Modelling and Control of Nonlinear and Distributed Parameter Systems: The port-Hamiltonian Approach

Inputs and Outputs

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Port-Hamiltonian systems with inputs and outputs

We are interested in boundary controls and boundary observations.

$$\frac{\partial x}{\partial t}(\zeta,t) = \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)[\mathcal{H}x(t)]$$

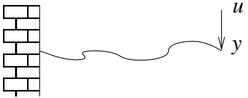
$$u(t) = W_{B,1}\begin{bmatrix}(\mathcal{H}x)(b,t)\\(\mathcal{H}x)(a,t)\end{bmatrix},$$

$$0 = W_{B,2}\begin{bmatrix}(\mathcal{H}x)(b,t)\\(\mathcal{H}x)(a,t)\end{bmatrix},$$

$$y(t) = W_C\begin{bmatrix}(\mathcal{H}x)(b,t)\\(\mathcal{H}x)(a,t)\end{bmatrix}.$$

Port-Hamiltonian systems with inputs and outputs

Example: Wave equation



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\begin{aligned}
\frac{\partial^2 w}{\partial t^2}(\zeta, t) &= \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta, t)\right] \\
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0 &= \frac{\partial w}{\partial t}(0, t) \\
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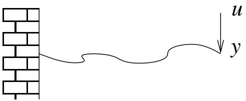

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Question: Is this a well-posed linear system?

Well-posedness of port-Hamiltonian systems

State space $X = L^2((a,b);\mathbb{R}^n)$ with (the energy) norm

$$\|f\|_X^2 = \frac{1}{2}\int_a^b f(\zeta)^T \mathcal{H}(\zeta) f(\zeta) d\zeta.$$

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Definition

The port-Hamiltonian system is called well-posed, if

- $Ax = P_1 \frac{d}{d\zeta}[\mathcal{H}x] + P_0[\mathcal{H}x]$ with domain
  $$D(A) = \left\{ x \in X \mid \frac{d}{d\zeta}\mathcal{H}x \in X, \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} = 0 \right\}$$
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  is the generator of a C_0-semigroup on X.
- There are $t_0, m_{t_0} > 0$:
  $$\|x(t_0)\|_X^2 + \int_0^{t_0} \|y(t)\|^2 dt \le m_{t_0} \left[\|x(0)\|_X^2 + \int_0^{t_0} \|u(t)\|^2 dt \right]$$

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Theorem (Z, Le Gorrec, Maschke, Villegas '10)

If $Ax = \left(P_1 \frac{d}{d\zeta} + P_0\right)[\mathcal{H}x]$ *generates a* C_0*-semigroup, then the port-Hamiltonian system is well-posed.*

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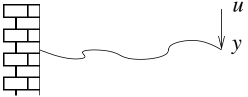
Remark: We even have a regular system.

Example: Wave equation



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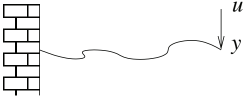

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$$P_1\mathcal{H} = \begin{bmatrix} 0 & T \\ \frac{1}{\rho} & 0 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma \\ \frac{1}{\rho} & \frac{1}{\rho} \end{bmatrix} \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix} \begin{bmatrix} \frac{1}{2\gamma} & \frac{\rho}{2} \\ -\frac{1}{2\gamma} & \frac{\rho}{2} \end{bmatrix} = S^{-1}\Delta S,$$

with $\gamma > 0$ und $\gamma^2 = \frac{T}{\rho}$.

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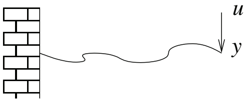

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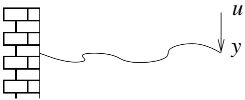

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So if T and ρ are continuously differentiable, then the controlled wave equation is well-posed.

Inputs and Outputs

Transfer Functions

Introduction

The aim of this part is to define transfer function for systems described by partial differential equations.
We derive these transfer functions via a very simple calculation.
For port-Hamiltonian systems we show that the energy/power balance induces properties on the transfer function.

Transfer function for an o.d.e.

Consider the simple system described by the ordinary differential equation

$$\dot{y}(t) + 5y(t) = 3u(t),$$

the transfer function of this system is given by

$$G(s) = \frac{3}{s+5}.$$

How do you come to this?

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- Exponential solutions.

Exponential solutions

One way for obtaining the transfer function of

$$\dot{y}(t) + 5y(t) = 3u(t)$$

is to take $u(t) = e^{st}$, $s \in \mathbb{C}$, and to try to find a solution of the same format, i.e., $y(t) = \alpha e^{st}$.

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$$s\alpha + 5\alpha = 3.$$

If $s \neq -5$, this is solvable;

$$\alpha = \frac{3}{s+5}.$$

Exponential solutions

So if we want to find an exponential solution

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- It is possible for all $s \in \mathbb{C}$ except for $s = -5$.
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- The α equals
$$\alpha = \frac{3}{s+5}.$$
- We call this the transfer function at s.

Transfer function via exponential solutions

Definition

Given an (abstract) differential equation in the variables $(u(t), z(t), y(t))$, where $u(t)$, $z(t)$, and $y(t)$ take their values in the (Hilbert) spaces U, Z, and Y, respectively.
Let $s \in \mathbb{C}$. If for every $u_0 \in U$, there exists a unique solution of the form $(u_0 e^{st}, z_0 e^{st}, y_0 e^{st})$, and the mapping $u_0 \mapsto y_0$ is linear and bounded, then this mapping is called the transfer function at s, and will be denoted by $G(s)$. $\square$

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We call a solution of the form $(u_0 e^{st}, z_0 e^{st}, y_0 e^{st})$ an exponential solution.

Transfer function for state linear systems

Consider the state differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

with A, B, C, and D matrices.
Let $s \in \mathbb{C}$, and $u_0 \in U$. We try to find a solution of the form $(u(t), x(t), y(t)) = (u_0 e^{st}, x_0 e^{st}, y_0 e^{st})$.

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Since e^{st} is never zero, this is equivalent to:

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This clearly defines a bounded linear mapping from u_0 to y_0, and so the transfer function at s is given by

$$G(s) = C(sI - A)^{-1}B + D.$$

This holds for all
$s \in \rho(A) := \{s \in \mathbb{C} \mid (sI - A)^{-1}$ exists as bounded operator$\}$.

Example

We take a heated bar. We heat it uniformly at one half, and we measure (half) the average temperature in the other half;

$$\begin{aligned}\frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial^2 x}{\partial \zeta^2}(\zeta,t) + \mathbb{1}_{[\frac{1}{2},1]}(\zeta)u(t)\\ \frac{\partial x}{\partial \zeta}(0,t) &= \frac{\partial x}{\partial \zeta}(1,t) = 0\\ y(t) &= \int_0^{\frac{1}{2}} x(\zeta,t)d\zeta.\end{aligned}$$

We obtain the transfer function.

Transfer function

We try to find an exponential solution of the p.d.e. This gives the following equations

$$\begin{aligned} s x_0(\zeta) e^{st} &= \frac{d^2 x_0}{d\zeta^2}(\zeta) e^{st} + \mathbb{1}_{[\frac{1}{2},1]}(\zeta) u_0 e^{st} \\ \frac{dx_0}{d\zeta}(0) e^{st} &= \frac{dx_0}{d\zeta}(1) e^{st} = 0 \\ y_0 e^{st} &= \int_0^{\frac{1}{2}} x_0(\zeta) e^{st} d\zeta. \end{aligned}$$

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The first two lines represent an o.d.e. with boundary conditions.

Transfer function

The solution of

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is given as

$$\begin{aligned} x_0(\zeta) &= \cosh(\sqrt{s}\zeta)x_0(0) - \\ &\quad \frac{1}{\sqrt{s}} \int_0^\zeta \sinh(\sqrt{s}(\zeta - \xi))\, \mathbb{1}_{[1/2,1]}(\xi)u_0 d\xi \end{aligned}$$

with

$$x_0(0) = \frac{\sinh(\sqrt{s}/2)u_0}{s \sinh(\sqrt{s})} = \frac{u_0}{2s\cosh(\sqrt{s}/2)}.$$

Transfer function

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Hence the transfer function is given by

$$G(s) = \frac{\tanh(\sqrt{s}/2)}{2s\sqrt{s}}.$$

Transfer function, remark

If you write the solution of the o.d.e.

$$\begin{aligned} sx_0(\zeta) &= \frac{d^2x_0}{d\zeta^2}(\zeta) + \mathbb{1}_{[\frac{1}{2},1]}(\zeta)u_0 \\ \frac{dx_0}{d\zeta}(0) &= \frac{dx_0}{d\zeta}(1) = 0 \end{aligned}$$

as a Fourier cosine series, then you find another expression for the transfer function. Namely,

$$G(s) = \frac{1}{4s} - 2\sum_{n=1}^{\infty} \frac{\sin(n\pi\frac{1}{2})^2}{n^2\pi^2(s+n^2\pi^2)}.$$

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However, the transfer function is unique, and so we find that

$$\frac{\tanh(\sqrt{s}/2)}{2s\sqrt{s}} = G(s) = \frac{1}{4s} - 2\sum_{n=1}^{\infty} \frac{\sin(n\pi\frac{1}{2})^2}{n^2\pi^2(s+n^2\pi^2)}.$$

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Note that it is (almost) the same as the engineering trick of replacing derivative with respect to time by an s.
We can do that for systems with control and observation at the boundary.

Transfer function, boundary control and observation

Example

Consider the system with boundary control and observation

$$\begin{aligned}\frac{\partial w}{\partial t}(\zeta,t) &= \frac{\partial w}{\partial \zeta}(\zeta,t)\\ w(1,t) &= u(t)\\ y(t) &= w(0,t).\end{aligned}$$

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Substituting exponential functions for all signals, gives

$$\begin{aligned}sx_0(\zeta)e^{st} &= \frac{dx_0}{d\zeta}(\zeta)e^{st}\\ x_0(1)e^{st} &= u_0 e^{st}\\ y_0 e^{st} &= x_0(0)e^{st}.\end{aligned}$$

Thus

Example of transfer function with boundary control and observation

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This is an ordinary differential equation with given (end) condition, u_0 and unknown (initial) condition, y_0.

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$$\begin{aligned} s x_0(\zeta) &= \frac{dx_0}{d\zeta}(\zeta) \\ x_0(1) &= u_0 \\ y_0 &= x_0(0). \end{aligned}$$

This is an ordinary differential equation with given (end) condition, u_0 and unknown (initial) condition, y_0.
The solution equals $x_0(\zeta) = e^{s(\zeta-1)}u_0$. Thus $y_0 = e^{-s}u_0$.
The transfer function equals

$$G(s) = e^{-s} \qquad s \in \mathbb{C}.$$

□

Bode and Nyquist plots

Similar like for rational function, we can draw the Bode and Nyquist plot of general transfer functions

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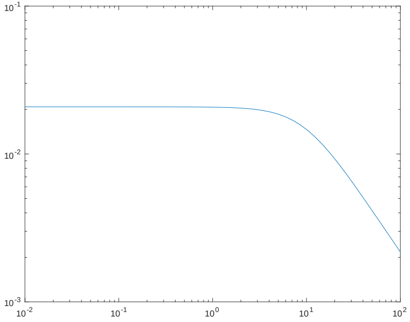
For instance the Bode magnitude plot of

$$G(s) = \frac{\tanh(\sqrt{s}/2)}{2s\sqrt{s}} - \frac{1}{4s}$$



Transfer functions for pH systems

Consider the port-Hamiltonian system with input and outputs

$$\frac{\partial x}{\partial t}(\zeta,t) = \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)[\mathcal{H}x(t)]$$

$$u(t) = W_{B,1}\begin{bmatrix}(\mathcal{H}x)(b)\\(\mathcal{H}x)(a)\end{bmatrix}, \quad 0 = W_{B,2}\begin{bmatrix}(\mathcal{H}x)(b)\\(\mathcal{H}x)(a)\end{bmatrix},$$

$$y(t) = W_C\begin{bmatrix}(\mathcal{H}x)(b)\\(\mathcal{H}x)(a)\end{bmatrix}$$

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\end{aligned}$$

Assume that the energy balance can be expressed in the inputs and outputs. That is

$$\dot{H}(t)=\left[u(t)^\top, y(t)^\top\right] Q \begin{bmatrix}u(t)\\y(t)\end{bmatrix}$$

with Q a symmetric matrix.

Transfer functions for pH systems

Since exponential solutions are solutions, the power balance

$$\dot{H}(t) = \left[u(t)^\top, y(t)^\top\right] Q \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$$

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Transfer functions for pH systems

Hence for the exponential solution the power balance can be written as

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&= \left[u_0^*, u_0^* G(s)^*\right] Q \begin{bmatrix} u_0 \\ G(s) u_0 \end{bmatrix} e^{2\mathrm{Re}(s)t}.
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Transfer functions for pH systems

Since

$$H(t) = \|x(t)\|_X^2 = \langle x(t), x(t) \rangle_X,$$

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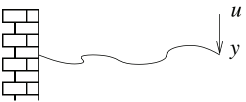
Or equivalently:

$$2\mathrm{Re}(s)\|x_0\|_X^2 = \dot{H}(0) = \left[u_0^*, u_0^* G(s)^*\right] Q \begin{bmatrix} u_0 \\ G(s)u_0 \end{bmatrix}.$$

Transfer functions for pH systems

Example: Wave equation



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\frac{\partial^2 w}{\partial t^2}(\zeta,t) &= \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right] \\
u(t) &= T(1)\frac{\partial w}{\partial \zeta}(1,t), \quad 0 = \frac{\partial w}{\partial t}(0,t) \\
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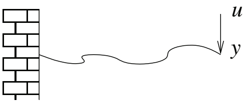

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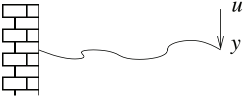

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$$\dot{H}(t) = u(t)y(t) = \left[u(t)^*, y(t)^*\right]\begin{bmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}u(t)\\ y(t)\end{bmatrix}.$$

Transfer function for the vibrating string system

From the general result we find

$$2\mathrm{Re}(s)\|x_0\|_X^2 \quad = \quad \left[u_0^*, u_0^* G(s)^*\right] \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ G(s)u_0 \end{bmatrix}$$

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Since $\|x_0\|_X^2 \geq 0$ and $|u_0|^2 > 0$, we find that for $\mathrm{Re}(s) > 0$ there holds

$$\mathrm{Re}(G(s)) \geq 0$$

Thus G is positive real. □