Tutorial 5: Stability, boundary control

Timoshenko beam: The Timoshenko beam equation incorporates shear and rotational inertia effects (cf. Figure 1) which makes it a more precise model than *Euler Bernoulli* model or *Rayleigh* models.

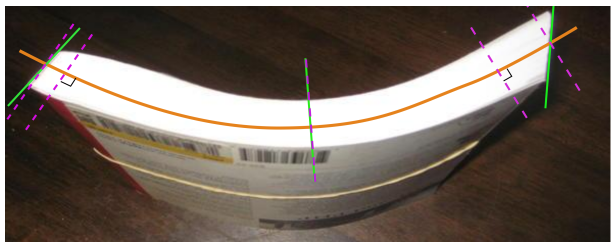

Figure 1: Timoshenko beam.@Wikipedia

It is well known on the following form

$$\begin{aligned}
\rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) &= \frac{\partial}{\partial \zeta}\left[K(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta,t)-\phi(\zeta,t)\right)\right], \quad \zeta\in(a,b),\ t\geq 0,\\
I_\rho(\zeta)\frac{\partial^2 \phi}{\partial t^2}(\zeta,t) &= \frac{\partial}{\partial \zeta}\left(EI(\zeta)\frac{\partial \phi}{\partial \zeta}(\zeta,t)\right)+K(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta,t)-\phi(\zeta,t)\right),
\end{aligned} \tag{1}$$

$w(\zeta,t)$ is the transverse displacement of the beam and $\phi(\zeta,t)$ is the rotation angle of a filament of the beam.

1. Use the following extensive variables as state variables:

$$\begin{aligned}
x_1(\zeta,t) &= \frac{\partial w}{\partial \zeta}(\zeta,t)-\phi(\zeta,t) && \text{shear displacement}\\
x_2(\zeta,t) &= \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t) && \text{momentum}\\
x_3(\zeta,t) &= \frac{\partial \phi}{\partial \zeta}(\zeta,t) && \text{angular displacement}\\
x_4(\zeta,t) &= I_\rho(\zeta)\frac{\partial \phi}{\partial t}(\zeta,t) && \text{angular momentum}
\end{aligned}$$

   to derive the port Hamiltonian formulation of this system.

2. Define the associated boundary port variables.

3. We consider that the beam is clamped on one side and controlled using force and torque proportional to velocity and angular velocity

respectively on the other side. Make explicit the input and output mappings that can be used for the feedback.

4. Write the closed loop system as a boundary control system.

5. Give the conditions such that the closed loop system is exponentially stable.