

Tutorial 3

1. Consider a vibrating string of length one whose position is fixed at both ends. The model is given as

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2}(\zeta, t) &= c^2 \frac{\partial^2 w}{\partial \zeta^2}(\zeta, t) + \mathbb{1}_{[1/6, 1/3]}(\zeta)u_1(t) + \sin(\pi\zeta)u_2(t), \\ w(\zeta, 0) &= w_0(\zeta), \quad \frac{\partial w}{\partial t}(\zeta, 0) = w_1(\zeta) \\ w(0, t) &= 0, \quad w(1, t) = 0. \end{aligned} \tag{1}$$

$w(\zeta, t)$ represents the deviation from the rest position at spatial point $\zeta \in [0, 1]$ and time $t \geq 0$, $w_0(\zeta)$ the initial profile and $w_1(\zeta)$ is the initial velocity. Furthermore, c is a positive constant and u_1, u_2 denote the inputs.

- (a) Formulated (1) as the abstract differential equation on the state space $X = L^2(0, 1)$

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

- (b) Show that A generates a contraction semigroup on the energy space, and prove that B is a bounded, linear operator.
- (c) Now we add the output

$$y(t) = \int_0^{\frac{1}{3}} w(\zeta, t) d\zeta$$

Write this as $Cx(t)$ with C a bounded linear operator. (hard)

2. Consider the following p.d.e.

$$\frac{\partial w}{\partial t}(\zeta, t) = c \frac{\partial w}{\partial \zeta}(\zeta, t), \quad \zeta \in [0, \ell], \quad t \geq 0, \tag{2}$$

with $c < 0$ and boundary control

$$w(0, t) - w(\ell, t) = u(t) \quad t \geq 0, \tag{3}$$

and boundary measurement

$$w(\ell, t) = y(t) \quad t \geq 0, \tag{4}$$

- (a) Formulate (2)–(4) as a controlled port-Hamiltonian system and show that it is a well-defined input output system.

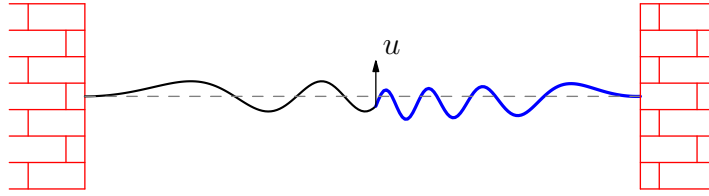


Figure 1: Vibrating string with control.

- (b) Can the change of energy be expressed in the input and output?
- 3. Consider two vibrating strings which are fixed at the boundary, and there is a control force in the middle, see Figure 1. Furthermore, we measure the velocity in the connection.
 - (a) Formulate the above as a controlled port-Hamiltonian system and show that it is a well-defined input output system.
 - (b) Can the change of energy be expressed in the input and output?