Modelling and Control of Nonlinear and Distributed Parameter Systems: The port-Hamiltonian Approach

Solutions, existence and properties

Hans Zwart

University of Twente and Eindhoven University of Technology, The Netherlands

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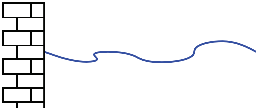
Introduction, wave equation



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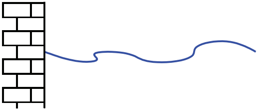

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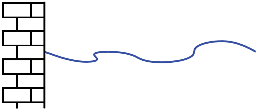

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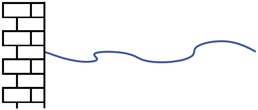

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The wave equation, state differential equation

So with the variables $x_1 = \rho \frac{\partial w}{\partial t}$ and $x_2 = \frac{\partial w}{\partial \zeta}$ the p.d.e. becomes

$$\frac{\partial}{\partial t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (\zeta, t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{=P_1} \frac{\partial}{\partial \zeta} \left(\underbrace{\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}}_{=\mathcal{H}} x(\zeta, t) \right).$$

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We have generalised this to our class of first order port-Hamiltonian equations.

Port-Hamiltonian partial differential equations

Our model class are p.d.e.'s of the form

$$\frac{\partial x}{\partial t}(\zeta, t) \quad = \quad \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right) [\mathcal{H}x(\zeta, t)]$$

with

- $x(\zeta, t) \in \mathbb{R}^n$, $\zeta \in [a, b]$, $t \geq 0$
- P_1 is an invertible, symmetric real $n \times n$-matrix,
- P_0 is a skew-symmetric real $n \times n$-matrix,
- $\mathcal{H}(\zeta)$ is a symmetric, invertible $n \times n$-matrix with $mI \leq \mathcal{H}(\zeta) \leq MI$ for some $m, M > 0$.

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The energy/Hamiltonian is defined as

$$H(t) = H(x(\cdot,t)) = \frac{1}{2}\int_a^b x(\zeta,t)^T \mathcal{H}(\zeta) x(\zeta,t) d\zeta.$$

Power balance

For the Port-Hamiltonian p.d.e. with energy/Hamiltonian

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it is not hard to show that along solutions; homework

$$\dot{H}(t) = \frac{dH}{dt}(x(\cdot,t)) = \frac{1}{2}\left[(\mathcal{H}x)^T(\zeta,t)P_1(\mathcal{H}x)(\zeta,t)\right]_a^b$$

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Thus the change of internal energy goes via the boundary of the spatial domain, i.e. power balance.

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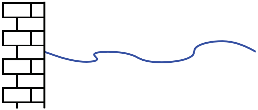

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with ρ is the mass density, and T is Young's modulus.

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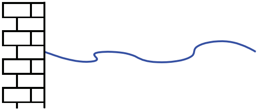

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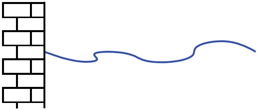

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This indicates that our states must be functions (of the spatial variable)

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$$\frac{1}{2} \int_0^1 \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} d\zeta.$$

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Since $MI > \mathcal{H}(\zeta) > mI$, we see that finite energy condition implies that the state should satisfy for all $t \geq 0$:

$$\int_0^1 \left\| \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} \right\|^2 d\zeta < \infty$$

The wave equation, state and state space

The functions $[0, 1] \ni \zeta \mapsto f(\zeta) \in \mathbb{R}^2$ which satisfy

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This norm is linked with the inner product

$$\langle f, g\rangle_X = \frac{1}{2}\int_0^1 f(\zeta)^\top \mathcal{H}(\zeta) g(\zeta) d\zeta$$

We see that $\|f\|_X^2 = \langle f, f\rangle_X$.

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So based on the energy of our system, we have chosen our state space as $X = L^2((0,1);\mathbb{R}^2)$ with the inner product

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Note that we already rewrote the p.d.e. model of the vibrating string in our state variables.

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The wave equation, change of view point

Instead of seeing the state as a function of time and space, we see it as a function of time (which at each time depends on the spatial variable). So

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Thus (short hand) $x(t) = T(t)x_0$. What properties do we expect from the solution mapping $T(t)$?

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- $T(t)x_0 \in X$ for all $x_0 \in X$. Thus $\|x(t)\|_X < \infty$ whenever $\|x_0\|_X < \infty$. In particular, $\|x(t)\|_X \leq m(t)\|x_0\|$ for some function $m(t)$.

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- For all $x_0 \in X$ there holds

$$\lim_{t \downarrow 0} \|T(t)x_0 - x_0\|_X = 0 \quad \text{or} \quad \lim_{t \downarrow 0} T(t)x_0 = x_0.$$

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This the strong continuity.
It tells that the solution becomes more and more the initial state when time get smaller and smaller.

Semigroup

We introduce some notation. $\mathcal{L}(X)$ denotes the set of linear and bounded operators from X to X. Thus if $Q \in \mathcal{L}(X)$, then

- $Q(\alpha x_0 + \beta \tilde{x}_0) = \alpha Q(x_0) + \beta Q(\tilde{x}_0)$, and
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Definition

A strongly continuous semigroup (C_0-semigroup) is an operator valued function, $(T(t))_{t \geq 0}$, from $[0, \infty)$ to $\mathcal{L}(X)$ which satisfies

- $T(0) = I$
- $T(t)T(s) = T(t+s), \quad t, s \in [0, \infty)$
- For all $x_0 \in X$ there holds

$$\lim_{t \downarrow 0} T(t)x_0 = x_0.$$

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Semigroup, example

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Let A be a (square) matrix, then

$$T(t) := e^{At}$$

is a C_0-semigroup on the state space $X = \mathbb{R}^n$. Homework

Contraction semigroup

Definition

The C_0-semigroup $(T(t))_{t\geq 0}$ is contraction semigroup if

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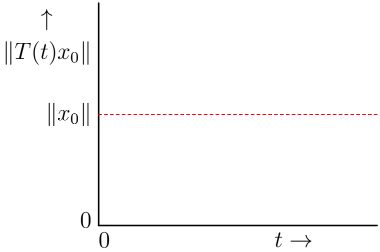

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For our port-Hamiltonian equation we have that the state is directly linked to the energy. So the p.d.e. must tell us what happens with the energy.
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So the boundary conditions must tell us what happens with this term.

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We need to add boundary conditions to this p.d.e. That are conditions in $x(\zeta, t)$ for ζ equal to a or b.

Port-Hamiltonian partial differential equations

Given our port-Hamiltonian partial differential equation

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We answer this question by using semigroup theory. However, we do it only for contraction semigroups.

Port-Hamiltonian p.d.e., existence of solutions

Theorem (Le Gorrec, Maschke & Zwart '05, Jacob & Zwart '11)

Assume the (standard) conditions on P_0, P_1 and $\mathcal{H}$. Assume further that W_B is a $n \times 2n$ matrix of full rank.

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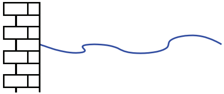
The solution map is a unitary C_0-group (i.e. $\|T(t)x_0\| = \|x_0\|$, $\forall x_0, \forall t$) if and only if

$$\dot{H} = 0.$$

Example: the wave equation



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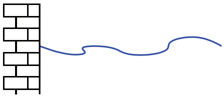

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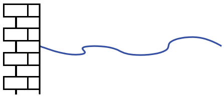

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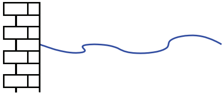

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Now we check the conditions.

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- If $0 < m \leq T(\zeta), \rho(\zeta)^{-1} \leq M$ for all ζ, then $\mathcal{H}(\zeta) = \begin{bmatrix} \rho(\zeta)^{-1} & 0 \\ 0 & T(\zeta) \end{bmatrix}$ satisfy $mI_2 \leq \mathcal{H}(\zeta) \leq MI_2$.

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- W_B has rank 2.
- $\dot{H} = 0$.

Thus the solution map is a unitary group on the energy space.

Summary

What we have introduced is:

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- Question: How to formulate state/state space for a partial differential equation?
  We study an example first.

Example (Transport equation)

On the spatial domain $[0, 1]$ consider the p.d.e.

$$\begin{aligned}
\frac{\partial w}{\partial t}(\zeta, t) &= \frac{\partial w}{\partial \zeta}(\zeta, t), \qquad \zeta \in [0, 1],\ t \geq 0, \\
w(1, t) &= 0 \\
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- If we now introduce $\dot{x}(t)=\frac{\partial w}{\partial t}(\cdot,t)$ and $Ax(t)=\frac{\partial w}{\partial \zeta}(\cdot,t)$, then the p.d.e. becomes

$$\dot{x}(t)=Ax(t).$$

State differential equation

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can with $x(t) = w(\cdot,t)$, $\dot{x}(t) = \frac{\partial w}{\partial t}(\cdot,t)$, and $Ax(t) := \frac{\partial w}{\partial \zeta}(\cdot,t)$, be written as abstract differential equation:

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Another problem: The (spatial) derivative does not exist for all $x(t)\in L^2(0,1)$.

More on A

We see that A is a mapping working for a fixed t, i.e., so for $f \in L^2(0,1)$ we can define Af as

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Summary on A

So the p.d.e.

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is written as the abstract differential equation:

$$\dot{x}(t) = Ax(t), \qquad x(0)=x_0=w_0$$

with $x(t)=w(\cdot,t)\in X=L^2(0,1)$, and

$$\left(Af\right)(\zeta) = \frac{df}{d\zeta}(\zeta)$$

with domain:

$$D(A)=\{f\in X \mid \frac{df}{d\zeta}\in X, f(1)=0\}.$$

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Given our port-Hamiltonian partial differential equation with boundary conditions

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)[\mathcal{H}(\zeta)x(\zeta,t)] \\ 0 &= W_B \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix} \end{aligned}$$

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- As state we choose $x(t) = x(\cdot, t)$.
- As state space we choose the energy space, i.e., $X = L^2((0, 1); \mathbb{R}^n)$ with inner product

$$\langle f, g \rangle_X = \frac{1}{2} \int_a^b f(\zeta)^\top \mathcal{H}(\zeta) g(\zeta) d\zeta.$$

Port-Hamiltonian p.d.e., state space formulation

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$$\dot{x}(t) = Ax(t),$$

where

$$Ax = \left(P_1 \frac{d}{d\zeta} + P_0\right)[\mathcal{H}x]$$

with domain

$$D(A) = \left\{ x \in X \mid \frac{d}{d\zeta}(\mathcal{H}x) \in X, W_B \begin{bmatrix} \mathcal{H}(b)x(b) \\ \mathcal{H}(a)x(a) \end{bmatrix} = 0 \right\}.$$

A and $T(t)$

We have now written our p.d.e.'s as

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What is their relation?
We begin by studying the question when X is finite-dimensional.

Finding A

Let A be given as

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix},$$

then

$$e^{At} = \begin{pmatrix} e^t & e^{3t} - e^t \\ 0 & e^{3t} \end{pmatrix}.$$

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Answer Evaluate the derivative of the semigroup at $t = 0$.
Since $\frac{d}{dt} e^{At} = A e^{At}$, we have

$$\frac{d}{dt} e^{At} \mid_{t=0} = A.$$

A and $T(t)$

Theorem

Assume that $(T(t))_{t\geq 0}$ is the solution map of our p.d.e., then for those $x_0 \in X$ for which the following limit exists

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A is named the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$. □

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Lemma

If $x_0 \in D(A)$, then for $t > 0$, $T(t)x_0$ is differentiable, and

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For $x_0 \in X$, $T(t)x_0$ is called a weak solution. $\square$

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We do this for a special class of C_0-semigroups.

Recall: Contraction semigroup

Definition

The C_0-semigroup $(T(t))_{t\geq 0}$ is contraction semigroup if

$$\|T(t)x_0\| \leq \|x_0\| \qquad \text{for all } t \geq 0 \text{ and for all } x_0 \in X.$$

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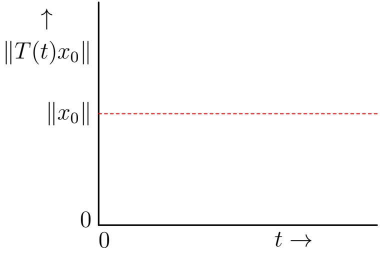

Contraction semigroup

We known that

$$\|T(t)x_0\|^2 = \langle T(t)x_0, T(t)x_0 \rangle.$$

For $x_0 \in D(A)$, we have that the derivative of $T(t)x_0$ equals $AT(t)x_0$.
So if we differentiate $\|T(t)x_0\|^2$, we find

$$\frac{d}{dt}\|T(t)x_0\|^2 = \langle AT(t)x_0, T(t)x_0 \rangle + \langle T(t)x_0, AT(t)x_0 \rangle.$$

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Now we choose $t = 0$. We know that $T(0)x_0 = x_0$. Thus at time equal to zero, we find

$$\frac{d}{dt}\left(\|T(t)x_0\|^2\right)\Big|_{t=0} = \langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle.$$

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So if $T(t)$ is a contraction semigroup, then

$$\langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle = \frac{d}{dt}\|T(t)x_0\|^2 \,|_{t=0} \leq 0.$$

This has to hold for all $x_0 \in D(A)$.

Contraction semigroup

Theorem (Lumer-Phillips)

Let A be a densely defined operator, then A generates a contraction semigroup on X if and only if

1. $\langle Ax_0, x_0\rangle + \langle x_0, Ax_0\rangle \leq 0$ *for all* $x_0 \in D(A)$.
2. *The range of* $A - I$ *is the whole of* X.

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Condition 1 comes from $\frac{d}{dt}\|T(t)x_0\|^2 \leq 0$. So for pH this is equivalent to $\dot{H}(t) \leq 0$. Note that Condition 2 seems to be missing in our existence theorem for pH systems.

Contraction semigroup

Example

Consider on the state space $X = L^2(0,1)$ the operator A which is given as

$$Af = \frac{df}{d\zeta}, \qquad \zeta \in [0,1]$$

with the domain

$$D(A) = \Big\{ f \in L^2(0,1) \mid f \text{ is absolutely continuous}, \frac{df}{d\zeta} \in L^2(0,1) \text{ and } f(1) = 0 \Big\}.$$

Let us check the properties:

Example: Contraction semigroup

- A is densely defined in $L^2(0,1)$.

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$$=$$

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$$\begin{aligned}\langle Ax, x\rangle + \langle x, Ax\rangle \\ &= \int_0^1 \frac{dx}{d\zeta}(\zeta)\overline{x(\zeta)}d\zeta + \int_0^1 x(\zeta)\overline{\frac{dx}{d\zeta}(\zeta)}d\zeta\end{aligned}$$

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- A is densely defined in $L^2(0,1)$.
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\langle Ax, x\rangle + \langle x, Ax\rangle \\
&= \int_0^1 \frac{dx}{d\zeta}(\zeta)\overline{x(\zeta)}d\zeta + \int_0^1 x(\zeta)\overline{\frac{dx}{d\zeta}(\zeta)}d\zeta \\
&= \int_0^1 \frac{d}{d\zeta}\left[x(\zeta)\overline{x(\zeta)}\right] d\zeta \\
&= |x(\zeta)|^2\Big|_0^1 \\
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- $$\begin{aligned}\langle Ax, x\rangle + \langle x, Ax\rangle \\ &= \int_0^1 \frac{dx}{d\zeta}(\zeta)\overline{x(\zeta)}d\zeta + \int_0^1 x(\zeta)\overline{\frac{dx}{d\zeta}(\zeta)}d\zeta \\ &= \int_0^1 \frac{d}{d\zeta}\left[x(\zeta)\overline{x(\zeta)}\right] d\zeta \\ &= |x(\zeta)|^2\Big|_0^1 \\ &= 0 - |x(0)|^2 \leq 0.\end{aligned}$$
- To see if the range of $(A - I)$ is everything, we have for every $f \in L^2(0,1)$ to solve $(A-I)x = f$.

Example: Contraction semigroup

Solving $(A - I)x = f$ means solving

$$\frac{dx}{d\zeta}(\zeta) - x(\zeta) = f(\zeta), \qquad \zeta \in (0, 1)$$

with boundary condition $x(1) = 0$.

Example: Contraction semigroup

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with boundary condition $x(1) = 0$. The solution of this differential equation with the given boundary value is

$$x(\zeta) = -\int_{\zeta}^{1} e^{\zeta - \xi} f(\xi) d\xi.$$

□

Example: Contraction semigroup

Conclusion:

$$Af = \frac{df}{d\zeta}, \qquad \zeta \in [0,1]$$

with the domain

$$D(A) \;=\; \left\{ f \in L^2(0,1) \mid \frac{df}{d\zeta} \in L^2(0,1) \text{ and } f(1) = 0 \right\}$$

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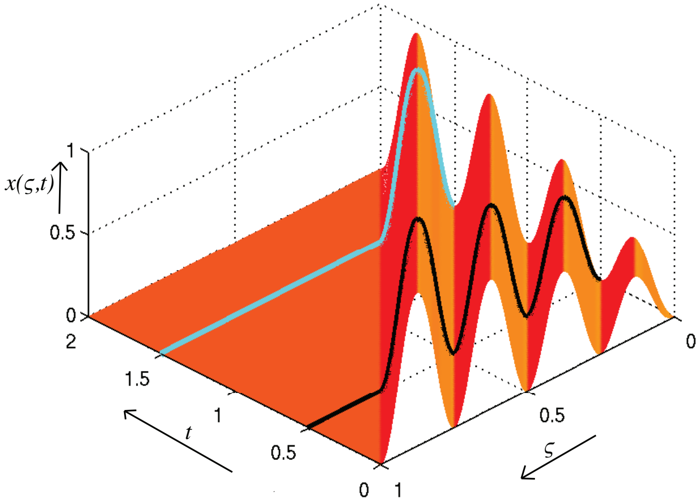
Note that A can also been seen as a pH system! Homework

States



Port-Hamiltonian Systems

Inputs and Outputs

Port-Hamiltonian systems with inputs and outputs

We are interested in boundary controls and boundary observations.

$$\frac{\partial x}{\partial t}(\zeta, t) = \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)[\mathcal{H}x(t)]$$

$$u(t) = W_{B,1}\begin{bmatrix}(\mathcal{H}x)(b)\\(\mathcal{H}x)(a)\end{bmatrix}, 0 = W_{B,2}\begin{bmatrix}(\mathcal{H}x)(b)\\(\mathcal{H}x)(a)\end{bmatrix}, y(t) = W_C\begin{bmatrix}(\mathcal{H}x)(b)\\(\mathcal{H}x)(a)\end{bmatrix}$$

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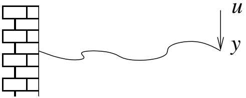
Example: Wave equation



$$\begin{aligned} \frac{\partial^2 w}{\partial t^2}(\zeta, t) &= \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left[T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t)\right] \\ u(t) &= T(1) \frac{\partial w}{\partial \zeta}(1, t), \quad 0 = \frac{\partial w}{\partial t}(0, t) \\ y(t) &= \frac{\partial w}{\partial t}(1, t) \end{aligned}$$

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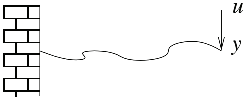

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Question: Is this a well-posed linear system?

Well-posedness of port-Hamiltonian systems

State space $X = L^2((a,b);\mathbb{R}^n)$ with (the energy) norm

$$\|f\|_X^2 = \frac{1}{2}\int_a^b f(\zeta)^T \mathcal{H}(\zeta) f(\zeta) d\zeta.$$

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The port-Hamiltonian system is called well-posed, if

- $Ax = P_1 \frac{d}{d\zeta}\left[\mathcal{H}x\right] + P_0\left[\mathcal{H}x\right]$ with domain
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  is the generator of a C_0-semigroup on X.
- There are $t_0, m_{t_0} > 0$:
  $$\|x(t_0)\|_X^2 + \int_0^{t_0} \|y(t)\|^2 dt \le m_{t_0} \left[\|x(0)\|_X^2 + \int_0^{t_0} \|u(t)\|^2 dt \right]$$

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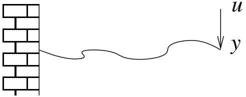
Remark: We even have a regular system.

Example: Wave equation



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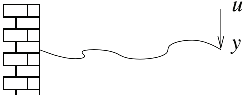

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$$P_1\mathcal{H} = \begin{bmatrix} 0 & T \\ \frac{1}{\rho} & 0 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma \\ \frac{1}{\rho} & \frac{1}{\rho} \end{bmatrix} \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix} \begin{bmatrix} \frac{1}{2\gamma} & \frac{\rho}{2} \\ -\frac{1}{2\gamma} & \frac{\rho}{2} \end{bmatrix} = S^{-1}\Delta S,$$

with $\gamma > 0$ und $\gamma^2 = \frac{T}{\rho}$.

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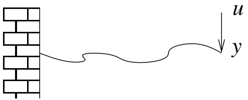

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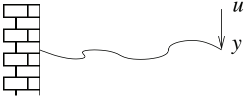

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So if T and ρ are continuously differentiable, then the controlled wave equation is well-posed.

Exercises

1. Show that for a pH system there holds:

$$\dot{H}(t) = \frac{dH}{dt}(x(\cdot,t)) = \frac{1}{2}\left[(\mathcal{H}x)^T(\zeta,t)P_1(\mathcal{H}x)(\zeta,t)\right]_a^b$$

2. Show that e^{At} is a C_0-semigroup, when A is a (square) matrix.
3. Show that $Af = \frac{df}{d\zeta}$ with domain $D(A) = \{f \in L^2(0,1) \mid f$ is such that $\frac{df}{d\zeta} \in L^2(0,1)$ and $f(1) = 0\}$ can be associated to a pH system.

Exercise

4
- a Show that the connected wave equations shown below can be written as a pH system,
- b Show that for no force ($u = 0$) we have that the solution map is a contraction semigroup.
- c Assume that we measure the velocity of the (vertical moving) middle bar. Show that the system is well-posed.