Modelling and Control of Distributed Parameter Systems: A port-Hamiltonian Approach

Abstract Differential Equations

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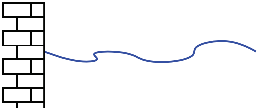
April 8, 2024

Introduction

In this first part we have seen models of physical systems, like that of the vibrating string



$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right].$$

In this part we investigate existence of solutions for (linear, time-invariant) partial differential equations.
We begin by identifying the state and state space.

Introduction: state and state space

Idea behind the state: The state is that which you have to know now to predict/know the future behaviour.

Example

The state for the mass-spring-damper system

$$M\ddot{y}(t) + D\dot{y}(t) + Ky(t) = 0$$

is the position: $y(t)$ and momentum $M\dot{y}(t)$ (or velocity $\dot{y}(t)$).
So if we have one mass, then the state space equals $X = \mathbb{R}^2$. □

What is the state for vibrating string

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left[T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) \right]?$$

Introduction: state and state space for p.d.e.

As state for the vibrating string we can take

$$w(\zeta,t) \text{ and } \frac{\partial w}{\partial t}(\zeta,t), \qquad \zeta \in (0,1)$$

or

$$\frac{\partial w}{\partial \zeta}(\zeta,t) \text{ and } \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t) \qquad \zeta \in (0,1).$$

or ......

Important: The state becomes/is a function of the spatial variable. Hence the state space is a space consisting out of functions. For instance, $X = L^2((a,b);\mathbb{R}^2)$.

$$L^2((a,b);\mathbb{R}^n) = \left\{ f : (a,b) \mapsto \mathbb{R}^n \mid \int_a^b \|f(\zeta)\|^2 d\zeta < \infty \right\}.$$

More on $L^2((a,b);\mathbb{R}^n)$

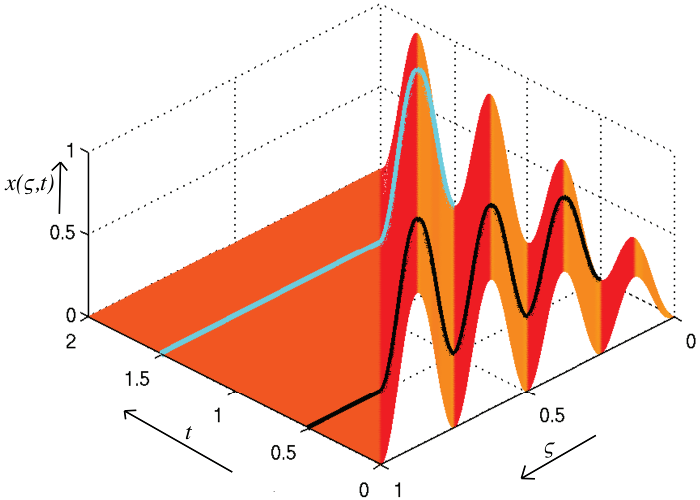
$L^2((a,b);\mathbb{R}^n)$ is a Hilbert space. That is

- There exists an inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle = \int_a^b g(\zeta)^\top f(\zeta) d\zeta.$$

  An inner product is a mapping from $X \times X$ to $\mathbb{R}$ satisfying
  - $\langle \alpha f + \beta h, g \rangle = \alpha \langle f, g \rangle + \beta \langle h, g \rangle$, $\alpha, \beta \in \mathbb{R}$;
  - $\langle f, g \rangle = \langle g, f \rangle$;
  - For $f \neq 0$, $\langle f, f \rangle > 0$.
- The norm $\| \cdot \|$ on an inner product space is given as $\|f\|^2 = \langle f, f \rangle$.
- If $\|f_n - f_m\| \to 0$ as $n, m \to \infty$, then there exists an $f \in X$ such that $\|f - f_n\| \to 0$ as $n \to \infty$.

States



Abstract state formulation

So we have now the idea that the state is (at every time) a function of the spatial variable, but how to write a p.d.e. in a state space form with such a state.
We consider an example first.

Example (Transport equation)

On the spatial domain $[0,1]$ consider the p.d.e.

$$\begin{aligned} \frac{\partial w}{\partial t}(\zeta,t) &= \frac{\partial w}{\partial \zeta}(\zeta,t), \qquad \zeta\in[0,1],\ t\geq 0,\\ w(1,t) &= 0,\\ w(\zeta,0) &= w_0(\zeta) \qquad \text{(given)}. \end{aligned}$$

□

As state $x(t)$ we choose w at the time t.

- So $x(t) = w(\cdot,t)$, or $(x(t))(\zeta) = w(\zeta,t)$.
- As state space we choose $L^2(0,1)$.
- If we now introduce $\dot{x}(t) = \frac{\partial w}{\partial t}(\cdot,t)$ and $Ax(t) = \frac{\partial w}{\partial \zeta}(\cdot,t)$, then the p.d.e. becomes

$$\dot{x}(t) = Ax(t).$$

State differential equation

So the p.d.e.

$$\begin{aligned} \frac{\partial w}{\partial t}(\zeta,t) &= \frac{\partial w}{\partial \zeta}(\zeta,t), \qquad \zeta\in[0,1],\ t\geq 0,\\ w(1,t) &= 0,\\ w(\zeta,0) &= w_0(\zeta) \qquad \text{(given)}. \end{aligned}$$

can with $x(t)=w(\cdot,t)$, $\dot{x}(t)=\frac{\partial w}{\partial t}(\cdot,t)$, and $Ax(t):=\frac{\partial w}{\partial \zeta}(\cdot,t)$, be written as the abstract differential equation:

$$\dot{x}(t)=Ax(t).$$

Where is the boundary condition?
Another problem: The derivative does not exist for all $x(t)\in L^2(0,1)$.

More on A

We see that A is a mapping working for a fixed t, i.e., so for $f \in L^2(0,1)$ we can define Af as

$$(Af)(\zeta) = \frac{df}{d\zeta}(\zeta)$$

We want that A maps into X, and so we only take the derivative of $f \in X$ when the answer lies in X again. So

$$D(A) = \{f \in X \mid \frac{df}{d\zeta} \in X, f(1) = 0\}.$$

Since the boundary condition is an essential part of the p.d.e. and since it is a condition in the spatial direction, It is added to the domain of A.

Summary on A

So the p.d.e.

$$\begin{aligned}\frac{\partial w}{\partial t}(\zeta,t) &= \frac{\partial w}{\partial \zeta}(\zeta,t), \qquad \zeta\in[0,1],\ t\geq 0,\\ w(1,t) &= 0\\ w(\zeta,0) &= w_0(\zeta)\end{aligned}$$

is written as the abstract differential equation:

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 = w_0$$

with $x(t) = w(\cdot,t) \in X = L^2(0,1)$, and

$$\left(Af\right)(\zeta) = \frac{df}{d\zeta}(\zeta) \quad \text{with domain:}$$

$$D(A) = \{f \in X \mid \frac{df}{d\zeta} \in X, f(1) = 0\}.$$

Solutions of p.d.e.'s

Consider a p.d.e. such as

$$\frac{\partial w}{\partial t}(\zeta, t) = \frac{\partial w}{\partial \zeta}(\zeta, t), \qquad \zeta \in [0, 1],\ t \geq 0,$$

with boundary condition $w(1, t) = 0$. We say that $w(\zeta, t)$ is a classical solution if it is differentiable with respect to the time and spatial variable, satisfies the p.d.e., and satisfies the boundary condition.

However, to study general existence of (linear) p.d.e.'s this concept is not sufficient.

Solutions of p.d.e.'s

Consider the p.d.e.

$$\frac{\partial w}{\partial t}(\zeta,t) = \frac{\partial w}{\partial \zeta}(\zeta,t), \quad \zeta \in [0,1],\ t \geq 0, \quad w(1,t) = 0.$$

We take a smooth test function $\phi(\zeta)$ and integrate over the spatial domain.

$$\begin{aligned}
\int_0^1 \phi(\zeta)\frac{\partial w}{\partial t}(\zeta,t)d\zeta &= \int_0^1 \phi(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)d\zeta \quad \text{(p.d.e.)} \\
\text{(int. by parts)} \quad &= [\phi(\zeta)w(\zeta,t)]_0^1 - \int_0^1 \dot{\phi}(\zeta)w(\zeta,t)d\zeta \\
\text{(b.c.)} \quad &= -\phi(0)w(0,t) - \int_0^1 \dot{\phi}(\zeta)w(\zeta,t)d\zeta.
\end{aligned}$$

If we take test functions with $\phi(0) = 0$, we find

Solutions of p.d.e.'s

$$\frac{d}{dt}\int_0^1 \phi(\zeta)w(\zeta,t)d\zeta = \int_0^1 \phi(\zeta)\frac{\partial w}{\partial t}(\zeta,t)d\zeta = -\int_0^1 \dot{\phi}(\zeta)w(\zeta,t)d\zeta.$$

Integrate this expression with respect to time

$$\int_0^1 \phi(\zeta)w(\zeta,t_f)d\zeta - \int_0^1 \phi(\zeta)w(\zeta,0)d\zeta = -\int_0^{t_f}\int_0^1 \dot{\phi}(\zeta)w(\zeta,t)d\zeta.$$

You see there are no derivatives of w taken anymore.
Now we call $w(\zeta,t)$ a weak or mild solution of the p.d.e. if the above equation is satisfied for all smooth test functions ϕ satisfying $\phi(0)=0$.

Weak and classical solutions of p.d.e.'s

It is easy to see that a classical solution is always a weak solution, but the converse need not to hold.
We will now study when a p.d.e. has a weak solution.
Note there is a difference between knowing the existence of a solution and having the form/expression of the solution. The expression for the solution can be hard/impossible to find. So we concentrate on existence.
We concentrate on solutions satisfying the additional property that

$$\|x(t)\| \leq \|x_0\| \quad \forall t > 0 \qquad \text{(contraction)}$$

Existence of solution, contractions

Theorem (Lumer-Phillips)

Let A be a densely defined operator, then $\dot{x}(t) = Ax(t), x(0) = x_0$ has for every $x_0 \in X$ a unique weak solution satisfying $\|x(t)\| \leq \|x_0\|$ for all $t \geq 0$ if and only if

1. $\langle Ax_0, x_0 \rangle + \langle x_0, Ax_0 \rangle \leq 0$ *for all* $x_0 \in D(A)$.
2. *The range of $A - I$ is the whole of X.*

□

Example

Consider on the state space $X = L^2(0,1)$ the operator A which is given as

$$Af = \frac{df}{d\zeta}, \qquad \zeta \in [0,1]$$

with the domain

$$D(A) = \Big\{ f \in L^2(0,1) \mid f \text{ is absolutely continuous, } \frac{df}{d\zeta} \in L^2(0,1) \text{ and } f(1) = 0 \Big\}.$$

Let us check the properties:

Example: Contractive weak solution

- A is densely defined in $L^2(0,1)$.
- $$\begin{aligned}
&\langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle \\
&\qquad = \int_0^1 \frac{dx_0}{d\zeta}(\zeta)\overline{x_0(\zeta)}d\zeta + \int_0^1 x_0(\zeta)\overline{\frac{dx_0}{d\zeta}(\zeta)}d\zeta \\
&\qquad = \int_0^1 \frac{d}{d\zeta}\left[x_0(\zeta)\overline{x_0(\zeta)}\right]d\zeta \\
&\qquad = |x_0(\zeta)|^2\Big|_0^1 \\
&\qquad = 0 - |x_0(0)|^2 \leq 0.
\end{aligned}$$
- To see if the range of $(A - I)$ is everything, we have for every $f \in L^2(0,1)$ to solve $(A - I)z = f$.

Example: Contractive weak solution

Solving $(A - I)z = f$ means solving

$$\frac{dz}{d\zeta}(\zeta) - z(\zeta) = f(\zeta), \qquad \zeta \in (0,1)$$

with boundary condition $z(1) = 0$. The solution of this differential equation with the given boundary value is

$$z(\zeta) = -\int_{\zeta}^{1} e^{\zeta - \tau} f(\tau) d\tau.$$

Example

Conclusion: The abstract differential equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

with

$$Af = \frac{df}{d\zeta}, \qquad \zeta \in [0,1]$$

and domain

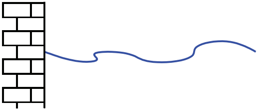
$$D(A) = \left\{ f \in L^2(0,1) \mid \frac{df}{d\zeta} \in L^2(0,1) \text{ and } f(1) = 0 \right\}$$

possesses for every $x_0 \in X = L^2(0,1)$ a unique weak solution in X which is not growing in norm. □

Port-Hamiltonian Systems

Homogeneous equation

The wave equation



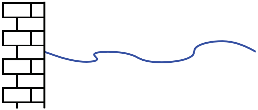
$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left[T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) \right].$$

We want to write this p.d.e. as a state differential equation, $\dot{x}(t) = Ax(t)$. Therefor we need

- The state x
- The state space X.
- The "system" operator A with its domain $D(A)$.

To answer the first two questions, we look at the energy associated to vibrating string.

The wave equation, energy



$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right].$$

The energy is given by

$$H(t) = \frac{1}{2}\int_0^1 \rho(\zeta)\left(\frac{\partial w}{\partial t}(\zeta,t)\right)^2 + T(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta,t)\right)^2 d\zeta$$

with ρ is the mass density, and T is Young's modulus.
This looks like an $L^2((0,1);\mathbb{R}^2)$-norm (squared) in the variables $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial \zeta}$.
This indicates a choice for the state variables. We choose $x_1 := \rho\dfrac{\partial w}{\partial t}$ (the momentum), $x_2 := \dfrac{\partial w}{\partial \zeta}$ (the strain).

The wave equation, state

With the choice $x_1 := \rho \frac{\partial w}{\partial t}$ (the momentum), $x_2 := \frac{\partial w}{\partial \zeta}$ (the strain), the energy

$$H(t) = \frac{1}{2} \int_0^1 \rho(\zeta) \left(\frac{\partial w}{\partial t}(\zeta, t) \right)^2 + T(\zeta) \left(\frac{\partial w}{\partial \zeta}(\zeta, t) \right)^2$$

becomes

$$H(t) = \frac{1}{2} \int_0^1 \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} d\zeta.$$

The wave equation, state and state space

Based on the (quadratic) expression of the energy

$$H(t) = \frac{1}{2}\int_0^1 \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} d\zeta,$$

we choose as state space

$$X = L^2((0,1);\mathbb{R}^2)$$

with inner product

$$\begin{aligned} \langle f, g\rangle_X &= \frac{1}{2}\int_0^1 \begin{bmatrix} f_1(\zeta) \\ f_2(\zeta) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} \begin{bmatrix} g_1(\zeta) \\ g_2(\zeta) \end{bmatrix} d\zeta \\ &= \frac{1}{2}\int_0^1 f(\zeta)^\top \mathcal{H}(\zeta) g(\zeta) d\zeta. \end{aligned}$$

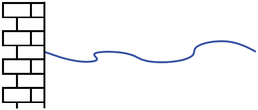
The wave equation, state and state space

With the inner product

$$\langle f, g\rangle_X = \frac{1}{2}\int_0^1 f(\zeta)^\top \mathcal{H}(\zeta) g(\zeta) d\zeta$$

we see that $\|f\|_X^2$ is precisely the energy.
So our state X is also called the energy space, i.e, the space consisting of all states/shapes/$\cdots\cdots$ with finite energy.
Next we rewrite the p.d.e. model of the vibrating string in our state variables.

The wave equation, state differential equation



$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right]$$

With the state variables $x_1 = \rho\frac{\partial w}{\partial t}$ and $x_2 = \frac{\partial w}{\partial \zeta}$ we can write the above p.d.e. as

$$\begin{aligned}\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(\zeta,t) &= \begin{bmatrix} \rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) \\ \frac{\partial^2 w}{\partial t \partial \zeta}(\zeta,t) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right] \\ \frac{\partial}{\partial \zeta}\left[\frac{\partial w}{\partial t}\right](\zeta,t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial \zeta}\left(\underbrace{\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}}_{=\mathcal{H}} x(\zeta,t)\right).\end{aligned}$$

The wave equation, state differential equation

With the state variables $x_1 = \rho \frac{\partial w}{\partial t}$ and $x_2 = \frac{\partial w}{\partial \zeta}$ we can write the above p.d.e. as

$$\frac{\partial}{\partial t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (\zeta, t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{= P_1} \frac{\partial}{\partial \zeta} \left(\underbrace{\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}}_{= \mathcal{H}} x(\zeta, t) \right).$$

We generalise this to our class of first order port-Hamiltonian equations.

Port-Hamiltonian partial differential equations

Our model class are p.d.e.'s of the form

$$\frac{\partial x}{\partial t}(\zeta, t) \quad = \quad \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right) [\mathcal{H} x(\zeta, t)]$$

with

- $x(\zeta, t) \in \mathbb{R}^n$, $\zeta \in [a, b]$, $t \geq 0$
- P_1 is an invertible, symmetric real $n \times n$-matrix,
- P_0 is a skew-symmetric real $n \times n$-matrix,
- $\mathcal{H}(\zeta)$ is a symmetric, invertible $n \times n$-matrix with $mI \leq \mathcal{H}(\zeta) \leq MI$ for some $m, M > 0$.

The energy/Hamiltonian is defined as

$$H(t) = H(x(\cdot, t)) = \frac{1}{2} \int_a^b x(\zeta, t)^T \mathcal{H}(\zeta) x(\zeta, t) d\zeta.$$

Power balance

For the Port-Hamiltonian p.d.e. with energy/Hamiltonian

$$H(x(\cdot,t)) = \frac{1}{2}\int_a^b x(\zeta,t)^T \mathcal{H}(\zeta) x(\zeta,t) d\zeta,$$

it is not hard to show that along solutions; homework

$$\dot{H}(t) = \frac{dH}{dt}(x(\cdot,t)) = \frac{1}{2}\left[(\mathcal{H}x)^T(\zeta,t) P_1 (\mathcal{H}x)(\zeta,t) \right]_a^b$$

Thus the change of internal energy goes via the boundary of the spatial domain, i.e. power balance.

Port-Hamiltonian partial differential equations

Given our port-Hamiltonian partial differential equation

$$\frac{\partial x}{\partial t}(\zeta, t) = \left(P_1 \frac{\partial}{\partial \zeta} + P_0 \right) [\mathcal{H}(\zeta) x(\zeta, t)]$$

with the properties on P_0, P_1 and $\mathcal{H}$.
We need to add boundary conditions to this p.d.e. That are conditions in $x(\zeta, t)$ for ζ equal to a or b.
We write these boundary conditions as

$$W_B \begin{bmatrix} \mathcal{H}(b) x(b, t) \\ \mathcal{H}(a) x(a, t) \end{bmatrix} = 0.$$

with W_B a matrix.
Question: Which boundary conditions lead to unique (weak) solutions?
We answer this question only for the (important) contractive case.

Port-Hamiltonian p.d.e., state space

Given our port-Hamiltonian partial differential equation with boundary conditions

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)[\mathcal{H}(\zeta)x(\zeta,t)] \\ 0 &= W_B \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix} \end{aligned}$$

with the properties on P_0, P_1 and $\mathcal{H}$.

- As state we choose $x(t) = x(\cdot,t)$.
- As state space we choose the energy space, i.e., $X = L^2((0,1);\mathbb{R}^n)$ with inner product

$$\langle f, g\rangle_X = \frac{1}{2}\int_a^b f(\zeta)^\top \mathcal{H}(\zeta) g(\zeta) d\zeta.$$

Port-Hamiltonian p.d.e., state space formulation

With the state $x(t) = x(\cdot, t)$ and $X = L^2((0,1);\mathbb{R}^n)$ our port-Hamiltonian p.d.e. with boundary conditions;

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)[\mathcal{H}(\zeta)x(\zeta,t)] \\ 0 &= W_B \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix} \end{aligned}$$

becomes

$$\dot{x}(t) = Ax(t),$$

where

$$Ax = \left(P_1 \frac{d}{d\zeta} + P_0\right)[\mathcal{H}x]$$

with domain

$$D(A) = \left\{ x \in X \mid \frac{d}{d\zeta}(\mathcal{H}x) \in X, W_B \begin{bmatrix} \mathcal{H}(b)x(b) \\ \mathcal{H}(a)x(a) \end{bmatrix} = 0 \right\}.$$

Port-Hamiltonian p.d.e., state space formulation

To check whether the abstract differential equation $\dot{x}(t) = Ax(t)$ has unique weak solution, we recall

Theorem (Lumer-Phillips)

Let A be a densely defined operator, then $\dot{x}(t) = Ax(t), x(0) = x_0$ has for every $x_0 \in X$ a unique weak solution satisfying $\|x(t)\| \leq \|x_0\|$, $t \geq 0$ if and only if

1. $\langle Ax_0, x_0 \rangle + \langle x_0, Ax_0 \rangle \leq 0$ *for all* $x_0 \in D(A)$.
2. *The range of* $A - I$ *is the whole of* X.

□

For our class of port-Hamiltonian p.d.e.'s we now have

Port-Hamiltonian p.d.e., existence of solutions

Theorem (Le Gorrec, Maschke & Z. '05, Jacob & Z. '11)

Assume the (standard) conditions on P_0, P_1 *and* $\mathcal{H}$. *Assume further that* W_B *is a* $n \times 2n$ *matrix of full rank.*
Then $\dot{x}(t) = Ax(t), x(0) = x_0$ *has for every* $x_0 \in X$ *a unique weak solution satisfying* $\|x(t)\| \leq \|x_0\|$, $t \geq 0$, *if and only if*

$$\langle Ax_0, x_0 \rangle + \langle x_0, Ax_0 \rangle \leq 0 \text{ for all } x_0 \in D(A).$$

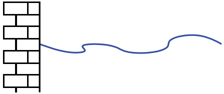
The latter is equivalent to $\dot{H}(0) \leq 0$ *for all* $x_0 \in D(A)$.
Moreover, $\dot{x}(t) = Ax(t), x(0) = x_0$ *has for every* $x_0 \in X$ *a unique weak solution satisfying* $\|x(t)\| = \|x_0\|$, $t \in \mathbb{R}$, *if and only if*

$$\langle Ax_0, x_0 \rangle_X + \langle x_0, Ax_0 \rangle = 0 \text{ for all } x_0 \in D(A)$$

or (equivalently) $\dot{H}(0) = 0$ *for all* $x_0 \in D(A)$.

Hence only the simple condition of L-P theorem needs to be checked.

Example: the wave equation



$$\begin{aligned}\frac{\partial^2 w}{\partial t^2}(\zeta,t) &= \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right]\\ \frac{\partial w}{\partial t}(0,t) &= T(1)\frac{\partial w}{\partial \zeta}(1,t) = 0\end{aligned}$$

We begin by writing the boundary conditions with the space variable $x_1 = \rho\frac{\partial w}{\partial t}$, $x_2 = \frac{\partial w}{\partial \zeta}$,

$$\begin{aligned}\begin{bmatrix}0\\0\end{bmatrix} &= \begin{bmatrix}T(1)\frac{\partial w}{\partial \zeta}(1,t)\\ \frac{\partial w}{\partial t}(0,t)\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}}_{=W_B}\begin{bmatrix}\frac{\partial w}{\partial t}(1,t)\\ T(1)\frac{\partial w}{\partial \zeta}(1,t)\\ \frac{\partial w}{\partial t}(0,t)\\ T(0)\frac{\partial w}{\partial \zeta}(0,t)\end{bmatrix}\\ &= \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\mathcal{H}(1)x(1,t)\\ \mathcal{H}(0)x(0,t)\end{bmatrix}.\end{aligned}$$

Example: the wave equation

Now we check the conditions.

- $P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an invertible 2×2 matrix $(n = 2)$.
- $P_0 = 0$, so skew-symmetric.
- If $0 < m \leq T(\zeta), \rho(\zeta)^{-1} \leq M$ for all ζ, then $\mathcal{H}(\zeta) = \begin{bmatrix} \rho(\zeta)^{-1} & 0 \\ 0 & T(\zeta) \end{bmatrix}$ satisfy $mI_2 \leq \mathcal{H}(\zeta) \leq MI_2$.
- W_B has rank 2.
- $\dot{H}(0) = 0$.

Thus our pH system $\dot{x}(t) = Ax(t), x(0) = x_0$ has for every $x_0 \in X$ a unique weak solution for $t \in \mathbb{R}$ with constant energy.

Solution operators

Beyond e^{At}.

Finite-dimensional

If we have a (finite-dimensional) abstract differential equation, such as

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \in \mathbb{R}^n$$

with A a given $n \times n$ matrix, then we know that the solution is given as

$$x(t) = e^{At}x_0.$$

For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix},$$

then

$$e^{At} = \begin{bmatrix} e^t & e^{3t} - e^t \\ 0 & e^{3t} \end{bmatrix}.$$

Question: Do we have something like this for our abstract differential equation on a Hilbert space X?

Properties of e^{At}

We know that e^{At} has the following properties:

- e^{At} is linear. That is because $\alpha x_0 + \beta \tilde{x}_0 \mapsto \alpha x(t) + \beta \tilde{x}(t)$.
- $e^{A0} = I$ (the identity). Because $x_0 = x(0) = e^{A0} x_0$.
- $e^{A(t_1+t_2)} = e^{At_1} e^{At_2}$, because of the time invariance of $\dot{x}(t) = Ax(t)$, any time may be chosen as initial time.

Two observations for the abstract differential equation $\dot{x}(t) = Ax(t)$, $x(0) = x_0 \in X$;

- We have (under some conditions) a solution $x(t), t \geq 0$, satisfying $x(0) = x_0$.
- The abstract differential equation is linear and time-invariant.

We introduce the solution map, that is the map from initial condition to state at time t.

Semigroup

We denote the state space by X. Thus our solution map

$$X \ni x_0 \mapsto x(t) = T(t)x_0 \in X$$

What can we say about this mapping when the underlying differential equation is linear and time-invariant?

Properties

- $T(t)$ is linear. That is $\alpha x_0 + \beta \tilde{x}_0 \mapsto \alpha x(t) + \beta \tilde{x}(t)$.
- $T(0) = I$ (the identity)
- $T(t_1 + t_2) = T(t_1)T(t_2)$, because of the time invariance, any time may be chosen as initial time.

It turned out that an additional property is needed.

Bounded operators

We assume that X, our state space, is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$.
We introduce some notation. $\mathcal{L}(X)$ denotes the set of linear and bounded operators from X to X.Thus if $Q \in \mathcal{L}(X)$, then

- $Q(\alpha x_0 + \beta \tilde{x}_0) = \alpha Q(x_0) + \beta Q(\tilde{x}_0)$, for all $x_0, \tilde{x}_0 \in X$ and $\alpha, \beta \in \mathbb{R}$, and
- there exists a $q \geq 0$ such that for all $x_0 \in X$,

$$\|Q(x_0)\| \leq q\|x_0\|.$$

The norm of $Q \in \mathcal{L}(X)$ is given as

$$||Q|| = \sup_{x_0 \in X, x_0 \neq 0} \frac{\|Qx_0\|}{\|x_0\|}.$$

Bounded operators, example

Example

Take $X = L^2(0,\infty)$ and $(Qf)(\zeta) = f(2\zeta+1) + 3f(\zeta)$.

- Linearity

$$\begin{aligned}(Q(\alpha f+\beta g))(\zeta) &= (\alpha f + \beta g)(2\zeta+1) + 3(\alpha f + \beta g)(\zeta) \\ &= \alpha f(2\zeta+1) + \beta g(2\zeta+1) + 3\alpha f(\zeta) + 3\beta g(\zeta) \\ &= \alpha\left[f(2\zeta+1) + 3f(\zeta)\right] + \beta\left[g(2\zeta+1) + 3g(\zeta)\right] \\ &= \alpha(Q(f))(\zeta) + \beta(Q(g))(\zeta).\end{aligned}$$

- Boundedness

$$\|Qf\| = \sqrt{\int_0^\infty (f(2\zeta+1) + 3f(\zeta))^2 d\zeta}$$

Bounded operators, example

We have

$$\begin{aligned}
\|Qf\| &= \sqrt{\int_0^\infty (f(2\zeta+1)+3f(\zeta))^2 d\zeta} \\
&\leq \sqrt{\int_0^\infty f(2\zeta+1)^2 d\zeta} + \sqrt{\int_0^\infty (3f(\zeta))^2 d\zeta} \\
&= \sqrt{\frac{1}{2}\int_1^\infty f(\tau)^2 d\tau} + 3\|f\| \\
&\leq \left[3+\frac{1}{2}\sqrt{2}\right]\|f\|.
\end{aligned}$$

So $Q \in \mathcal{L}(X)$. With some more work we can show that

$$\|Q\| = 3+\frac{1}{2}\sqrt{2}.$$

C_0-semigroups

Definition

A strongly continuous semigroup (C_0-semigroup) is an operator valued function, $(T(t))_{t\geq 0}$, from $[0,\infty)$ to $\mathcal{L}(X)$ which satisfies

- $T(0) = I$
- $T(t)T(s) = T(t+s), \quad t,s \in [0,\infty)$
- For all $x_0 \in X$ there holds

$$\lim_{t\downarrow 0} T(t)x_0 = x_0.$$

□

Semigroup

So the only "unexpected" property is

$$T(t)x_0 \to x_0 \quad \text{if} \quad t \downarrow 0$$

This the strong continuity.
It tells that the solution becomes more and more the initial state when time get smaller and smaller.

Contraction semigroup

In the previous we have shown existence of weak solutions with the additional property that

$$\|x(t)\| \leq \|x_0\| \text{ for all } x_0 \in X \quad \text{and } t \geq 0.$$

Using the semigroup $T(t)$ this means that

$$\|T(t)x_0\| \leq \|x_0\| \text{ for all } x_0 \in X \text{ and } t \geq 0.$$

Or equivalently,

$$\|T(t)\| \leq 1 \text{ for all } t \geq 0.$$

These C_0 semigroups have a special name.

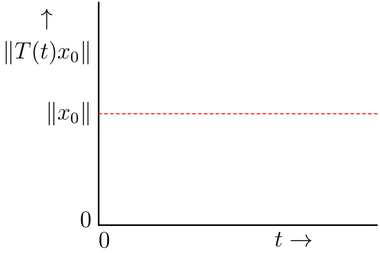
Contraction semigroup

Definition

The C_0-semigroup $(T(t))_{t\geq 0}$ is contraction semigroup if

$$\|T(t)x_0\| \leq \|x_0\| \qquad \text{for all } t \geq 0 \text{ and for all } x_0 \in X.$$

□



Summary: A and $T(t)$

Now we have of a p.d.e. two concepts

- $(T(t))_{t\geq 0}$; solution map, i.e., $x(t) = T(t)x_0$ is the solution, and
- A; used to rewrite the p.d.e. into $\dot{x}(t) = Ax(t)$.

Question: Can we find the one given the other?
Answer: From A to $T(t)$ very hard/ impossible. That is why we focus on existence.
To answer the other implication we look at the finite-dimensional case once more.

Finding A

Let A be given as

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix},$$

then

$$e^{At} = \begin{pmatrix} e^t & e^{3t} - e^t \\ 0 & e^{3t} \end{pmatrix}.$$

Problem: Suppose now that you know only e^{At}. How would you find A back?

Answer Evaluate the derivative of the semigroup at $t = 0$.

Since $\frac{d}{dt}e^{At} = Ae^{At}$, we have

$$\frac{d}{dt}e^{At} \mid_{t=0} = A.$$

A and $T(t)$

Theorem

Assume that $(T(t))_{t\geq 0}$ is the solution map of our p.d.e., then for those $x_0 \in X$ for which the following limit exists

$$\lim_{t\downarrow 0} \frac{T(t)x_0 - x_0}{t},$$

we have that

$$Ax_0 = \lim_{t\downarrow 0} \frac{T(t)x_0 - x_0}{t}.$$

Furthermore, $D(A)$ consists of precisely those $x_0 \in X$ for which the limit exists.
A is named the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$. □

A and $T(t)$

Lemma

If $x_0 \in D(A)$, then for $t > 0$, $T(t)x_0$ is differentiable, and

$$\frac{d}{dt}\left(T(t)x_0\right) = AT(t)x_0.$$

So $x(t) := T(t)x_0$ is the solution (classical) of

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

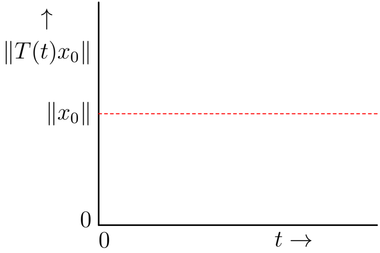
For $x_0 \in X$, $T(t)x_0$ is the weak solution. □

Contraction semigroup

Definition
The C_0-semigroup $(T(t))_{t\geq 0}$ is contraction semigroup if

$$\|T(t)x_0\| \leq \|x_0\| \qquad \text{for all } t \geq 0 \text{ and for all } x_0 \in X.$$

$\square$



What can we say about the A's of these semigroups?

Contraction semigroup

We known that

$$\|T(t)x_0\|^2 = \langle T(t)x_0, T(t)x_0 \rangle.$$

For $x_0 \in D(A)$, we have that the derivative of $T(t)x_0$ equals $AT(t)x_0$.
So if we differentiate $\|T(t)x_0\|^2$, we find

$$\frac{d}{dt}\|T(t)x_0\|^2 = \langle AT(t)x_0, T(t)x_0 \rangle + \langle T(t)x_0, AT(t)x_0 \rangle.$$

Contraction semigroup

So we know:

$$\frac{d}{dt}\|T(t)x_0\|^2 = \langle AT(t)x_0, T(t)x_0\rangle + \langle T(t)x_0, AT(t)x_0\rangle.$$

Now we choose $t = 0$. We know that $T(0)x_0 = x_0$. Thus at time equal to zero, we find

$$\frac{d}{dt}\left(\|T(t)x_0\|^2\right)\bigg|_{t=0} = \langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle.$$

So if $T(t)$ is a contraction semigroup, then

$$\langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle = \frac{d}{dt}\|T(t)x_0\|^2 \mid_{t=0} \leq 0.$$

This has to hold for all $x_0 \in D(A)$.

Thanks for your attention!