Modelling and Control of Distributed Parameter Systems: A port-Hamiltonian Approach

Stability

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Introduction

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Definition The system $\dot{x}(t) = Ax(t)$ is exponentially stable, if there exists a $M \geq 1$, $\omega < 0$ such that for all $x_0 \in X$ the following holds

$$\|x(t)\| \leq M e^{\omega t} \|x_0\|, \qquad t \geq 0.$$

Exponential stability for pH-systems

We return to our homogeneous pH system. That is, we consider

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}\left[\mathcal{H}(\zeta)x(\zeta,t)\right] + P_0\left[\mathcal{H}(\zeta)x(\zeta,t)\right] \tag{1}$$

with the boundary condition

$$W_B \begin{bmatrix} (\mathcal{H}x)(b,t) \\ (\mathcal{H}x)(a,t) \end{bmatrix} = 0, \tag{2}$$

As before/always we assume that the following hold:

- P_1 is an invertible, symmetric real $n \times n$ matrix;
- P_0 is an anti-symmetric real $n \times n$ matrix;
- For all $\zeta \in [a,b]$ the $n \times n$ matrix $\mathcal{H}(\zeta)$ is real, symmetric, and $mI \leq \mathcal{H}(\zeta) \leq MI$, for some $M, m > 0$ independent of ζ;
- W_B be a full rank real matrix of size $n \times 2n$.

Exponential stability for pH-systems

Theorem Consider the operator A associated with (1) and (2). Furthermore, we assume that next to the standard conditions the following is satisfied;

- $\mathcal{H}$ is continuously differentiable on the interval $[a, b]$.

Then, if for some positive constant k one of the following conditions is satisfied for all $x_0 \in D(A)$

$$\langle Ax_0, x_0\rangle_{\mathcal{H}} + \langle x_0, Ax_0\rangle_{\mathcal{H}} \leq -k\|(\mathcal{H}x_0)(b)\|^2 \tag{3}$$

$$\langle Ax_0, x_0\rangle_{\mathcal{H}} + \langle x_0, Ax_0\rangle_{\mathcal{H}} \leq -k\|(\mathcal{H}x_0)(a)\|^2, \tag{4}$$

the system is exponentially stable. □

Exponential stability for pH-systems

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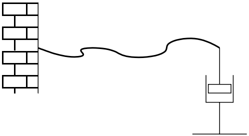
Similarly, the condition at $\zeta = a$.

Exponential stability for pH systems

Example: Damped wave equation



$$
\begin{aligned}
\frac{\partial^2 w}{\partial t^2}(\zeta,t) &= \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right] \\
\frac{\partial w}{\partial t}(1,t) &= -\alpha \cdot T(1)\frac{\partial w}{\partial \zeta}(1,t), \\
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Here is α a positive constant.

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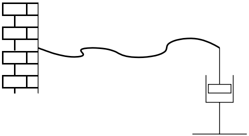

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\end{aligned}$$

Here is α a positive constant. We assume that ρ and T are continuous differentiable.

We check the contraction property. The condition on P's , $\mathcal{H}$, and number of boundary conditions are satisfied (check) and so we calculate the power balance (check)

$$\dot{H}(t) = -\alpha\left[T(1)\frac{\partial w}{\partial \zeta}(1,t)\right]^2 \leq 0.$$

So we have a contractive solution for every initial condition in X.

Exponential stability for pH systems

To conclude exponential stability, we need that

$$\dot{H}(t) \leq -k\|(\mathcal{H}x)(1,t)\|^2 = -k\left[\left(T(1)\frac{\partial w}{\partial \zeta}(1,t)\right)^2 + \frac{\partial w}{\partial t}(1,t)^2\right]$$

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Since we have that $\frac{\partial w}{\partial t}(1,t) = -\alpha \cdot T(1)\frac{\partial w}{\partial \zeta}(1,t)$, we find that

$$\begin{aligned}\left(T(1)\frac{\partial w}{\partial \zeta}(1,t)\right)^2 &+ \frac{\partial w}{\partial t}(1,t)^2 \\ &= \left(T(1)\frac{\partial w}{\partial \zeta}(1,t)\right)^2 + \alpha^2\left(T(1)\frac{\partial w}{\partial \zeta}(1,t)\right)^2 \\ &= (1+\alpha^2)\left(T(1)\frac{\partial w}{\partial \zeta}(1,t)\right)^2.\end{aligned}$$

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Combining this with the result on the previous slide, gives

Exponential stability for pH systems

$$\begin{aligned}\dot{H}(t) &= -\alpha\left[T(1)\frac{\partial w}{\partial \zeta}(1,t)\right]^2 \\ &= -\alpha \cdot \frac{1}{1+\alpha^2}\left[\left(T(1)\frac{\partial w}{\partial \zeta}(1,t)\right)^2 + \frac{\partial w}{\partial \zeta}(1,t)^2\right] \\ &= \frac{-\alpha}{1+\alpha^2}\|(\mathcal{H}x)(1,t)\|^2.\end{aligned}$$

Thus we can conclude exponential stability. □