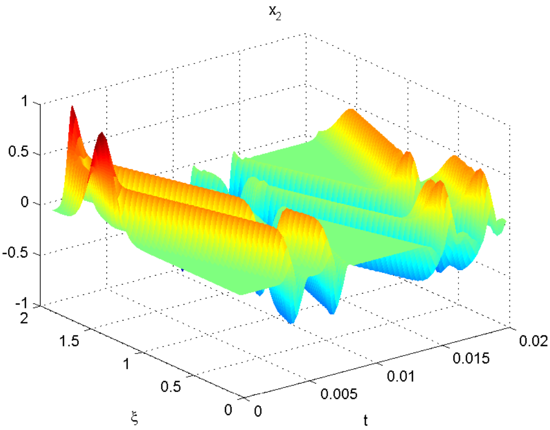

Figure: Open loop time response to initial conditions on x_2 with reflective boundary conditions.

Example

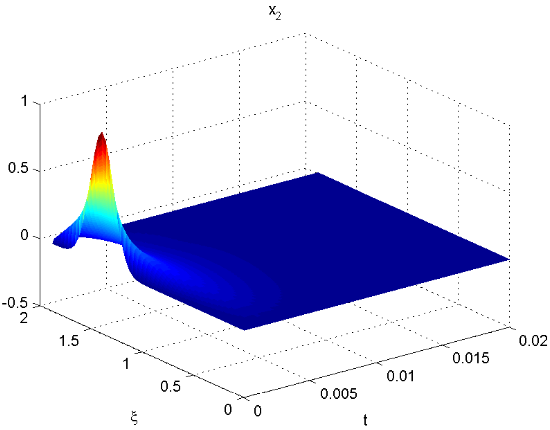

Figure: Closed loop time response to initial conditions on x_2 with reflective boundary conditions (with the dynamic feedback (44)).

Example

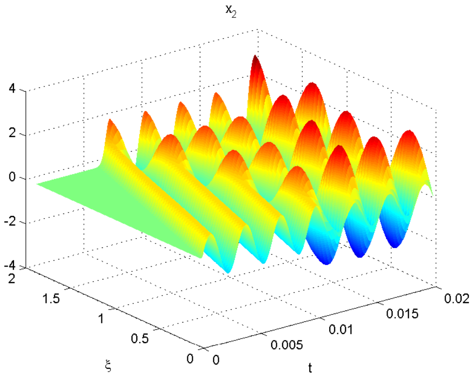

Figure: Time response to boundary input on x_2 with reflective boundary condition at point L in open loop.

Example

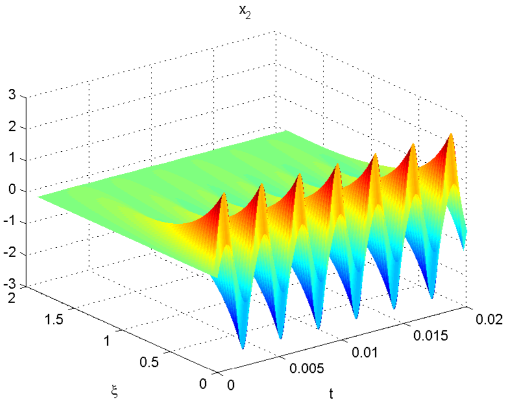

Figure: Time response to boundary input on x_2 with reflective boundary condition at point L in open loop (with the dynamic feedback (44)).

Example

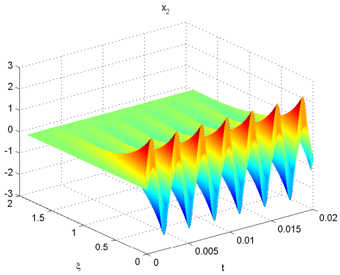

Figure: Time response to boundary input on x_2 with reflective boundary condition at point L in closed loop with a 10% (top) and 200% (bottom) variation of $\mathcal{L}_1$.

Example

The control strategy is now applied to the boundary control of the 2D wave equation

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Outline

Control of finite dimensional PHS

Stability of BCS

In-domain controlled Port Hamiltonian systems

Conclusion and future works

Conclusion and future work

Conclusion

In this talk we have:

- Given an overview on control design by energy shaping.
- Discussed the control design using structural invariants in the boundary control case.
- Discussed the distributed control design using structural invariants in the linear infinite dimensional case.
- Applied it to the 1D wave equation and checked the performances on the 2D wave equation.

Conclusion and future work

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In this talk we have:

- Given an overview on control design by energy shaping.
- Discussed the control design using structural invariants in the boundary control case.
- Discussed the distributed control design using structural invariants in the linear infinite dimensional case.
- Applied it to the 1D wave equation and checked the performances on the 2D wave equation.

Ongoing and future work

- Extension to the under actuated case.
- Use of observers to get rid of initialisation issues.
- Link with backstepping approaches.

Thank you for your attention !

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